Spacelike hypersurfaces in de Sitter space

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Abstract. In this paper, we study the close spacelike hypersurfaces in de Sitter space. Using Bonnet-Myer's theorem, we prove a rigidity theorem for spacelike hypersurfaces without the constancy condition on the mean curvature or the scalar curvature.

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1 Introduction

Let $\mathbb{N}_1^{n+1}(c)$ be an $(n+1)-$dimensional connected indefinite Riemannian manifold of index 1 and of constant curvature c. According to $c > 0, c = 0$ and $c < 0$, it is called de Sitter space, Minkowski space and anti-de Sitter space, respectively, and it is denoted by $\mathbb{S}_1^{n+1}(c)$, $\mathbb{R}_1^{n+1}$ and $\mathbb{H}_1^{n+1}(c)$ (see [20]). A hypersurface M of $\mathbb{N}_1^{n+1}(c)$ is said to be spacelike if the induced metric on M from that of the ambient space is positive definite. Since the importance of spacelike hypersurfaces in general relativity has been emphasized, spacelike hypersurfaces have been studied by many mathematicians.

E. Calabi [7] first studied the Bernstein problem for a maximal spacelike entire graph in $\mathbb{R}_1^{n+1}$ and proved that it has to be hyperplane, when $n \leq 4$. Later, S. Y. Cheng and S. T. Yau [14] proved that the conclusion remains true for all n. Further results about rigidity of maximal spacelike hypersurfaces in $\mathbb{N}_1^{n+1}(c)$ can be found in [9,11,13,16, etc.].

As a natural generalization, the rigidity phenomenon for spacelike hypersurfaces in $\mathbb{N}_1^{n+1}(c)$ with constant mean curvature has also been studied. A. J. Goddard conjectured in [17] that complete spacelike hypersurfaces of constant mean curvature in de Sitter space should be totally umbilical. This turned out to be true only for compact hypersurfaces, due to independent work of S. Montiel ([21]) and J. L. Barbosa and V. Oliker ([6]). On the other hand, K. Akutagawa [2] and J. Ramanathan [22] proved independently that a complete spacelike hypersurface in a de Sitter space with constant mean curvature is totally umbilical if the mean curvature H satisfied $H^2 \leq 1$ when $n = 2$ and $n^2H^2 \leq 4(n-1)$ when $n \geq 3$. One can find more results about rigidity of spacelike hypersurfaces with constant mean curvature in [4,18, etc.].

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Also, some rigidity theorems for hypersurfaces with constant scalar curvature have been proven. Q. M. Cheng and S. Ishikawa [12] have shown that compact spacelike hyperspaces in $\mathbb{S}_1^{n+1}(c)$ with constant scalar curvature $R < n(n-1)c$ must be totally umbilical. For additional results on rigidity of spacelike hyperspaces with constant scalar curvature, see [3,15,19,20, etc.].

In this paper, we prove a rigidity theorem of spacelike hyperspaces without the constancy condition on the mean curvature or the scalar curvature. To be precise, we have the following theorem.

Theorem 1.1. *Let M^n be an $n-$dimensional closed spacelike hypersurface in de Sitter space $\mathbb{S}_1^{n+1}(1)$. Let S and H be the squared norm of the second fundamental form and the mean curvature of M^n, respectively. Suppose that the fundamental group $\pi_1(M^n)$ of M^n is infinite and $S_1(n,H) < S \le S_2(n,H)$, where*

$$S_1(n,H) = n^2H^2 - n(n-1), \tag{1.1}$$

and

$$S_2(n,H) = \frac{n^2-2}{n-1}H^2 + 2. \tag{1.2}$$

Then S is constant, $S = S_2(n,H)$, and M^n is isometric to a Riemannian product $\mathbf{S}^{(n-1)}(c_1) \times \mathbf{H}^1(c_2)$, where $\frac{1}{c_1} + \frac{1}{c_2} = 1$, $c_1 > 0$ and $c_2 < 0$.

2 Preliminaries

Let M^n be an $n-$dimensional spacelike hypersurface in de Sitter space $\mathbb{S}_1^{n+1}(1)$. We choose a local field of semi-Riemannian orthonormal frames $e_1, \cdots, e_{n+1}$ in $\mathbb{S}_1^{n+1}(1)$ such that at each point of M^n, $e_1, \cdots, e_n$ span the tangent space of M^n and form an orthonormal frame there. We use the following convention on the range of indices:

$$1 \le A, B, C, \cdots \le n+1; 1 \le i, j, k, \cdots \le n. \tag{2.1}$$

Let $\omega_1, \cdots, \omega_{n+1}$ be its dual frame field so that the semi-Riemannian metric of $\mathbb{S}_1^{n+1}(1)$ is given by $d\bar{s} = \sum_i \omega_i^2 - \omega_{n+1}^2 = \sum_A \varepsilon_A \omega_A^2$, where $\varepsilon_i = 1$ and $\varepsilon_{n+1} = -1$. Then the structure equations of $\mathbb{S}_1^{n+1}(1)$ are given by

$$d\omega_A = -\sum_B \varepsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.2}$$

$$d\omega_{AB} = -\sum_C \varepsilon_C \omega_{AC} \wedge \omega_{BC} - \frac{1}{2}\sum_{C,D} \varepsilon_C \varepsilon_D \mathbf{K}_{ABCD} \omega_C \wedge \omega_D, \tag{2.3}$$

$$\mathbf{K}_{ABCD} = \varepsilon_A \varepsilon_B (\delta_{AD}\delta_{BC} - \delta_{AC}\delta_{BD}). \tag{2.4}$$

Restrict these form to M^n, we have $\omega_{n+1} = 0$, the Riemannian metric of M^n is written as $ds^2 = \sum_i \omega_i^2$. By Cartan's lemma, we have

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}. \tag{2.5}$$

From these formulas, we obtain the structure equations of M^n as follows.

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{2.6}$$

$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_k, \tag{2.7}$$

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.8}$$

where R_{ijkl} are the components of the curvature tensor of M^n and $h = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j$ is the second fundamental form of M^n. The squared norm S of h and the mean curvature H of M^n are given by

$$S = \sum_{i,j} h_{ij}^2, \quad nH = \sum_i h_{ii}, \tag{2.9}$$

respectively. Then the Ricci tensor and the scalar curvature R of M^n are given by

$$R_{ij} = (n-1)\delta_{ij} - nHh_{ij} + \sum_k h_{ik}h_{kj}, \tag{2.10}$$

$$R = n(n-1) - n^2H^2 + S, \tag{2.11}$$

respectively. The eigenvalues $\lambda_1, \cdots, \lambda_n$ of (h_{ij}) are the principal curvatures of M^n. Let h_{ijk} denote the covariant derivative of h_{ij}, defined by

$$\sum_k h_{ijk}\omega_k = dh_{ij} - \sum_k h_{kj}\omega_{ki} - \sum_k h_{ik}\omega_{kj}. \tag{2.12}$$

Then we have the Codazzi equation

$$h_{ijk} = h_{ikj}. \tag{2.13}$$

The Weyl curvature tensor $W = (W_{ijkl})$ of M^n is given by

$$\begin{aligned} W_{ijkl} = R_{ijkl} &- \frac{1}{n-2}(R_{ik}\delta_{jl} - R_{il}\delta_{jk} + \delta_{jl}\delta_{ik} - R_{jk}\delta_{il}) \\ &+ \frac{R}{(n-1)(n-2)}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \end{aligned} \tag{2.14}$$

The Beck curvature tensor $B = (B_{ijk})$ of M^n is defined by

$$B_{ijk} = \frac{1}{n-2}(R_{ijk} - R_{ikj}) - \frac{1}{2(n-1)(n-2)}(\delta_{ij}R_k - \delta_{ik}R_j), \tag{2.15}$$

where R_{ijk} are the components of the covariant derivative of the Ricci curvature tensor of M^n and $R_k = e_kR$. M^n is said to be locally conformally flat if, for each $x \in M^n$, there exists a conformal diffeomorphism of a neighborhood of x onto an open set of the $n-$dimensional Euclidean space. When $n \geq 4$, M^n is locally conformally flat if and only if $W_{ijkl} = 0$ and $B_{ijkl} = 0$ on M^n. When $n = 3$, we always have

$W_{ijkl} = 0$ on M^3 , then M^3 is locally conformally flat if and only if $B_{ijk} = 0$. Hence, if M^n is a locally conformally flat Riemannian manifold, then

$$R_{ijkl} = \frac{1}{n-2}(R_{ik}\delta_{jl} - R_{il}\delta_{jk} + \delta_{jl}\delta_{ik} - R_{jk}\delta_{il}) + \frac{R}{(n-1)(n-2)}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \tag{2.16}$$

and

$$R_{ijk} - \frac{1}{2(n-1)}\delta_{ij}R_k = R_{ikj} - \frac{1}{2(n-1)}\delta_{ik}R_j. \tag{2.17}$$

The following lemma is needed in the proof of Theorem 1.1.

Lemma 2.1. *(see [5]) If the Ricci curvature of a compact Riemannina manifold is non-negative and positive at a point, then the manifold carries a metric of positive Ricci curvature.*

3 Proof of the Theorem 1.1

Let us first choose a suitable orthonormal frame field $\{e_1, \cdots, e_n\}$, which diagonalizes the second fundamental form of M^n, so that $\sum_{i,j} h_{ij}\omega_i \otimes \omega_j = \sum_i \lambda_i \omega_i \otimes \omega_i$. By (2.7), we have

$$R_{ijkl} = (1 - \lambda_i\lambda_j)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \tag{3.1}$$

$$R_{ij} = 0, i \neq j, \tag{3.2}$$

and

$$R_{ii} = (n-1) - nH\lambda_i + \lambda_i^2, i = 1, \cdots, n. \tag{3.3}$$

Since, for any fixed $i \in \{1, \ldots, n\}$,

$$(nH - \lambda_i)^2 = (\sum_{k\neq i} \lambda_k)^2 \leq (n-1)\sum_{k\neq i} \lambda_k^2 = (n-1)(S - \lambda_i^2), \tag{3.4}$$

we have

$$n^2H^2 - (n-1)S - 2nH\lambda_i \leq -n\lambda_i^2 \leq 0. \tag{3.5}$$

Combining with $(\lambda_i - H)^2 \geq 0$, it is easy to see that

$$\lambda_i^2 - nH\lambda_i \geq nH\lambda_i - H^2 \geq \frac{n^2H^2 - (n-1)S}{2} - H^2 = \frac{(n^2-2)H^2 - (n-1)S}{2}. \tag{3.6}$$

Then by (3.3), we have

$$R_{ii} \geq (n-1) + \frac{(n^2-2)H^2 - (n-1)S}{2} = \frac{n-1}{2}\left(\frac{n^2-2}{n-1}H^2 + 2 - S\right). \tag{3.7}$$

Since the assumption that $S \leq S(n,H) = \frac{n^2-2}{n-1}H^2 + 2$, then

$$R_{ii} \geq 0, \forall i = 1, \ldots, n \ . \tag{3.8}$$

This, combined with (3.2), implies that the Ricci curvature of M^n is nonnegative. Since the fundamental group of M^n is infinite, by Bonnet-Myers theorem [10] and Lemma 2.1, we conclude that $\forall x \in M^n$, there exists a unit vector $X \in T_xM^n$, such that the Ricci curvature of M^n satisfies $Ric(X,X) = 0$. Note that the Ricci curvature attains its minimum and maximum in the principal directions. Without loss of generality, we can assume that $R_{nn} = 0$, at any fixed point $x \in M^n$. Therefore, when $i = n$, all of the above inequalities should be equalities at x, and $S(x) = S_2(n,H)(x)$. Consequently, we obtain that $\lambda_1(x) = \cdots = \lambda_{n-1}(x)$. Since $x \in M^n$ is arbitrary, one can deduces that there are at most two distinct principal curvatures on M^n. Now let us assume that $\lambda_1 = \cdots = \lambda_{n-1} = \lambda$ and $\lambda_n = \mu$. Since $R_{ii} \geq 0$ and one of $R_{11}, \cdots, R_{nn}$ is zero, by (3.3), we deduce that $\lambda\mu - 1 = 0$. Thus there are exactly two distinct principal curvatures, one of them is simple, and $S = S(n,H)$ holds on M^n. Next we are going to show that λ and μ are constant functions on M^n.

Note that M^n is a closed manifold with non-negative Ricci curvature, then the Riemannian universal covering space $\tilde{M}^n$ of M^n can be decomposed as $\mathfrak{M}^{n-s} \times \mathfrak{R}^s$ for some $s \in \{0, 1, \cdots, n\}$, where $\mathfrak{M}^{n-s}$ is a closed simply connected $(n-s)-$dimensional Riemannina manifold with non-negative Ricci curvature and $\mathfrak{R}^s$ is the $s-$dimensional Euclidean space with standard flat metric (see [23]). The infinity of fundamental group $\pi_1(M^n)$ implies that $\mathfrak{M}^{n-s} \times \mathfrak{R}^s$ is non-compact and so we have $s \geq 1$. Next we will conclude that $s = 1$. Since $\lambda_1 = \cdots = \lambda_{n-1} = \lambda$,$\lambda_n = \mu$ and $\lambda\mu - 1 = 0$ on M^n, by (3.1), we know that the sectional curvature $K(X \wedge Y)$ of the plane spanned by X and Y, is given by

$$\begin{aligned} K(X \wedge Y) &= \sum_{i,j,k,l} X_iY_jX_kY_lR_{ijkl} \\ &= \sum_{i,j,k,l} X_iY_jX_kY_l(1-\lambda_i\lambda_j)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \\ &= 1 - (\sum_i \lambda_i X_i^2)(\sum_j \lambda_j Y_j^2) + (\sum_i \lambda_i X_i Y_i)^2 \\ &= 1 - (\lambda(1-X_n^2) + \mu X_n^2)(\lambda(1-Y_n^2) + \mu Y_n^2) + (\lambda(-X_nY_n) + \mu X_nY_n)^2 \\ &= (1-\lambda^2)(1 - X_n^2 - Y_n^2), \end{aligned} \tag{3.9}$$

where $X = \sum_{i=1}^{n} X_ie_i, Y = \sum_{i=1}^{n} Y_ie_i \in T_xM^n$ with $|X| = |Y| = 1$, $< X, Y >= 0$.

On the other hand, note that

$$S = (n-1)\lambda^2 + \mu^2, \tag{3.10}$$

and

$$nH = (n-1)\lambda + \mu, \tag{3.11}$$

we have

$$\lambda^2 = \frac{n^2H^2 - S - 2(n-1)}{(n-1)(n-2)}. \tag{3.12}$$

Since the assumption that $S > n^2H^2 - n(n-1)$, then

$$\lambda^2 < 1. \tag{3.13}$$

Now we need the following proposition.

Proposition 3.1. *A vector $X \in T_xM^n$ with $|X| = 1$ satisfies the following condition*

$(*)$ if $Y \in T_xM^n$ with $< X, Y >= 0$, $|Y| = 1$, then $K(X \wedge Y) = 0$,

if and only if $X = \pm e_n(x)$.

Proof. If $X = \pm e_n(x)$ and Y satisfies $< X, Y >= 0, |Y| = 1$, then $Y = \sum_{i=1}^{n-1} Y_i e_i(x)$, $\sum_{i=1}^{n-1} Y_i^2 = 1$. Thus

$$\begin{aligned} K(X \wedge Y) &= \textstyle\sum_{i,j} Y_iY_jR_{ninj} = \sum_{i,j} Y_iY_j(1 - \lambda_i\lambda_n)(\delta_{nn}\delta_{ij} - \delta_{ni}\delta_{nj}) \\ &= \textstyle\sum_{i\neq n} Y_i^2(1 - \lambda_i\lambda_n) = \sum_{i\neq n} Y_i^2(1 - \lambda\mu) = 0 \end{aligned} \tag{3.14}$$

On the other hand, if $(*)$ is satisfied and let $X = ae_n(x) + W$, where $< W, e_n(x) >= 0, W \neq 0$. Take a vector $Z = \sum_{i=1}^{n-1} Z_ie_i(x) \in T_xM^n$ satisfying $|Z| = 1, \langle Z, W\rangle = 0$. Then $\langle Z, X\rangle = 0$. By (3.9) and (3.13), we have

$$K(X, Z) = (1 - \lambda^2)(1 - a^2) = (1 - \lambda^2)|W|^2 > 0, \tag{3.15}$$

which leads to a contradiction. Thus X is parallel to $e_n(x)$, we know that $X = \pm e_n(x)$, since $|X| = 1$. Proposition 3.1 is proved. □

Let us go on the proof of Theorem 1.1. Since M and $\mathfrak{M}^{n-s} \times \mathfrak{R}^s$ are locally isometric, it follows from Proposition 3.1 that $s = 1$. Next we claim that the sectional curvature of $\mathfrak{M}^{n-1}$ is constant and $\mathfrak{M}^{n-1}$ is isometric to an $(n-1)-$dimsional sphere.

Case 1 : suppose that $n \geq 4$. Let $\pi : \mathfrak{M}^{n-1} \times \mathfrak{R} \to M$ be the natural projection, then π is a local isometry. For any $x \in \mathfrak{M}^{n-1}$, let $\{u_1, \cdots, u_{n-1}\}$ be an orthonormal base of $T_x\mathfrak{M}^{n-1}$. Since $T_{(x,0)}(\mathfrak{M}^{n-1} \times \mathfrak{R}) = T_x\mathfrak{M}^{n-1} \times T_0\mathfrak{R}$, we know that $\{(u_1, 0), ..., (u_{n-1}, 0), (\mathbf{0}, 1)\}$ is an orthonormal base of $T_{(x,0)}(\mathfrak{M}^{n-1} \times \mathfrak{R})$, where $\mathbf{0}$ is the zero-vector of $T_x\mathfrak{M}^{n-1}$. Observe that for any $v \in T_{(x,0)}(\mathfrak{M}^{n-1} \times \mathfrak{R})$ with $< v, (\mathbf{0}, 1) >= 0$, $|v| = 1$, $\mathfrak{K}((\mathbf{0}, 1) \wedge v) = 0$, where $\mathfrak{K}$ denotes the sectional curvature of $\mathfrak{M}^{n-1} \times \mathfrak{R}$. It follows that $K((d\pi_{(x,0)}(\mathbf{0}, 1)) \wedge X) = 0$, where $X \in T_{\pi(x,0)}M^n$ with $|X| = 1$ and $< (d\pi_{(x,0)}(\mathbf{0}, 1)), X >= 0$. Thus by Proposition 3.1 we know that $d\pi_{(x,0)}(\mathbf{0}, 1) = \pm e_n(y)$ and $d\pi_{(x,0)}(u_j, 0) \in span\{e_1(y), \cdots, e_{n-1}(y)\}$, for any $j = 1, \cdots, n-1$, where $y = \pi(x, 0)$. Since $\mathfrak{K}((u_i, 0) \wedge (u_j, 0))= K((d\pi_{(x,0)}(u_i, 0)) \wedge (d\pi_{(x,0)}(u_j, 0)) = 1 - \lambda^2(y)$, $i \neq j \in \{1, \cdots, n-1\}$, then for any $x \in \mathfrak{M}^{n-1}$ and two-dimensional plane $P \subset T_x\mathfrak{M}^{n-1}$, the sectional curvature $\mathrm{K}(P)$ of $\mathfrak{M}^{n-1}$ on P must satisfy $\mathrm{K}(P) = \mathfrak{K}(P) = g(x) > 0$, where $g(x) = 1 - \lambda^2(\pi(x, 0))$ is a function on $\mathfrak{M}^{n-1}$. Note that $dim(\mathfrak{M}^{n-1}) \geq 3$, by the well-known Schur Lemma ([8]), we know that the sectional curvature of $\mathfrak{M}^{n-1}$ is constant and positive.

Case 2 : Now suppose that $n = 3$. Let us first show that (2.16) is satisfied and so M^3 is a locally conformally flat manifold. In fact, since $n = 3$, by taking the covariant derivatives of (2.9), we have

$$R_{ijk} = -3H_kh_{ij} - 3Hh_{ijk} + \sum_l(h_{ilk}h_{lj} + h_{il}h_{ljk}). \tag{3.16}$$

Then by (2.10) ,we get

$$R_k = -18HH_k - S_k. \tag{3.17}$$

Thus

$$\begin{aligned} B_{ijk} &= (R_{ijk} - R_{ikj}) - \frac{1}{4}(\delta_{ij}R_k - \delta_{ik}R_j) \\ &= -3H_k h_{ij} + 3H_j h_{ik} - \sum_l (h_{ilj}h_{lk} - h_{ilk}h_{lj}) \\ &+ \frac{1}{4}(\delta_{ij}(18HH_k - S_k) - \delta_{ik}(18HH_j - S_j)). \end{aligned} \tag{3.18}$$

Let $h_{ij} = \lambda_i \delta_{ij}$, we have

$$S_k = 2\sum_{i,j} h_{ij}h_{ijk} = 2\sum_i \lambda_i h_{iik}. \tag{3.19}$$

Hence

$$\begin{aligned} B_{ijk} &= -3H_k(\lambda_i - \frac{3}{2}H)\delta_{ij} + 3H_j(\lambda_i - \frac{3}{2}H)\delta_{ik} \\ &- h_{ikj}(\lambda_k - \lambda_j) - \frac{1}{4}(\delta_{ij}S_k - \delta_{ik}S_j) \\ &= 3(H_j\delta_{ik} - H_k\delta_{ij})(\lambda_i - \frac{3}{2}H) + h_{ikj}(\lambda_j - \lambda_k) \\ &+ \frac{1}{2}\sum_l (\delta_{ik}h_{llj} - \delta_{ij}h_{llk})\lambda_l. \end{aligned} \tag{3.20}$$

Let $h_{kj} = \lambda_k \delta_{kj}$ in (2.11), we get

$$h_{ijk} = \delta_{ij}e_k\lambda_i - (\lambda_i - \lambda_j)\omega_{ij}(e_k), \tag{3.21}$$

and so we have

$$h_{iik} = e_k\lambda_i. \tag{3.22}$$

Since $h_{ijk} = h_{ikj}$, if i, j, k are all distinct, then by (3.21), we have

$$(\lambda_i - \lambda_j)\omega_{ij}(e_k) = (\lambda_i - \lambda_k)\omega_{ik}(e_j). \tag{3.23}$$

Since $\lambda_1 = \lambda_2 = \lambda, \lambda_3 = \mu$, from (3.20), (3.21) and (3.23), when i, j, k are all distinct, we conclude that

$$B_{ijk} = (\lambda_j - \lambda_k)h_{ikj} = (\lambda_j - \lambda_k)(\lambda_k - \lambda_i)\omega_{ik}(e_j) = (\lambda_j - \lambda_k)(\lambda_j - \lambda_i)\omega_{ij}(e_k) = 0. \tag{3.24}$$

By (3.20), it is easy to see that $B_{iii} = 0$. Let $\mathfrak{A}$ be the Weingarten operator defined by the second fundamental form, that is, for any $x \in M$ and all $X, Y \in T_xM$, $\mathfrak{A} : T_xM \to$

T_xM, $<\mathfrak{A}X, Y>= h(X,Y)$. Let $\mathfrak{D}_x(\lambda)=\{X\in T_xM : \mathfrak{A}X=\lambda_x X\}$ and $\mathfrak{D}(\lambda)$ be the assignment of $\mathfrak{D}_x(\lambda)$ to each point $x\in M$. Since the multiplicity of the principal curvature λ is greater than 1, then $\mathfrak{D}(\lambda)$ is a completely integrable distribution on M and that λ is constant on each leaf of $\mathfrak{D}(\lambda)$. Thus we have $e_1\lambda=e_2\lambda=0$. If $i\neq j$, by (3.20) and (3.22), we have

$$\begin{aligned} B_{iij} &= -3H_j(\lambda_i-\frac{3}{2}H)+h_{iij}(\lambda_i-\lambda_j)-\frac{1}{2}\sum_l h_{llj}\lambda_l \\ &= -(2e_j\lambda+e_j\mu)(\lambda_i-\lambda-\frac{1}{2}\mu)+(e_j\lambda_i)(\lambda_i-\lambda_j)-\frac{1}{2}\sum_l h_{llj}\lambda_l. \end{aligned} \tag{3.25}$$

Then

$$B_{112}=-(e_2\mu)(-\frac{1}{2}\mu)-\frac{1}{2}(e_2\mu)\mu=0, \tag{3.26}$$

$$B_{113}=-(2e_3\lambda+e_3\mu)(-\frac{1}{2}\mu)+(e_3\lambda)(\lambda-\mu)-(e_3\lambda)\lambda-\frac{1}{2}(e_3\mu)\mu=0. \tag{3.27}$$

Similarly, we have

$$B_{221}=B_{223}=B_{331}=B_{332}=0, B_{ijj}=B_{iji}=0, i\neq j. \tag{3.28}$$

Therefore M^3 is locally conformally flat and so is $\tilde{M}^3$. Recall that $\tilde{M}^3=\mathfrak{M}^2\times\mathfrak{R}$, where the Gaussian curvature $\mathfrak{k}$ of $\mathfrak{M}^2$ is positive. Next we will use the same notations R_{ijkl} and R_{ij}, etc. to denote the components of the curvature tensor and the Ricci curvature tensor, etc. of $\tilde{M}^3$, respectively. Take an orthonormal local frame field $\{v_1,v_2,v_3\}$ of $\tilde{M}^3$ such that v_1 and v_2 are tangent to $\mathfrak{M}^2$. Since $\tilde{M}^3=\mathfrak{M}^2\times\mathfrak{R}$, we have

$$\begin{aligned} R_{11}=R_{22}=R_{1212}+R_{1313}=R_{1212}=\mathfrak{k}, \\ R_{12}=R_{13}=R_{23}=R_{33}=0, R=2\mathfrak{k}. \end{aligned} \tag{3.29}$$

Let $i=j=1, k=2$ in (2.16), then

$$R_{112}-R_{121}=\frac{1}{2}R_2. \tag{3.30}$$

By the definition of covariant derivative, we get

$$R_{112}=(dR_{11})(v_2)-\sum_{l=1}^{3}R_{l1}\omega_{l1}(v_2)-\sum_{l=1}^{3}R_{l2}\omega_{l2}(v_2)=v_2R_{11}=v_2\mathfrak{k}, \tag{3.31}$$

$$R_{121}=(dR_{12})(v_1)-\sum_{l=1}^{3}R_{l2}\omega_{l1}(v_1)-\sum_{l=1}^{3}R_{1l}\omega_{l2}(v_1)=0. \tag{3.32}$$

Thus, $v_2\mathfrak{k} = \frac{1}{2}v_2\mathfrak{k}$, and so $v_2\mathfrak{k} = 0$. Similarly, we have $v_1\mathfrak{k} = 0$. Therefore $\mathfrak{k}$ is a constant function.

Hence, for any $n \geq 3$, $\mathfrak{M}^{n-1}$ is a sphere, which implies that the scalar curvature of $\mathfrak{M}^{n-1}$ is constant. But the scalar curvature of $\mathfrak{M}^{n-1}$ is given by $r = (n-1)(n-2)(1-\lambda^2)$. Thus λ is a constant and so is $\mu = \frac{1}{\lambda}$. That is, M^n is an isoparametric spacelike hypersurface in $\mathbb{S}_1^{n+1}(1)$ with two distinct principal curvatures one of which is simple.

According to the congruence theorem of N. Abe etc.(see [1]), we know that M^n is isometric to a Riemannian product $\mathbf{S}^{(n-1)}(c_1) \times \mathbf{H}^1(c_2)$, where $\frac{1}{c_1} + \frac{1}{c_2} = 1$, $c_1 > 0$ and $c_2 < 0$.

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References

[1] N. Abe, N. Koike and S. Yamaguchi, *Congruence theorems for proper semi-Riemannian hypersurfaces in a real space forms*, Yokohama Math. J. 35 (1987), 123-136.

[2] K. Akutagawa, *On spacelike hypersurfaces with constant mean curvature in the de Sitter space*, Math. Z. 196 (1987), 13-19.

[3] H. Alencar, M. do Carmo and G. Colares, *Stable hypersurfaces with constant scalar curvature*, Math. Z. 213 (1993), 117-131.

[4] J. A. Aledo, L. J. Alias and A. Romero, *Integral formulas for compact space-like hypersurfaces in de Sitter space: applications to the case of constant higher order mean curvature*, J. Geom. Phys. 31 (1999), 195-208.

[5] T. Aubin, *Some nonlinear problems in Riemannian geometry*, Springer-Verlag, Berlin, 1998.

[6] J. L. M. Barbosa and V. Oliker, *Spacelike hypersurfaces with constant mean curvature in Lorentz space*, Matem. Contemp. 4 (1993), 27-44.

[7] E. Calabi, *Examples of Bernstein problems for nonlinear equations*, Proc. Sympos. Pure Math. 15 (1970), 223-230.

[8] M. do Carmo, *Riemannian geometry*, Birkhäuser, Boston, 1992.

[9] R. M. B. Chaves, L. A. M. Sousa Jr. and B. C. Valério, *New characterizations for hyperbolic cylinders in anti-de Sitter spaces*, J. Math. Anal. Appl. 393 (2012), 166-176.

[10] J. Cheeger and D. Ebin, *Comparison theorems in Riemannian geometry*, North-Holland, New York, 1975.

[11] Q. M. Cheng, *Complete maximal spacelike hypersurfaces of $H_1^4(c)$*, Manuscripta Math. 82 (1994), 149-160.

[12] Q. M. Cheng and S. Ishikawa, *Spacelike hypersurfaces with constant scalar curvature*, Manuscripta Math. 95 (1998), 499-505.

[13] Q. M. Cheng and Y. J. Suh, *Maximal spacelike hypersurfaces in $H_1^{n+1}(-1)$ with zero Gauss-Kronecker curvature*, J. Korean Math. Soc. 43 (2006), 147-157.

[14] S. Y. Cheng and S. T. Yau, *Maximal spacelike hypersurfaces in the Lorentz-Minkowski space*, Ann. of Math. 104 (1976), 407-419.

[15] Y. Chu and S. Zhai, *On spacelike hypersurfaces with constant scalar curvature in the anti-de Sitter space*, Differential Geometry and its Applications 29 (2011), 737-746.

[16] T. Ishihara, *Maximal spacelike submanifolds of a pseudo-Riemannian space form of constant curvature*, Michigan Math. J. 35 (1988), 345-352.

[17] A. J. Goddard, *Some remarks on the existence of spacelike hypersurfaces of constant mean curvature*, Math. Proc. Cambridge Phil. Soc. 82 (1977), 489-495.

[18] J. Liu and Y. Gao, *The Riemannian product structures of spacelike hypersurfaces with constant k-th mean curvature in the de Sitter spaces*, J. Math. Anal. Appl. 378 (2011), 109-116.

[19] X. Liu, *Spacelike hypersurfaces of constant scalar curvature in the de Sitter space*, Atti Sem. Mat. Fir. Univ. Modena 48 (2000), 99-106.

[20] X. Liu, *Spacelike hypersurfaces of constant scalar curvature*, Manuscripta Math. 105 (2001), 367-377.

[21] S. Montiel, *An integral inequality for compact spacelike hypersurfaces in the de Sitter space and applications to the case of constant mean curvature*, Indiana Univ. Math. J. 37 (1988), 909-917.

[22] J. Ramanathan, *Complete spacelike hypersurfaces of constant mean curvature in de Sitter space*, Indiana University Math. J. 36 (1987), 349-359.

[23] T. Sakai, *Riemannian geometry* (Translated from the Japanese original by the author), American Mathematical Society, Providence,1996.

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