On singular non-holonomic geometry

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Abstract. The aim of the paper is to prove that every smooth generalized vector subbundle $\mathcal{D}$ of a vector bundle E is the image of a smooth endomorphism Φ on the fibers of E that induces authomorphisms of fibers of $\mathcal{D}$ and is called here a natural anchor. Two new constructions of finite sets of smooth generators of the fibers of $\mathcal{D}$ are obtained, using any finite set of generators of the module $\Gamma(E)$. Natural anchors on generalized subbundles and almost regular Dirac subbundle of the Pontryagin vector bundle $P(E)$ are constructed.

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1 Introduction

A generalized vector subbundle $\mathcal{D}$, according to [2], or a singular subbundle in [9] of a vector bundle $\pi : E \to M$ is a collection of vector subspaces $\mathcal{D}_x$ of fibers E_x in all the points x of the base M. Then $\mathcal{D}$ is a smooth one if every vector in its fiber is a restriction of a smooth local section that is a section of the generalized vector subbundle as well. Basic aspects of smooth generalized vector subbundles, or smooth singular subbundles, are discussed in [1, 2, 3, 9]. According to [2], a cosmooth vector subbundle $\mathcal{D}$ of a vector bundle E is a generalized vector subbundle of E such that its annihilator $\mathcal{D}^\perp \subset E^*$ is smooth. But other generalized vector subbundles can be associated with a given $\mathcal{D}$. For example, we prove that if $\mathcal{D}$ is smooth, then the set $End_{\mathcal{D}}(E)$, of endomorphisms of the fibers of E that have their image in the fibers of $\mathcal{D}$, is a smooth generalized vector subbundle of $End(E)$, the vector bundle of endomorphisms on the fibers of E (Proposition 3.5). In Theorem 4.1, the main result of the paper, we prove in fact that this smooth subbbundle has a global smooth section, called a natural anchor, having its image onto the fibers of $\mathcal{D}$. Considering smooth morphisms of smooth vector subbundles as restrictions of morphisms of vector bundles, we prove that the natural anchors we construct are restricting to automorphisms of $\mathcal{D}$. Moreover, if $\mathcal{D}'' \subset \mathcal{D}' \subset E$ are smooth (we say that $\mathcal{D}''$ is a smooth subbundle of $\mathcal{D}'$), then the natural anchors Φ' and Φ'' constructed

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on $\mathcal{D}'$ and $\mathcal{D}''$ respectively have the property that $\Phi'' \circ \Phi'$ restricts on $\mathcal{D}'$ to an onto map on the fibers of $\mathcal{D}''$ (Proposition 4.6).

These facts can be extended to constructions in tensor vector bundles or in the duality smooth-cosmooth, but these facts are beyond the scope of this work, where we focus mainly on some geometric properties derived from orthogonal projections on the fibres of smooth generalized vector subbundles.

There are some important differences between the regular vector subbundles and their generalized smooth counterparts. One of the major differences is the smoothness of orthogonality. The basic constructions in the paper use a scalar product g on the fibres of E. The g-orthogonality related to a generalized vector subbundle is somewhat different from the classical notion of orthogonality. For example, the orthogonal $\mathcal{D}^{\perp_g} \subset E$ of the smooth $\mathcal{D} \subset E$ is isomorphic with $\mathcal{D}^{\perp}$ and consequently it is a cosmooth generalized vector subbundle. Instead of this, we consider the smooth orthogonal $\mathcal{D}^{\vdash_g}$ as the smooth one generated by $\mathcal{D}^{\perp_g}$ and the smooth orthogonal completion $\mathcal{D}^{\models_g} = \mathcal{D} + \mathcal{D}^{\vdash_g}$; some of properties of $\mathcal{D}^{\vdash_g}$ and $\mathcal{D}^{\models_g}$ are stated in Propositions 2.1 and 4.7. For example, the subset of M of maximal dimension for $\mathcal{D}^{\models_g}$is a dense open subset of the base M, where the fibers of $\mathcal{D}^{\models_g}$ and E are the same. Since in the case when $\mathcal{D}$ is singular, the smooth orthogonal can not be equal with the cosmooth orthogonal, then $\mathcal{D}^{\models_g}$ can not be E. A special natural sum anchor is considered on $\mathcal{D}^{\models_g}$ in Proposition 4.7. In the case of a regular vector subbundle $\mathcal{D} \subset E$, the constructions fit in the well-known case: $\mathcal{D}^{\vdash_g} = \mathcal{D}^{\perp_g}$ and $\mathcal{D}^{\models_g} = E$. (See 4. of Proposition 2.1.) These results give Propositions 4.8 and 4.9, where the Pontryagin bundle $P(E) = E \oplus E^*$ of the vector bundle E is used as support for Dirac type subbundles. Related to almost regular Dirac structures of the Pontryagin vector bundle $P(E)$ of a vector bundles there is proved that there are two natural transverse anchors that give an adapted anchor for $\mathcal{D}$ (Proposition 4.8) and that a Dirac vector subbundle can be induced on a dense open subset of the base (Proposition 4.9). We notice that the basic orthogonal used in this paper comes from a Riemannian metric on the fibers of E; it is different from the orthogonal considered in [4], according to the canonical pseudo-Riemannian metric of signature (k, k) in the fibers of the Pontryagin bundle, where k is the dimension of the fibers of E.

Some basic tools used in the paper are classical results of Whitney and properties of extension of smooth sections on closed subsets (see [6, 7, 10]), but in a slight different form. A first fact is the existence of a test function φ_{M_0} for any closed subset $M_0 \subset M$, i.e. a positive smooth real function on M, bounded by $[0, 1]$ and having the set of zeros exactly M_0, where all its differentials also vanish (Proposition 3.3). An other basic ingredient used in the paper, in the spirit of [5, 6], is a smooth section s_{M_0} defined on a closed subsets M_0 of the base M of E (the section is smooth if it is locally the restriction of a local smooth section of the vector bundle E on an open subset of M) and the property of s_{M_0} to extend to a smooth section on the whole M (Proposition 3.4).

In the points of $\Sigma_{\min}$, i.e. of minimal dimension for $\mathcal{D}$, the orthogonal projection on the fibres of $\mathcal{D}$ is smooth, extending to a global endomorphism (Proposition 3.6). We prove also that for any constant dimension level $\Sigma_r \subset M$ of $\mathcal{D}$, the orthogonal projection on the fibres of $\mathcal{D}$ is smooth, modulo a test function, extending also to a global endomorphism (Proposition 3.6). The sum of all the endomorphisms corresponding to all the constant dimension level of $\mathcal{D}$ give the natural anchor Φ of Theorem 4.1. The surjection of Φ on $\mathcal{D}$, as well as of the restriction $\Phi'' \circ \Phi' : \mathcal{D}' \to \mathcal{D}''$, for $\mathcal{D}'' \subset \mathcal{D}' \subset E$, given by Proposition 4.6, follow using some technical Lemmas 4.2 and 4.3.

Using the above construction, a corresponding finite set of smooth generators of the fibres of $\mathcal{D}$ can be considered, starting from a given finite set of generators of $\Gamma(E)$ (Proposition 3.9 and Corollary 4.5). This result is not new, proved in [2, Theorem 4.1] and in [9, Theorem 1.]. But our proof uses different arguments, relating a finite set of global generators of $\Gamma(E)$ with a set of global generators on the fibres of $\mathcal{D}$, using orthogonal projections (in Proposition 3.9) or sums of orthogonal projections (in Corollary 4.5), both modulo multiplications with test functions. The number of generators we obtain in Corollary 4.5 is equal to the number of the smooth global generators of $\Gamma(E)$, as in [9, Theorem 1.]. In [2], this number is multiplied by the maximal rank of the fibers of $\mathcal{D}$. Notice that in [2] it is pointed out that in general, when $\mathcal{D}$ is not a regular vector subbundle, there is not a finite set of generators for $\Gamma(\mathcal{D})$.

The content of the paper is as follows. In section 2 we give some preliminaries, consisting of basic definitions and notations, as well as a short view of orthogonality concerning generalized vector subbundles. In section 3 we prove the existence of test functions and we use them to construct certain global smooth prolongations and smooth projections. Using these constructions, the main Theorem 4.1 and some related properties are proved in the last section.

2 Generalities

A vector bundle is denoted by $\pi : E \to M$, or E for short, when no confusion is possible.

A *generalized vector subbundle* (a g.v.s. for short), according to [2] (or a *singular vector subbundle* in [9]) of the vector bundle E is a subset $\mathcal{D} \subset E$ such that there is an assignment of a vector subspace $\mathcal{D}_x = \mathcal{D} \cap E_x \subset E_x = \pi^{-1}(x) \subset E$ to every $x \in M$.

A vector $X_x \in \mathcal{D}_x$ is *allowed* if there is a smooth section Y of $\mathcal{D}_{|U_x}$ on an open neighborhood U_x of x, such that $Y_x = X_x$. Denote by $\mathcal{A}(\mathcal{D})_x \subset \mathcal{D}_x$ the set of allowed vectors in x. The null vector $\bar{0}_x \in E_x$ is obviously allowed since $\bar{0}_x \in \mathcal{A}(\mathcal{D})_x$, thus $\mathcal{A}(\mathcal{D})_x$ is non-void. It is easy to see that $\mathcal{A}(\mathcal{D}) = \underset{x\in M}{\cup} \mathcal{A}(\mathcal{D})_x$ is a generalized vector subbundle. A g.v.s. $\mathcal{D}$ is *smooth* if $\mathcal{A}(\mathcal{D}) = \mathcal{D}$. In general, for an arbitrary $\mathcal{D}$, $\mathcal{A}(\mathcal{D})$ is smooth, according to its construction.

Let us observe that if $\mathcal{D}_1$ and $\mathcal{D}_2$ are smooth g.v.s.'s of E, then $\mathcal{D}_1 + \mathcal{D}_2 \subset E$ is also a smooth g.v.s.

Let E and E' be two vector bundles over the same base M. If $\mathcal{D} \subset E$ and $\mathcal{D}' \subset E'$ are g.v.s.'s, a morphism of g.v.s. is $f : \mathcal{D} \to \mathcal{D}'$ covered by a collection of linear maps $f_x : \mathcal{D}_x \to \mathcal{D}'_x$, $(\forall)x \in M$, such that there is a linear morphism of vector bundles $F : E \to E'$ that restricts to f in every fiber. An isomorphism is obviously given by the existence of a couple of inverse morphisms $f : \mathcal{D} \to \mathcal{D}'$ and $f^{-1} : \mathcal{D}' \to \mathcal{D}$ that are mutually inverse. Notice that in this case the extending linear morphisms F of f and F' of f^{-1} may be not necessary isomorphisms, but an isomorphism of E and E' induces an isomorphism of g.v.s.

Let $\mathcal{D} \subset E$ be a g.v.s. of E. Consider a Riemannian metric g on the fibers of E and let $\mathcal{D}^{\perp_g} \subset E$ be the orthogonal g.v.s., i.e. $\mathcal{D}_x^{\perp_g} = (\mathcal{D}_x)^{\perp_g}$, $(\forall)x \in M$. Then $(\forall)x \in M$, the vector space $\mathcal{D}_x^{\perp_g}$ is canonically isomorphic with the annihilator $\mathcal{D}_x^{\perp} = \{\omega_x \in E_x^*, \omega_x(X_x) = 0, (\forall) X_x \in \mathcal{D}_x\}$, via the musical isomorphism $\cdot^{\#} : E \to E^*$ given by the metric g.

Let us observe that two orthogonal g.v.s., corresponding to two different Riemannian metrics, are both isomorphic to the annihilator, thus they are isomorphic.

A morphism $f : \mathcal{D} \to \mathcal{D}'$ of g.v.s., induced by $F : E \to E'$, restricts to a morphism $f_{\mathcal{A}} : \mathcal{A}(\mathcal{D}) \to \mathcal{A}(\mathcal{D}')$, induced by the same F.

Also, if $\mathcal{D}$ and $\mathcal{D}'$ are isomorphic, then $\mathcal{A}(\mathcal{D})$ and $\mathcal{A}(\mathcal{D}')$ are isomorphic via the same linear morphisms.

If $\mathcal{D} \subset E$ is a smooth g.v.s. and g is a Riemannian metric on the fibers of E, then we say that

– $\mathcal{D}^{\perp_g}$ is a *cosmooth orthogonal* of $\mathcal{D}$,
– $\mathcal{D}^{\vdash_g} = \mathcal{A}(\mathcal{D}^{\perp_g}(\mathcal{D})) \subset E$ is a *smooth orthogonal* of $\mathcal{D}$ and
– $\mathcal{D}^{\models_g} = \mathcal{D} + \mathcal{D}^{\vdash_g}$ is a *smooth orthogonal completion* of $\mathcal{D}$.

Proposition 2.1. *For a smooth $\mathcal{D}$, the following properties hold:*

1. *– the smooth orthogonal of $\mathcal{D}^{\models_g}$ is null (i.e. $(\mathcal{D}^{\models_g})^{\vdash_g} = \bar{0}$) and consequently*
2. *– the smooth orthogonal completion of $\mathcal{D}^{\models_g}$ is $\mathcal{D}^{\models_g}$ itself (i.e. $(\mathcal{D}^{\models_g})^{\models_g} = \mathcal{D}^{\models_g}$);*
3. *– the smooth orthogonal completion $\mathcal{D}^{\models_g} = \mathcal{D} + \mathcal{D}^{\vdash_g}$ has the property that its maximal dimension of the fibers is $m = \dim M$ and is taken on an open dense subset of M;*
4. *– in the case when $\mathcal{D}$ has a regular dimension r, then $\mathcal{D}^{\perp_g} = \mathcal{D}^{\vdash_g}$ and $\mathcal{D}^{\models_g} = E$.*

Notice that a simple consequence of 3. of Proposition 2.1 above is that given the smooth generalized vector subbundles $\{\mathcal{D}_i\}_{i=\overline{1,n}}$ of some vector bundles over M, then the intersection $\Sigma = \bigcap\limits_{i=1}^{n} \Sigma^i_{\max}$ of maximal dimensions $\{\Sigma^i_{\max}\}$ of orthogonal completions $\left\{\mathcal{D}_i^{\models_g}\right\}$ is an open dense subset of M.

For a generalized vector subbundle $\mathcal{D}$, we denote by $r(x) = \dim \mathcal{D}_x$, for $x \in M$, $\mathcal{R} = \{r(x) : x \in M\}$. If $S \subset M$, then $\mathcal{D}_S = \bigcup\limits_{x \in S} \mathcal{D}_x$ is the restriction of $\mathcal{D}$ to S. Consider

$$\mathcal{R} = \{r(x) : x \in M\} = \{r_i\}_{i=\overline{0,k}}, \tag{2.1}$$

where

$$r_{\min} = r_0 < r_1 < \cdots < r_k = r_{\max}. \tag{2.2}$$

For $r_i \in \mathcal{R}$, we denote by

$$\Sigma_{r_i} = \{x \in M : \dim \mathcal{D}_x = r_i\}, \tag{2.3}$$

and also $\Sigma_{<r_i} = \{x \in M : \dim \mathcal{D}_x < r_i\}$, $\Sigma_{\le r_i} = \{x \in M : \dim \mathcal{D}_x \le r_i\} = \Sigma_{r_i} \cup \Sigma_{<r_i}$, $\Sigma_{>r_i} = \{x \in M : \dim \mathcal{D}_x > r_i\}$, $\Sigma_{\ge r_i} = \{x \in M : \dim \mathcal{D}_x \ge r_i\} = \Sigma_{r_i} \cup \Sigma_{>r_i}$. We say that the subset $\Sigma_{r_{\min}}$ is the *minimal set* and $\Sigma_{r_{\max}}$ is the *maximal set*. The subsets $\Sigma_{<r_i}$ and $\Sigma_{\le r_i}$ are closed and their complements, the sets $\Sigma_{\ge r_i}$ and $\Sigma_{>r_i}$ are open in

M. The subset $\Sigma_{r_i} \subset \Sigma_{\geq r_i}$ is the minimal subset of $\mathcal{D}_{|\Sigma_{\geq r_i}}$ and $\Sigma_{>r_i}$ is void if $i = k$ and is equal to $\Sigma_{\geq r_{i+1}}$ if $0 \leq i < k$.

Let us denote by $\Sigma_i^{\mathcal{D}}$ the set Σ_i of $\mathcal{D}$, and $\mathcal{R}^{\mathcal{D}} = \{r^{\mathcal{D}}(x) : x \in M\} = \{r_i\}_{i=\overline{0,k}}$, where $k = \max \mathcal{R}^{\mathcal{D}}$. The set $\Sigma_k^{\mathcal{D}} \subset M$ is open; if it is also a dense set in M, then the set $\Sigma_{m-k}^{\mathcal{D}^{\vdash_g}}$ contains the set $\Sigma_k^{\mathcal{D}}$, thus it is also a dense set in M, thus $m - r_k$ is the maximal dimension of the g.v.s. $\mathcal{D}^{\vdash_g}$. For example, it is the case when $\mathcal{D}$ is tangent to the leaves of a singular Riemannian foliation.

3 Sections and test functions

We say that a real function $\varphi \in \mathcal{F}(M)$ is a *test function* for a closed set $M_0 \subset M$ if $\varphi(x) = 0$ iff $x \in M_0$, its values are in $[0,1]$ and all its differentials vanish in every $x \in M_0$. The proof of the next simple result follows by induction on the order of the partial derivatives in a coordinate chart.

Lemma 3.1. *Let $\psi_0 : I\!R \to [0,1]$ be smooth such that $\psi_0(t) = 0$ iff $t = 0$ and all the derivatives of ψ_0 vanish in $t = 0$. Then for every function $f : M \to I\!R$ the function $F = \psi_0 \circ f$ has the same zeros as f and all the differentials of F vanish in its zeros.*

Let $\pi : E \to M$ be a vector bundle that is a vector subbundle of a trivial vector bundle $M \times I\!R^m \to M$. A section $s \in \Gamma(E)$ can be considered as a smooth function $s : M \to I\!R^m$ and it is *bounded* if its values are bounded according to the canonical Euclidean norm $|\cdot|$ of $I\!R^m$. If φ is a test function for a closed subset $M_0 \subset M$ and v' is a bounded section (as above) from $\Gamma(E_{|M\setminus M_0})$, then defining $v = \varphi v'$ on M_1 and $v = 0$ on M_0, then v is smooth on M. For this, one can use a simple result that asserts that *bounded$\cdot$* (*smooth zero*) = *smooth zero*. More precisely we have:

Lemma 3.2. *1. Let $U \subset I\!R^n$ be open, $x_0 \in U$ and $\varphi, \psi : U \to I\!R$, such that φ vanishes together with all its partial derivatives in x_0 and ψ is bounded on U. Then the function $\varphi_0 = \varphi \cdot \psi$ vanishes as well together with all its partial derivatives in x_0.*

2. Let us suppose that a real function φ is smooth on U and all its partial derivatives vanish on the set U_0 of the zeros of φ. Consider also a real and bounded function ψ on U that is smooth on $U \setminus U_0$. Then the function $\varphi \cdot \psi$ is smooth on U.

Given a closed subset $M_0 \subset M$, it is a classical result, due to Whitney that there is a smooth real function having M_0 the set of zeros (the proof combines the embedding theorem with his extension theorems in [10]). An other proof can be found for example in [7, Proposition IV.1.1], and we can give a slight different one, but using a similar way, inspired by [2, Section 4].

We say that two test function φ and φ' are *equivalent* if there is a real function ψ on M that is null in no point, such that $\varphi' = \psi\varphi$. It is easy to see that the zero set of φ and φ' is the same close subset of M. The existence of test functions is an important tool used in the sequel.

Proposition 3.3. *Let M be a differentiable manifold and $M_0 \subset M$ be a closed subset. Then there is a test function for M_0.*

Notice that the construction performed in the proof above is too theoretical to be effectively used. More simple constructions, applicable in some particular cases, can be considered. For example, the following construction.

Let us consider the open set $M_1 = M|M_0$. There is an open cover $\mathcal{U}_1$ of M_1 that consists of the open sets $U \subset M_1$ such that there is a local chart (V, θ), $\bar{U} \subset V \subset M_1$, $\theta(U) = B(\bar{0}, 1) \subset I\!\!R^n$. For such an U, consider a bump function $f_U \in \mathcal{F}(M)$ such that $0 < f_U(x) \leq 1$, $(\forall)x \in U$ and $f_U(x) = 0$, $(\forall)x \in M \backslash U$. Since M_1 is paracompact, one can find an at most countable open refinement $\{W_n\}_{n \geq 1}$ of $\mathcal{U}$, that is locally finite. Considering $W_n \subset U_n$ and $f_n = F_{U_n}$, $(\forall)x \in I\!\!N$, we define $\psi(x)$ as the sum of all $f_n(x)$, such that $x \in W_n$; the sum is finite, since the open cover $\{W_n\}_{n=1,\ldots}$ is locally finite. From its construction, the function ψ is locally differentiable on $M \backslash M_0$ and null outside the complement of the closure of $M \backslash M_0$. Thus ψ is differentiable iff it is differentiable on $\partial(M \backslash M_0) = \partial M_0$. It is the case, for example, when every point $x_0 \in \partial M_0$ belongs to a finite number of sets from $\{\partial \bar{U}_n\}_{n \geq 1}$. Also using Lemma 3.1, we obtain a test function for M_0.

Let E be a vector bundle over the base M, $M_0 \subset M$ be a closed subset and $\pi : E \to M$ be the canonical projection. Let us denote by $E_{M_0} = \underset{x \in M_0}{E_x}$ and $\pi_{M_0} : E_{M_0} \to M_0$ the restriction of π to M_0. Notice that, in general, M_0 and E_{M_0} need not to be manifolds. A section s_0 of E_{M_0} is usually defined as a map $M_0 \ni x \stackrel{s_0}{\to} s_0(x) \in E_x$; it is *smooth* if every point $x \in M_0$ has an open neighborhood $U_0 \subset M$ and there is smooth section $S_0 : U_0 \to E_{U_0}$ such that $S_{0|M_0} = s_0$, i.e. the restriction of S_0 to $U_0 \cap M_0$ is s_0 (se [6] for more details).

Proposition 3.4. *Let $M_0 \subset M$ be a closed subset and $s_0 : M_0 \to E_{M_0}$ be a smooth section. Then there is a smooth section $S_0 : M \to E$ that extends s_0.*

Let $\mathcal{D} \subset E$ be a smooth generalized subbundle. The subset $\Sigma_{\min} \subset M$ of minimal dimension of the fibers of $\mathcal{D}$ is closed, since its complement is open. If $E \to M$ is a vector bundle, denote by $End(E) \to M$ the vector bundle of linear endomorphisms on the fibers of E.

Let us consider the generalized vector subbundle $End_{\mathcal{D}}(E) \subset End(E)$ of endomorphisms that send in every point the fibers of E in the fibers of $\mathcal{D}$. We investigate in that follows the existence of some smooth sections of this generalized vector subbundle. Notice that if g is a Riemannian metric in the fibers of E, then the orthogonal projection on the fibers of $\mathcal{D}$ is, in general, non-smooth in ordinary sense. For example, the orthogonal projection of $I\!\!R^4 = TI\!\!R^2$ on the smooth subbundle $\mathcal{D} \subset TI\!\!R^2$ generated by the vector field $C = y\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial y}$ is not smooth in origin.

Proposition 3.5. *Let us assume that $\mathcal{D} \subset E$ is smooth. Then:*

a)The level sets (2.3) of $End_{\mathcal{D}}(E)$ are the same as that of $\mathcal{D}$, only dimensions (2.2) are the dimensions of $\mathcal{D}$ multiplied by the dimensions of the fibers of E.

b) The generalized vector subbundle $End_{\mathcal{D}}(E) \subset End(E)$ is smooth.

We consider now orthogonal projections.

Proposition 3.6. *Let g be a scalar product on the fibers of E .*

1. *The orthogonal projection on the fibres of minimal dimension is a smooth section $P_0 : \Sigma_{\min} \to (End(E))_{\Sigma_{\min}}$.*
2. *Let $s \in \Gamma(E)$ be a section on E and consider the section $s_0 : \Sigma_{\min} \to E_{\Sigma_{\min}}$, $s_{0,x} = P_{0,x}(s_x)$, (i.e. $s_{0,x}$ is the orthogonal projection on s_x on $\mathcal{D}_x$), $(\forall)x \in \Sigma_{\min}$. Then s_0 is smooth.*

In the next Proposition we extend explicitly orthogonal projections from $E_{\Sigma_{\min}}$ on $\mathcal{D}_{\Sigma_{\min}}$.

Proposition 3.7. *Let g be a scalar product on the fibers of E, $\mathcal{D} \subset E$ be a smooth generalized subbundle and . Then the following assertions hold true.*

1. *There is an endomorphism Π_0 on the fibers of E such that*
   - *(a) Π_0 restricts on $\Sigma_{\min}$ to the orthogonal projection P_0 of the fibers of $E_{\Sigma_{\min}}$ on the fibers of $\mathcal{D}_{\Sigma_{\min}}$;*
   - *(b) in the points of $x \in M \backslash \Sigma_{\min}$ where $\Pi_{0,x}$ is non null, $\Pi_{0,x}$ is a linear positive combination of orthogonal projections on subspaces of $\mathcal{D}_x$ of dimension $r_{\min}$.*
2. *If $\{s_i\}_{i=\overline{1,k}} \subset \Gamma(E)$ is a global system of a module generators, then the set of sections $\{s'_i = \Pi_0(s_i)\}_{i=\overline{1,k}} \subset \Gamma(E)$ has the property that the set $\{s'_{i,x}\}_{i=\overline{1,k}}$ span $\mathcal{D}_x$, $(\forall)x \in \Sigma_{\min}$.*

We say that:

$\{u_i\}_{i=\overline{0,k}}$ is a *set of test functions* for $\mathcal{D}$ if $u_0 = 1$ and u_i is a test function for the closed subset $\Sigma_{\leq r_{i-1}} \subset M$, $(\forall)i = \overline{0,k}$;

two sets of test functions $\{u_i\}_{i=\overline{0,k}}$ and $\{v_i\}_{i=\overline{0,k}}$ for $\mathcal{D}$ are *equivalent* (or $\{u_i\}_{i=\overline{0,k}}$ ˜ $\{v_i\}_{i=\overline{0,k}}$ for brief) if the test functions u_i and v_i are equivalent, for every $i = \overline{1,k}$.

According to Proposition 3.5, a set of test function is good for $\mathcal{D}$ iff it is good for $End_{\mathcal{D}}(E)$.

We say that a section $S \in \Gamma(E)$ is *bounded* if there is an embedding $E \xrightarrow{I} \Theta^m(M) = M \times I\!R^m$ and a $K > 0$ such that $\sum\limits_{i=1}^{m} \left|(I \circ S)^i(x)\right|^2 \leq K$. For example, on the vector bundle $\Theta^m(M)$, the canonical sections $\{\bar{e}_i\}_{i=\overline{1,m}}$ are bounded. The property of a section to be bounded does not depend on the embedding of E, but on the image $S(M) \subset E$ that must be relative compact. This follows using the Borel-Lebesgue criteria: a relative compact subset of $I\!R^m$ is just a bounded subset. This boundedness property of a set of generators of $\Gamma(E)$ is not restrictive, since any set of generators $\{S_\alpha\}_{\alpha=\overline{1,s}}$, $S_i : M \to E$ can be replaced with a new set of bounded generators $\{\varphi S_\alpha\}_{\alpha=\overline{1,s}}$. Indeed, consider an embedding $E \xrightarrow{I} \Theta^m(M) = M \times I\!R^m$ and take

$$\varphi = 1/\left(1 + \sum_{i=1}^{m}\sum_{\alpha=1}^{s}\left|(I \circ S_\alpha)^i(x)\right|^2\right). \tag{3.1}$$

We prove now that any bounded section on E can be orthogonally projected on the fibers of $\mathcal{D}$, modulo a test function, to a smooth section of a constant level set Σ_r.

Proposition 3.8. *Let g be a scalar product on the fibers of E, $\mathcal{D} \subset E$ be a smooth generalized subbundle of E and $\{u_i\}_{i=\overline{0,k}}$ be a set of test functions for $\mathcal{D}$. Then the following assertions hold true.*

1. *There are* $\{u_i\}_{i=\overline{0,k}} \sim \{v_i\}_{i=\overline{0,k}}$ *such that for every* $i = \overline{0,k}$ *there is an endomorphism* Π_i *on the fibers of* E *such that*
   (a) Π_i *restricts on* Σ_{r_i} *to* $v_{i|\Sigma_{r_i}} \cdot P_i$, *where* P_i *is the orthogonal projection of the fibers of* $E_{\Sigma_{r_i}}$ *on the fibers of* $\mathcal{D}_{\Sigma_{r_i}}$ *and*
   (b) *in the points of* $x \in M \backslash \Sigma_{r_i}$ *where* $\Pi_{i,x}$ *is non null,* $\Pi_{i,x}$ *is a linear positive combination of orthogonal projections on subspaces of* $\mathcal{D}_x$ *of dimension* r_i.
2. *If* $S \in \Gamma(E)$, *then there are* $\{u_i\}_{i=\overline{0,k}} \sim \{v_i\}_{i=\overline{0,k}}$ *and smooth sections* $\{\tilde{S}_i = \Pi_i(S)\}_{i=\overline{0,s}} \in \Gamma(\mathcal{D})$ *that have the property that* $\hat{S}_{i,x} = v_i(x) pr_x(S_x)$, $x \in \Sigma_{r_i}$, *where* $pr_x : E_x \to \mathcal{D}_x$ *is the orthogonal projection according to* g, $(\forall)\ i = \overline{0,k}$.

In this stage we can use 2. of Proposition 3.8 to obtain a first finite set of generators for fibers of $\mathcal{D}$.

Proposition 3.9. *Let* g *be a scalar product on the fibers of* E, $\{s_\alpha\}_{\alpha=\overline{1,s}} \subset \Gamma(E)$ *be a global system of generators,* $\mathcal{D} \subset E$ *be a smooth generalized subbundle and* $\{u_i\}_{i=\overline{0,k}}$ *be a set of test functions for* $\mathcal{D}$. *Then there is a set of global generators* $\{S_{\alpha,i}\}_{\alpha=\overline{1,s},i=\overline{0,k}}$ *of the fibers of* $\mathcal{D}$ *and a set* $\{v_i\}_{i=\overline{0,s}}$ *of test functions for* $\mathcal{D}$, *equivalent with* $\{u_i\}_{i=\overline{0,k}}$, *such that if* $x \in \Sigma_{r_i}$, *then* $S_{\alpha,i,x} = v_i(x) pr_x(s_{\alpha,x})$, *where* $pr_x : E_x \to \mathcal{D}_x$ *is the orthogonal projection according to* g, $(\forall)\ \alpha = \overline{1,s}, i = \overline{0,k}$.

In fact, Proposition 3.9 above or Corollary 4.5 in the next section contain each new proofs of the following result.

Theorem 3.10. *[2, Theorem 4.1], [9, Theorem 1] If* E *is a vector bundle over a connect manifold* M *and* $\mathcal{D}$ *is a smooth generalized subbundle of* E, *then* $\mathcal{D}$ *is globally finitely generated.*

Example. Let us consider $\mathcal{D} = \mathcal{D}(\Gamma_0)$ on $I\!R$ given by $\Gamma_0 \subset \mathcal{X}(I\!R)$ generated by $X_0 = \varphi_0 \frac{d}{dt}$, where

$$\varphi_0(t) = \begin{cases} e^{-\frac{1}{t}} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}, \tag{3.2}$$

Then $\Sigma_1 = (0, +\infty)$ and $\Sigma_0 = (-\infty, 0]$. A test function for Σ_0 is $u_0 = \varphi_0$ and a generator for $\mathcal{D}_{\Sigma_1}$ is the bounded section $s_0 = \frac{d}{dt}$. The section $s = \varphi_0 s_0$ is a generator for $\mathcal{D}$. See [8] for more examples.

Notice that the result in Proposition 3.9 gives $s \cdot (k+1)$ generators, where s is the number of generators of sections of E and $k+1$ is the number of constant level sets of $\mathcal{D}$). In Corollary 4.5 of next section we use a natural anchor to construct s generators, the same number as in [9, Introduction]. The number of generators constructed in [2, Section 4] is $s \cdot \text{maxdim}(\mathcal{D})$. An upper estimate on s is $(1 + \dim M) \cdot rank\ E$ for a general E; in the case $E = TM$, the Whitney embedding theorem gives an upper estimate for s as $2 \dim M$ (see [2] and [9] for more details).

4 Natural anchors

We say that a smooth endomorphism in the fibers of E (i.e. $\Phi \in End(E)$) having its images the fibers of $\mathcal{D}$ is a *natural anchor* for $\mathcal{D}$. The following Theorem contains the main result of the paper.

Theorem 4.1. *If M is a connect manifold, E is a vector bundle over M and $\mathcal{D}$ is a smooth generalized subbundle of E, then there is a smooth natural anchor for $\mathcal{D}$.*

Lemma 4.2. *Let $W_1, \ldots, W_n \subset V$ be vector subspaces and g be a (positive definite) scalar product on V. Let us denote by $\Pi_i : V \to W_i$ the orthogonal projection, $W = W_1 + \cdots + W_n$ and let $\alpha_1, \ldots \alpha_n$ be some strict positive real scalars. Then the linear map*

$$\Pi : V \to W, \Pi(\bar{x}) = \alpha_1 \Pi_1(\bar{x}) + \cdots + \alpha_n \Pi_n(\bar{x})$$

is a surjection.

Now we can prove the following statement..

Lemma 4.3. *Let us assume the settings of Lemma 4.2 and, additionally, one of the subspaces $W_1, \ldots, W_n \subset V$ is equal to $W = W_1 + \cdots + W_n$. Then the following assertions hold true:*

1. *The restriction of Π to W induces a linear automorphism of W.*
2. *If $W_0 \subset V$ is a vector subspace such that $W \subset W_0$, then the restriction of Π to W_0 induces an onto linear map (from W_0 on W).*

Let $\mathcal{D}'$ and $\mathcal{D}''$ be two smooth g.v.s. of two vector bundle E' and E'' respectively. A smooth (generalized) morphism of $\mathcal{D}''$ and $\mathcal{D}'$ is a map $f : \mathcal{D}'' \to \mathcal{D}'$ such that it is the restriction to $\mathcal{D}''$ of a vector bundle map $F : E'' \to E'$.

Corollary 4.4. *There is a natural anchor of $\mathcal{D}$ that restricts to a linear automorphism of the fibers of $\mathcal{D}$.*

Corollary 4.5. *Let $\{s_\alpha\}_{\alpha=\overline{1,s}} \subset \Gamma(E)$ be a global system of generators and $\mathcal{D} \subset E$ be a smooth generalized subbundle. Then there is a set $\{\Phi(S_\alpha)\}_{\alpha=\overline{1,s}} \subset \Gamma(\mathcal{D}) \subset \Gamma(E)$ of a global set of generators of the fibers of $\mathcal{D}$, where Φ is a natural anchor.*

We say that $\mathcal{D}''$ is a *subbundle* of $\mathcal{D}'$ if $\mathcal{D}'' \subset \mathcal{D}'$, i.e. $\mathcal{D}''_x \subset \mathcal{D}'_x$, $(\forall)x \in M$. The inclusion $\mathcal{I} : \mathcal{D}'' \to \mathcal{D}'$ is the restriction of the identity $I \in End(E)$, thus $\mathcal{I}$ is a smooth morphism. provided that $\mathcal{D}'$ and $\mathcal{D}''$ are smooth.

Proposition 4.6. *Assume that $\mathcal{D}'' \subset \mathcal{D}'$ are smooth g.v.s.'s. Then there are two natural anchors Φ' and Φ'' on $\mathcal{D}'$ and $\mathcal{D}''$ respectively such that $P = \Phi'' \circ \Phi'$ restricts to a smooth morphism $p : \mathcal{D}' \to \mathcal{D}''$ that is onto on fibers.*

As we have already remarked in 3. of Proposition 2.1, it follows that the sets of maximal dimensions $\Sigma'_{\max}$ and $\Sigma''_{\max}$ of $(\mathcal{D}')^{\models_g}$ and $(\mathcal{D}'')^{\models_g}$ respectively are open dense subsets of M, thus their intersection is also an open dense subset of M.

According to Theorem 4.1, some natural anchors of $\mathcal{D}$ and $\mathcal{D}^{\vdash_g}$ always exist and their sum gives a natural anchor of the smooth orthogonal completion $\mathcal{D}^{\models_g}$. Using also Proposition 2.1, some properties can be summarized in the following statement.

Proposition 4.7. *Let g be a scalar product on the fibers of E, $\mathcal{D}^{\vdash_g} = \mathcal{A}(\mathcal{D}^{\perp_g}(\mathcal{D})) \subset E$ be the smooth orthogonal of a smooth generalized vector subbundle $\mathcal{D}$. If Φ and $\Phi^{\vdash_g} \in End(E)$ are natural anchors of $\mathcal{D}$ and $\mathcal{D}^{\vdash_g}$ respectively, then*

1. *$\Phi \circ \Phi^{\vdash_g} = \Phi^{\vdash_g} \circ \Phi = 0$, i.e. Φ and $\Phi^{\vdash_g}$ are transverse,*

2. $\Phi^{\models_g} = \Phi + \Phi^{\vdash_g}$ *is a natural anchor of the smooth orthogonal completion* $\mathcal{D}^{\models_g}$,

3. $\Phi^{\models_g}(\mathcal{D}) = \mathcal{D}$ *and* $\Phi^{\models_g}\left(\mathcal{D}^{\vdash_g}\right) = \mathcal{D}^{\vdash_g}$ *and*

4. $\Phi^{\models_g}$ *induces on the dense open subset* $\Sigma^{\models_g}_{\max} \subset M$ *an automorphism of the fibres of* E *and a natural splitting* $E_x = \mathcal{D}_x \oplus \mathcal{D}_x^{\vdash_g}$, $(\forall)x \in \Sigma^{\models_g}_{\max}$.

These can be related to some g.v.s.'s in the Pontryagin bundle $P(E) = E \oplus E^*$, where the non-degenerate quadratic form $\varepsilon : P(E) \to \mathcal{F}(M)$, $\varepsilon(X, \omega) = \omega(X)$ can be considered (see [4] for more details). The quadratic form ε has the signature (k, k), where k is the dimension of the fibers of E. Considering a smooth g.v.s. $\mathcal{D} \subset E$ and a Riemannian metric g in the fibers of E, there are canonical isomorphisms:

– $\mathcal{D}^{\perp_g} \cong \mathcal{D}^{\perp} \subset E^*$, $\mathcal{D}^{\vdash_g} \cong \mathcal{A}(\mathcal{D}^{\perp}) \overset{not.}{=} \mathcal{D}^{\vdash} \subset E^*$ and

– $\mathcal{D}^{\models_g} \cong \mathcal{D} + \mathcal{D}^{\vdash} \overset{not.}{=} \mathcal{D}^{\models} \subset P(E)$.

We stress that the above orthogonals are related to the Riemannian metric g and they are different from the orthogonal considered in [4], according to the canonical pseudo-Riemannian metric ε.

We say that a g.v.s. $\mathcal{D} \subset P(E)$ is of *Dirac type* if it is smooth and the restriction of ε to $\mathcal{D}$ is null. For example, every smooth g.v.s. of E or E^* can be considered as smooth g.v.s. of $P(E)$ of Dirac type; we say that they are *pure*. We say that $\mathcal{D}$ *allows a pure decomposition* if $\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2$, where $\mathcal{D}_1 \subset E$ and $\mathcal{D}_2 \subset E^*$, call here as the *pure decomposition*. We say also that a Dirac type $\mathcal{D} \subset P(E)$ is *Dirac almost regular* if it allows a pure decomposition and $\mathcal{D} = \mathcal{D}_1^{\models}$ or $\mathcal{D} = \mathcal{D}_2^{\models}$, where $\mathcal{D}_1 \subset E$ and $\mathcal{D}_2 \subset E^*$ are its pure components. According to [4], a *Dirac vector subbundle* $\mathcal{D} \subset P(E)$ is just a vector subbundle of Dirac type and of maximal dimensions of fibers, i.e. the dimension of the fibers of E. in this case the Dirac pure components $\mathcal{D}_1$ and $\mathcal{D}_2$ are vector subbundles of E and E^* respectively and $\mathcal{D}$ is Dirac regular.

If $\mathcal{D}_0 \subset E^*$ is a smooth g.v.s., then using Theorem 4.1 there is a natural anchor $\Phi_0 \in End(E^*)$ for $\mathcal{D}_0$, i.e. $\Phi_0(E^*) = \mathcal{D}_0$. Considering the dual endomorphism $\Phi = \Phi_0^* \in End(E)$, then the image of Φ is a smooth g.v.s. $\mathcal{D} \subset E$. Let $\mathcal{D} \subset E$ be of Dirac almost regular and $\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2$ a pure decomposition. We can consider some natural anchors $\Phi_1 \in End(E)$ and $\Phi_2 \in End(E^*)$ for $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively, i.e. $\Phi_1(E) = \mathcal{D}_1$ and $\Phi_2(E) = \mathcal{D}_2$. Then the natural anchors Φ_1,Φ_2^* are transverse, i.e. $\Phi_1 \circ \Phi_2^* = \Phi_2^* \circ \Phi_1 = 0$, thus their images have a null intersection. Using also Proposition 4.7, the following assertion holds true.

Proposition 4.8. *If* $\mathcal{D} \subset P(E)$ *is Dirac almost regular, then there are two natural transverse anchors* Φ_1, $\Phi_2 \in End(E)$ *such that* $\Phi_1 + \Phi_2^* \in End(P(E))$ *is an adapted anchor for* $\mathcal{D}$.

Using 3. of Propositions 4.7, Proposition 4.8 and the above observations, the following assertion holds true.

Proposition 4.9. *If* $\mathcal{D} \subset P(E)$ *is Dirac almost regular, then there is an open dense subset of* M *where the restriction of* $\mathcal{D}$ *is a Dirac vector subbundle.*

Some constructions and results presented in the paper for smooth vector subbundles can be translated, by duality, to cosmooth vector subbundles.

References

[1] F. Bullo, A. D. Lewis, *Geometric control of mechanical systems: Modeling, analysis, and design for simple mechanical control systems*, Texts in Applied Mathematics, 49, Springer-Verlag, New York, 2005.

[2] L. D. Drager, J. M. Lee, E. Park, K. Richardson, *Smooth vector subbundles are finitely generated*, Ann. Glob. Anal. Geom. 41, 3 (2012), 357-369.

[3] A. Isidori, *Nonlinear Control Systems*, 3-rd ed., Springer Verlag, New York, 1995.

[4] M. Jotz, T. S. Ratiu, J. Śniatycki, *Singular reductions of Dirac structures*, Trans. Amer. Math. Soc. 363 (2011), 2967-3013.

[5] S. Lang, *Differential and Riemannian Manifolds,* 3-rd ed., Springer Verlag, New York, 1995.

[6] J. M. Lee, *Introduction to Smooth Manifolds*, Springer Verlag, New York, 2003.

[7] M. Muger, *An Introduction to Differential Topology, de Rham Theory and Morse Theory*, http://www.math.ru.nl/˜mueger/diff_notes.pdf (2005).

[8] M. Popescu, P. Popescu, *Generalized subbundles of vector bundles*, Proceedings of The International Conference "Differential Geometry and Dynamical Systems"(DGDS-2011) October 6-9, 2011, Bucharest, ROMANIA, Geometry Balkan Press, BSG Proceedings 19 (2012), 146-151.

[9] H. J. Sussmann, *Smooth distributions are globally finitely spanned*, Analysis and design of nonlinear control systems, In honor of Alberto Isidori, Astolfi, Alessandro (ed.) et al., Berlin: Springer, 2008, 3-8.

[10] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. 36 (1934), 63-89.

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