Connections on 2-osculator bundles of infinite dimensional manifolds

A. Suri and H. Abedi

Abstract. The geometry of the second order osculating bundle Osc^2M, is in many cases determined by its spray and the associated nonlinear connection. For a Banach manifold M, we firstly endow Osc^2M with a fiber bundle structure over M. Three different concepts which are used in many finite dimensional literatures, that is the horizontal distributions, nonlinear connections and sprays are studied in detail and their close interaction is revealed. Moreover we propose a special lift for a connection on the base manifold to Osc^2M.

M.S.C. 2010: 58A05; 58B20.
Key words: Banach manifold; osculating bundle; connection map; nonlinear spray; connection.

1 Introduction

The second-order osculator bundle of a smooth Banach manifold M, denoted by Osc^2M, consists of the space of all equivalence classes of curves on M which agree up to their accelerations. This natural extension of the tangent bundle TM was studied by numerous authors in finite and infinite dimensional cases ([1], [6], [3], [7], etc.).

There are two different approaches in higher order geometry literatures. The first one considers Osc^2M as a fibre bundle over M, and the research mainly focuses on the study of Lagrangians, Finsler structure, second order differential equations, sprays and prolongations on Osc^2M. This approach includes many references which consider finite dimensional manifolds ([1], [2], [6] and their references). The second approach includes those works which introduce the second order tangent bundle as a vector bundle over M and is the subject of study for both finite and infinite dimensional cases ([3], [4], [7]).

As a part of a continuous research in higher order geometry, we shall extend the first framework to the infinite dimensional case of Banach manifolds. Moreover our results are susceptible to be extended to the non-Banach case. However a connection between the two approaches appeared in [8] and reveals the necessity of further research on this subject.

Balkan Journal of Geometry and Its Applications, Vol.19, No.1, 2014, pp. 107-116.

In the present paper, we first endow Osc^2M with a Banach manifold structure which simultaneously offers a fibre bundle structure for $\pi : Osc^2M \longrightarrow M$. Then we deal with different geometric tools, mainly related to connections, on this bundle. We first introduce the notion of horizontal distribution on Osc^2M, and further the connection maps as well as their correspondence with horizontal distributions and nonlinear connections.

The next part deals with sprays on Osc^2M. For a Banach manifold we define the concept of the 2-spray as a vector field on Osc^2M that obeys a special condition determined by the Liouville vector field and a 2-tangent structure. The local behavior of a 2-spray and its relation with connection maps (and consequently nonlinear connections) are subsequently studied in detail. We finally introduce a special way to lift connections from the base manifold M to Osc^2M. All the maps and manifolds are assumed to be C^∞. However, if necessary, we may suppose less degrees of differentiability. Moreover, whether a partition of unity is needed, we consider our manifolds to be partitionable (see also [7]).

2 Preliminaries

Let M be a manifold modeled on a Banach space $\mathbb{E}$. For $x \in M$ define

$$C_x := \{f |\ f : (-\epsilon, \epsilon) \longrightarrow M\ ;\ f \text{ is smooth and } f(0) = x\}.$$

As a natural extension of the tangent bundle define the following equivalence relation. The curves $f, g \in C_x$ are said to be 2-equivalent iff $f'(0) = g'(0)$ and $f''(0) = g''(0)$ and we write $f \approx_x g$. Define $Osc^2_xM := C_x/\approx_x$ and the *second osculating bundle* of M to be $Osc^2M := \bigcup_{x\in M} Osc^2_xM$. Denote the representative of the equivalence class containing f with $[f]_x$ and the canonical projection $\pi : Osc^2M \longrightarrow M$ which sends $[f]_x$ to x.

Let $\mathcal{A} = \{(U_\alpha, \psi_\alpha)\}_{\alpha\in I}$ be a C^∞ atlas for M. For any $\alpha \in I$ define

$$\Psi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{E} \times \mathbb{E}[\gamma]_{x_0} \longmapsto \big((\psi_\alpha \circ \gamma)(0), (\psi_\alpha \circ \gamma)'(0), \frac{1}{2}(\psi_\alpha \circ \gamma)''(0)\big).$$

Theorem 2.1. *The family $\mathcal{B} = \{(\pi^{-1}(U_\alpha), \Psi_\alpha)\}_{\alpha\in I}$ defines a manifold structure for Osc^2M, which models it on $\mathbb{E} \times \mathbb{E} \times \mathbb{E}$.*

Proof. Clearly Ψ_α is well defined and $\bigcup_{\alpha\in I} \pi^{-1}(U_\alpha) = Osc^2M$. Ψ_α is surjective, since for any $(x, \xi_1, \xi_2) \in \psi_a(U_\alpha) \times \mathbb{E} \times \mathbb{E}$, the curve $\gamma := \psi_\alpha^{-1} \circ \bar{\gamma}$ with $\bar{\gamma}(t) = x + t\xi_1 + t^2\xi_2$ is mapped to (x, ξ_1, ξ_2) via Ψ_α. It is easily seen that Ψ_α is also injective. For any α and $\beta \in I$ with $U_{\alpha\beta} := U_\alpha \cap U_\alpha \neq \varnothing$ the overlap map

$$\Psi_\alpha \circ \Psi_\beta^{-1} : \psi_\beta(U_{\alpha\beta}) \times \mathbb{E} \times \mathbb{E} \longrightarrow \psi_\alpha(U_{\alpha\beta}) \times \mathbb{E} \times \mathbb{E}$$

is given by

$$\begin{aligned}
\Psi_\alpha \circ \Psi_\beta^{-1}(x, \xi_1, \xi_2) &= \Psi_\alpha([\gamma]_{x_0}) = \big((\psi_\alpha \circ \gamma)(0), (\psi_\alpha \circ \gamma)'(0), \tfrac{1}{2}(\psi_\alpha \circ \gamma)''(0)\big) \\
&= \big((\psi_\alpha \circ \psi_\beta^{-1} \circ \bar{\gamma})(0), (\psi_\alpha \circ \psi_\beta^{-1} \circ \bar{\gamma})'(0), \tfrac{1}{2}(\psi_\alpha \circ \psi_\beta^{-1} \circ \bar{\gamma})''(0)\big) \\
&= \big(\psi_{\alpha\beta}(x), d\psi_{\alpha\beta}(x)\xi_1, d\psi_{\alpha\beta}(x)\xi_2 + \tfrac{1}{2}d^2\psi_{\alpha\beta}(x)(\xi_1, \xi_1)\big)
\end{aligned}$$

where $\psi_\beta(x_0) = x$, $\psi_{\alpha\beta} := \psi_\alpha \circ \psi_\beta^{-1}$, $d^2 = d(d)$ means the second order differential, and $\bar{\gamma}(t) = x + t\xi_1 + t^2\xi_2$. □

Due to the transition functions of the bundle (π, Osc^2M, M), we can see that generally π is a smooth fibre bundle.

Remark 2.1. Osc^2M can be considered as a subbundle of the *fibre bundle* $\sigma : TTM \longrightarrow M$ with $\sigma(x, \xi, y, \eta) = x$. In fact Osc^2M is locally made of those elements (x, ξ, y, η) with the property $\xi = y$ ([7]). Moreover, (σ, TTM, M), and consequently (π, Osc^2M, M), admits a vector bundle structure if and only if M it is endowed with a linear connection [3, 7, 8].

By using the transition functions for the bundle $\pi : Osc^2M \longrightarrow M$, we can compute the transformation rule of natural charts for $TOsc^2M$ as follows

$$\begin{aligned} &T\Psi_\alpha \circ \Psi_\beta^{-1}(x, \xi_1, \xi_2; y, \eta_1, \eta_2) \\ &= \Big(\psi_{\alpha\beta}(x)\ ,\ d\psi_{\alpha\beta}(x)\xi_1\ ,\ d\psi_{\alpha\beta}(x)\xi_2 + \tfrac{1}{2}d^2\psi_{\alpha\beta}(x)(\xi_1, \xi_1) \\ &d\psi_{\alpha\beta}(x)y\ ,\ d\psi_{\alpha\beta}(x)\eta_1 + d^2\psi_{\alpha\beta}(x)(\xi_1, y)\ , \\ &d\psi_{\alpha\beta}(x)\eta_2 + d^2\psi_{\alpha\beta}(x)(\xi_2, y) + d^2\psi_{\alpha\beta}(x)(\xi_1, \eta_1) + \tfrac{1}{2}d^3\psi_{\alpha\beta}(x)(\xi_1, \xi_1, y)\Big). \end{aligned} \tag{2.1}$$

3 Distributions, connection maps and sprays

In this section we discuss in detail the relationship between various definitions of a nonlinear connections on $\pi : Osc^2M \longrightarrow M$. We shall hereinafter denote $\mathbf{Osc^2M}$ by $\mathbf{E}$.

3.1 Distributions

The vertical subbundle of $\pi : E \longrightarrow M$, denoted by $V\pi$, is $V\pi = \bigcup_{u\in M} V_u\pi$ where $V_u\pi = kerd_u\pi$ for $u \in E$. Locally, on a bundle chart $(\Psi, \pi^{-1}(U))$, $d\pi$ maps $(x, \xi_1, \xi_2, y, \eta_1, \eta_2)$ onto (x, y) where $x, y, \xi_i, \eta_i \in \mathbb{E}$ for $1 \le i \le 2$. It is easily seen that the elements of $V_u\pi$ locally have the form $(x, \xi_1, \xi_2, \eta_1, \eta_2)$ and $V\pi$ is a subbundle of $\tau_E : TE \longrightarrow E$ with fibres of type $\mathbb{E} \times \mathbb{E}$.

Definition 3.1. A nonlinear connection on π is a smooth subbunlde $H\pi$ of TE such that at every point $u \in E$, $V_u\pi \oplus H_u\pi = T_uE$.

Let $\nu : TE \longrightarrow V\pi$ and $h : TE \longrightarrow H\pi$ be the natural vector bundle projections. Smoothness of a nonlinear connection means that for any vector field X on E, $h \circ X$ is a smooth map. Let $\big(\pi^{-1}(U_\alpha), \Psi_\alpha\big)$ be a chart of E. Since $\nu_\alpha := \nu|_{U_\alpha}$ is continuous and linear on fibres, there exist the local maps $\overset{1}{N}_\alpha, \overset{2}{N}_\alpha : \Psi_\alpha(U_\alpha) \longrightarrow L(\mathbb{E}, \mathbb{E})$ given by

$$\begin{aligned} \nu_\alpha : U_\alpha \times \mathbb{E}^5 \quad &\longrightarrow \quad U_\alpha \times \mathbb{E}^4 \\ (x, \xi_1, \xi_2; y, \eta_1, \eta_2) \quad &\longmapsto \quad (x, \xi_1, \xi_2; 0, \eta_1 + \overset{1}{N}_\alpha (x, \xi_1, \xi_2)y, \eta_2 + \overset{2}{N}_\alpha (x, \xi_1, \xi_2)y) \end{aligned}$$

for any $(x, \xi_1, \xi_2; y, \eta_1, \eta_2) \in T_uE$. $\overset{1}{N}_\alpha$ and $\overset{2}{N}_\alpha$ are the local components of the connection for the given local chart (U_α, Ψ_α) and the sign "+" is conventional. Moreover, since $\nu \oplus h = id$, we have

$$h_\alpha(x, \xi_1, \xi_2; y, \eta_1, \eta_2) = \Big(x, \xi_1, \xi_2; y, -\overset{1}{N}_\alpha (x, \xi_1, \xi_2)y, -\overset{2}{N}_\alpha (x, \xi_1, \xi_2)y\Big).$$

The compatibility condition for $\{\overset{1}{N}_\alpha, \overset{2}{N}_\alpha\}$ and $\{\overset{1}{N}_\beta, \overset{2}{N}_\alpha\}$ on different charts $\big(\pi^{-1}(U_\alpha)$ $, \Psi_\alpha\big)$ and $\big(\pi^{-1}(U_\beta), \Psi_\beta\big)$ with $U_\alpha \cap U_\alpha \neq \varnothing$, is a consequence of the equality

$$\nu_\alpha \circ T(\Psi_\alpha \circ \Psi_\beta^{-1}) = T(\Psi_\alpha \circ \Psi_\beta^{-1}) \circ \nu_\beta.$$

A short computation shows that

$$d\psi_{\alpha\beta}(x)[\overset{1}{N}_\beta\ (u)y] = \overset{1}{N}_\alpha\ (u')d\psi_{\alpha\beta}(x)y + d^2\psi_{\alpha\beta}(x)(\xi_1, y) \tag{3.1}$$

and

$$\begin{aligned} d\psi_{\alpha\beta}(x)[\overset{2}{N}_\beta\ (u)y] + d^2\psi_{\alpha\beta}(x)\big(\xi_1, \overset{1}{N}_\beta\ (u)y\big) \quad &= \quad \overset{2}{N}_\alpha\ (u')y' + \frac{1}{2}d^3\psi_{\alpha\beta}(x)(\xi_1, \xi_1, y) \\ &\quad + d^2\psi_{\alpha\beta}(x)(\xi_2, y) \end{aligned} \tag{3.2}$$

where $u = (x, \xi_1, \xi_2)$ and $u' := \Big(\psi_{\alpha\beta}(x), d\psi_{\alpha\beta}(x)\xi_1, d\psi_{\alpha\beta}(x)\xi_2 + \frac{1}{2}d^2\psi_{\alpha\beta}(x)(\xi_1, \xi_1)\Big)$.

3.2 Connection maps

Another known and useful definition for connections due to different literatures is the concept of *connection map* [1]. We associate to a nonlinear connection on the 2-osculator bundle its connection map. It will be proved that the kernel of a connection map is a nonlinear connection.

As a first step we introduce on E the "2-tangent structure", introduced for the finite dimensional case by Miron [6].

Definition 3.2. A 2-tangent structure on E is a $C^\infty(E)$-linear map $J : \mathfrak{X}(E) \longrightarrow \mathfrak{X}(E)$ s.t. locally on a chart $(\Psi_\alpha, \pi^{-1}(U_\alpha))$,it is given by

$$J_\alpha(x, \xi_1, \xi_2; y, \eta_1, \eta_2) = (x, \xi_1, \xi_2; 0, y, \eta_1).$$

Proposition 3.1. *The map J is globally defined.*

Proof. It suffices to show that on the overlaps $J_\alpha \circ T\Psi_{\alpha\beta} = T\Psi_{\alpha\beta} \circ J_\beta$. Using the above definition and equation (2.1) we get

$$\begin{aligned} & J_\alpha \circ T\Psi_{\alpha\beta}(x, \xi_1, \xi_2; y, \eta_1, \eta_2) = \Big(\psi_{\alpha\beta}(x)\ ,\ d\psi_{\alpha\beta}(x)\xi_1\ ,\ d\psi_{\alpha\beta}(x)\xi_2 \\ & +\frac{1}{2}d^2\psi_{\alpha\beta}(x)(\xi_1, \xi_1);\ 0\ , d\psi_{\alpha\beta}(x)y\ ,\ d\psi_{\alpha\beta}(x)\eta_1 + d^2\psi_{\alpha\beta}(x)(\xi_1, y)\Big) \\ & = T\Psi_{\alpha\beta} \circ J_\beta(x, \xi_1, \xi_2; y, \eta_1, \eta_2), \end{aligned}$$

which means that J can be considered as a global map. □

Now we state the following

Definition 3.3. A connection map on $\pi : E \longrightarrow M$ is a vector bundle morphism

$$\mathcal{K} = (\mathcal{K}_1, \mathcal{K}_2) : (TE, \tau_E, E) \longrightarrow (TM \oplus TM, \tau_M \oplus \tau_M, M \oplus M)$$

such that $\mathcal{K}_2 \circ J = \mathcal{K}_1$ and $\mathcal{K}_2 \circ J^2 = \pi_*$.

The property $\mathcal{K}_1 \circ J = \pi_*$ is directly followows from the above definition, since

$$\pi_* = \mathcal{K}_2 \circ J^2 = (\mathcal{K}_2 \circ J) \circ J = \mathcal{K}_1 \circ J.$$

We now introduce the local representation of the connection map $\mathcal{K}$. On a chart (U, ϕ), since $\mathcal{K}$ is a vector bundle morphism, we have

$$\begin{aligned}
\mathcal{K}|_U(x,\xi_1,\xi_2,y,0,0) &= \big(x,\mathcal{K}_1|_U(x,\xi_1,\xi_2,y,0,0)\big) \oplus \big(x,\mathcal{K}_2|_U(x,\xi_1,\xi_2,y,0,0)\big)\\
&:= (x,\overset{1}{M}(x,\xi_1,\xi_2)y) \oplus (x,\overset{2}{M}(x,\xi_1,\xi_2)y)
\end{aligned}$$

where

$$\overset{i}{M}\colon U \times \mathbb{E} \times \mathbb{E} \longrightarrow L(\mathbb{E},\mathbb{E}) \qquad i = 1,2.$$

Using the properties of connection maps, we get

$$\begin{aligned}
\mathcal{K}|_U(x,\xi_1,\xi_2,0,\eta_1,0) &= \big(x,\mathcal{K}_1|_U \circ J(x,\xi_1,\xi_2,\eta_1,0,0)\big)\\
&\quad \oplus \big(x,\mathcal{K}_2|_U \circ J(x,\xi_1,\xi_2,\eta_1,0,0)\big)\\
&= \big(x,\pi_*(x,\xi_1,\xi_2,\eta_1,0,0)\big) \oplus \big(x,\mathcal{K}_1(x,\xi_1,\xi_2,\eta_1,0,0)\big)\\
&= (x,\eta_1) \oplus (x,\overset{1}{M}(x,\xi_1,\xi_2)\eta_1)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{K}|_U(x,\xi_1,\xi_2,0,0,\eta_2) &= \big(x,\mathcal{K}_1|_U \circ J(x,\xi_1,\xi_2,0,\eta_2,0)\big)\\
&\quad \oplus \big(x,\mathcal{K}_2|_U \circ J^2(x,\xi_1,\xi_2,\eta_1,0,0)\big)\\
&= \big(x,\pi_*(x,\xi_1,\xi_2,0,\eta_2,0)\big) \oplus \big(x,\pi_*(x,\xi_1,\xi_2,\eta_2,0,0)\big)\\
&= (x,0) \oplus (x,\eta_2).
\end{aligned}$$

As a consequence of the above computations, we have the following

Theorem 3.2. *The local expression of* $\mathcal{K} = (\mathcal{K}_1, \mathcal{K}_2)$ *is given by*

$$\begin{aligned}
\mathcal{K}_1(x,\xi_1,\xi_2;y,\eta_1,\eta_2) &= \big(x,\eta_1 + \overset{1}{M}(x,\xi_1,\xi_2)y\big)\\
\mathcal{K}_2(x,\xi_1,\xi_2;y,\eta_1,\eta_2) &= \big(x,\eta_2 + \overset{2}{M}(x,\xi_1,\xi_2)y + \overset{1}{M}(x,\xi_1,\xi_2)\eta_1\big)
\end{aligned}$$

for any $(x,\xi_1,\xi_2;y,\eta_1,\eta_2) \in T_uE$.

Proof. The result is a direct consequence of the above computations. More precisely, we have

$$\begin{aligned}
&\mathcal{K}|_U(x,\xi_1,\xi_2;y,\eta_1,\eta_2) = \mathcal{K}|_U\{(u;y,0,0) + (u;0,\eta_1,0) + (u;0,0,\eta_2)\}\\
&= \big(x,\overset{1}{M}(u)y\big) \oplus \big(x,\overset{2}{M}(u)y\big) + (x,\eta_1) \oplus (x,\overset{1}{M}(u)\eta_1) + (x,0) \oplus (x,\eta_2)\\
&= \big(x,\eta_1 + \overset{1}{M}(x,\xi_1,\xi_2)y\big) \oplus \big(x,\eta_2 + \overset{2}{M}(x,\xi_1,\xi_2)y + \overset{1}{M}(x,\xi_1,\xi_2)\eta_1\big).
\end{aligned}$$

□

To obtain the compatibility conditions for $\overset{1}{M}, \overset{2}{M}$ we verify the equality $T\psi_{\alpha\beta} \circ \mathcal{K}_{i\beta} = \mathcal{K}_{i\alpha} \circ T\Psi_{\alpha\beta}$, for $1 \leq i \leq 2$. A key step in obtaining these conditions is the equality (2.1). In fact

$$d^2\psi_{\alpha\beta}(x)(\xi_1, y) + \overset{1}{M}_\alpha (u')d\psi_{\alpha\beta}(x)y = d\psi_{\alpha\beta}(x) \overset{1}{M}_\beta (u)y \tag{3.3}$$

and

$$\begin{aligned} d\psi_{\alpha\beta}(x) \overset{2}{M}_\beta (x, \xi_1, \xi_2,)y &= \overset{2}{M}_\alpha (u')d\psi_{\alpha\beta}(x)y + \overset{1}{M}_\alpha (u')(d^2\psi_{\alpha\beta}(x)(\xi_1, y)) \\ &\quad + d^2\psi_{\alpha\beta}(x)(\xi_2, y) + \frac{1}{2}d^3\psi_{\alpha\beta}(x)(\xi_1, \xi_1, y) \end{aligned} \tag{3.4}$$

holds true.

Proposition 3.3. *Let $\mathcal{K}$ be a connection map on $\pi : E \longrightarrow M$. Then $\mathcal{K}$ determines a nonlinear connection for which $\overset{1}{N}_\alpha = \overset{1}{M}_\alpha$ and*

$$\overset{2}{N}_\alpha (x, \xi_1, \xi_2)y = \overset{2}{M}_\alpha (x, \xi_1, \xi_2)y - \overset{1}{M}_\alpha (x, \xi_1, \xi_2)[\overset{1}{M}_\alpha (x, \xi_1, \xi_2)y]. \tag{3.5}$$

Proof. These local components defined by the above equations produce a nonlinear connection if and only if they satisfy the equations (3.1) and (3.2). The compatibility condition for $\overset{1}{N}_\alpha$ and $\overset{1}{N}_\beta$ immediately follows from equation (3.3). The rest of the proof is a verification of (3.2) as follows.

$$\begin{aligned} & d\psi_{\alpha\beta}(x)[\overset{2}{N}_\beta (u)y] = d\psi_{\alpha\beta}(x) \overset{2}{M}_\alpha (u)y - d\psi_{\alpha\beta}(x)\{\overset{1}{M}_\alpha (u)[\overset{1}{M}_\alpha (u)y]\} \\ &= \overset{2}{M}_\alpha (u')d\psi_{\alpha\beta}(x)y + \overset{1}{M}_\alpha (u')(d^2\psi_{\alpha\beta}(x)(\xi_1, y)) + d^2\psi_{\alpha\beta}(x)(\xi_2, y) \\ &\quad + \frac{1}{2}d^3\psi_{\alpha\beta}(x)(\xi_1, \xi_1, y) - d^2\psi_{\alpha\beta}(x)\big(\xi_1, \overset{1}{M}_\beta (u)y\big) - \overset{1}{M}_\alpha (u')[d^2\psi_{\alpha\beta}(x)(\xi_1, y)] \\ &\quad + \overset{1}{M}_\alpha (u')[\overset{1}{M}_\alpha (u')d\psi_{\alpha\beta}(x)y] \\ &= \overset{2}{N}_\alpha (u')d\psi_{\alpha\beta}(x)y + d^2\psi_{\alpha\beta}(x)(\xi_2, y) + \frac{1}{2}d^3\psi_{\alpha\beta}(x)(\xi_1, \xi_1, y) \\ &\quad - d^2\psi_{\alpha\beta}(x)\big(\xi_1, \overset{1}{M}_\beta (u)y\big). \end{aligned}$$

□

The next two propositions reveal the mutual relation between connection maps and connections as distributions.

Proposition 3.4. *The kernel of $\mathcal{K}$ is $H\pi$ where $H\pi$ is the horizontal distribution determined by the components obtained form proposition (3.3).*

Proof. Let $X_u := (x, \xi_1, \xi_2; y, \eta_1, \eta_2) \in T_uE$. Then $\mathcal{K}(X_u) = 0$ if and only if $\eta_1 = - \overset{1}{M} (u)y$ and

$$\begin{aligned} \eta_2 &= - \overset{2}{M} (u)y - \overset{1}{M} (u)\eta_1 = - \overset{2}{M} (u)y + \overset{1}{M} (u)[\overset{1}{M} (u)y] \\ &= -\big(\overset{1}{N} (u)y + \overset{1}{N} (u)[\overset{1}{N} (u)y]\big) + \overset{1}{N} (u)[\overset{1}{N} (u)y] = \overset{2}{N} (u)y \end{aligned}$$

which means that $X_u \in H_u\pi$. Conversely, we suppose that $X_u \in H_u\pi$ or

$$X_u = (x, \xi_1, \xi_2; y, -\overset{1}{N}(u)y, \overset{2}{N}(u)y).$$

Then it is easily seen that $K(X_u) = 0$. □

Computation similar to that in theorem 3.5 shows that:

Proposition 3.5. *Let N be a nonlinear connection on π. Then one can associate a connection map with the local components $\overset{1}{M}_\alpha = \overset{1}{N}_\alpha$ and*

$$\overset{2}{M}_\alpha (x, \xi_1, \xi_2)y = \overset{2}{N}_\alpha (x, \xi_1, \xi_2)y + \overset{1}{N}_\alpha (x, \xi_1, \xi_2)[\overset{1}{N}_\alpha (x, \xi_1, \xi_2)y]. \tag{3.6}$$

3.3 2-Sprays and Nonlinear connections on Osc^2M

Another geometric tool ([5], [6]) is the concept of spray. Consider the Lioville vector field $\Gamma_2 : E \longrightarrow TE$ mapping (x, ξ_1, ξ_2) to $(x, \xi_1, \xi_2; 0, \xi_1, 2\xi_2)$. It is not hard to show that if M admits a partition of unity then Γ_2 is a global vector field.

Definition 3.4. A 2-spray on $\pi : E \longrightarrow M$ is a vector field S on E with the property $JS = \Gamma_2$.

Note that the notion of spray defined by Lang [5] contains the requirement of homogeneity for local components. More precisely Lang considered those homogeneous sprays, on TM, which associate to linear connections on M.

Theorem 3.6. *A 2-spray S on a chart $(\pi^{-1}(U_\alpha), \Psi_a)$ is locally given by*

$$S_\alpha(x, \xi_1, \xi_2) = \big(x, \xi_1, \xi_2; \xi_1, 2\xi_2, -3G_\alpha(x, \xi_1, \xi_2)\big)$$

for some smooth mapping $G_\alpha : U_\alpha \times \mathbb{E} \times \mathbb{E} \longrightarrow \mathbb{E}$.

Proof. Consider the chart $(\pi^{-1}(U_\alpha), \Psi_\alpha)$ for E and restriction of the vector field S, say S_a, to this chart. There exist the smooth functions $f_i : U_\alpha \times \mathbb{E} \times \mathbb{E} \longrightarrow \mathbb{E}$, $1 \leq i \leq 3$, such that

$$S_\alpha(x, \xi_1, \xi_2) = \Big(x, \xi_1, \xi_2, f_1(u), f_2(u), f_3(u)\Big) \quad ; \quad u = (x, \xi_1, \xi_2) \in U_\alpha \times \mathbb{E} \times \mathbb{E}.$$

Since $J \circ S = \Gamma_2$ then $f_1(x, \xi_1, \xi_2) = \xi_1$, $f_2(x, \xi_1, \xi_2) = 2\xi_2$ and $f_3(x, \xi_1, \xi_2) := -3G_\alpha(x, \xi_1, \xi_2)$ for some smooth function

$$G_\alpha : U_\alpha \times \mathbb{E} \times \mathbb{E} \longrightarrow \mathbb{E}.$$

The technical coefficient 3 is necessary to avoid extra heavy coefficients in the compatibility conditions (which holds in higher order geometry as well). To compute the compatibility condition for S_α and S_β on overlaps we first note that

$$\begin{aligned} S_\alpha \circ \Psi_{\alpha\beta}(x, \xi_1, \xi_2) \quad = \quad & \Big(\psi_{\alpha\beta}(x) \ , \ d\psi_{\alpha\beta}(x)\xi_1 \ , \ \overbrace{d\psi_{\alpha\beta}(x)\xi_2 + \frac{1}{2}d^2\psi_{\alpha\beta}(x)(\xi_1, \xi_1)}^{A} \ ; \\ & d\psi_{\alpha\beta}(x)\xi_1 \ , \ 2A \ , \ 3G_\alpha\big(\psi_{\alpha\beta}(x), d\psi_{\alpha\beta}(x)\xi_1, A\big)\Big). \end{aligned}$$

After some computations we obtain the following compatibility condition

$$
\begin{aligned}
3G_\alpha\big(\psi_{\alpha\beta}(x), d\psi_{\alpha\beta}(x)\xi_1, A\big) \quad &= \quad d^2\psi_{\alpha\beta}(x)(\xi_2,\xi_1) + \frac{1}{2}d^3\psi_{\alpha\beta}(x)(\xi_1,\xi_1,y) \\
&\quad -3d\psi_{\alpha\beta}(x)[G_\beta(x,\xi_1,\xi_2)] + d^2\psi_{\alpha\beta}(x)(\xi_1,2\xi_2) \\
&\quad = 3d^2\psi_{\alpha\beta}(x)(\xi_2,\xi_1) + \frac{1}{2}d^3\psi_{\alpha\beta}(x)(\xi_1,\xi_1,\xi_1) \\
&\quad -3d\psi_{\alpha\beta}(x)[G_\beta(x,\xi_1,\xi_2)].
\end{aligned} \tag{3.7}
$$

□

For a given 2-spray S on M, one can associate a connection map, and consequently a nonlinear connection on T^2M, in the following way.

Proposition 3.7. *Let S be a 2-spray with the local components $\{S_\alpha\}_{\alpha\in I}$. Then*

$$\overset{1}{M}_\alpha\,(x,\xi_1,\xi_2)y := \partial_3 G_\alpha(x,\xi_1,\xi_2)y \quad \text{and} \quad \overset{2}{M}_\alpha(x,\xi_1,\xi_2)y := \partial_2 G_\alpha(x,\xi_1,\xi_2)y \quad ;\alpha\in I$$

are the local components of a connection map on T^2M.

Proof. It is enough to show that $\overset{1}{M}$ and $\overset{2}{M}$ satisfy the equations (3.3) and (3.4) respectively. If $\alpha,\beta\in I$ and $U_\alpha\cap U_\beta\neq\varnothing$ then for every $(x,\xi_1,\xi_2,y)\in U_{\alpha\beta}\times\mathbb{E}^3$ we have

$$
\begin{aligned}
&d\psi_{\alpha\beta}(x)\,\overset{1}{M}_\beta\,(x,\xi_1,\xi_2)y = d\psi_{\alpha\beta}(x)G_\beta(x,\xi_1,\xi_2)y \\
&= d\psi_{\alpha\beta}(x)lim_{t\to 0}\Big(G_\beta(x,\xi_1,\xi_2+ty) - G_\beta(x,\xi_1,\xi_2)\Big)/t \\
&= lim_{t\to 0}\Big(d\psi_{\alpha\beta}(x)G_\beta(x,\xi_1,\xi_2+ty) - d\psi_{\alpha\beta}(x)G_\beta(x,\xi_1,\xi_2)\Big)/t \\
&\overset{*}{=} \partial_3 G_\alpha(u')d\psi_{\alpha\beta}(x)y + d^2\psi_{\alpha\beta}(x)(\xi_1,y) \\
&=\overset{1}{M}_\alpha\,(u')y + d^2\psi_{\alpha\beta}(x)(\xi_1,y).
\end{aligned}
$$

where in (*) we used the equation (3.7). In a similar way the compatibility conditions for $\overset{2}{M}$ can be proved. □

3.4 Lifting of connections

The aim of this section is to provide a way to lift a linear connection from the base manifold M to a connection map (and consequently to lead to a connection by proposition (3.3)) on the bundle $\pi : E \longrightarrow M$.

Theorem 3.8. *Let ∇ be a linear connection on M with the local components $\{\Gamma_\alpha\}_{\alpha\in I}$. Then there exists a nonlinear connection on E which only depends on ∇.*

Proof. Suppose that ∇ is a linear connection on M. For $\alpha\in I$ define

$$\overset{1}{M}_\alpha\,(x,\xi_1)y := \Gamma_\alpha(x)[\xi_1,y]$$
$$\overset{2}{M}_\alpha\,(x,\xi_1,\xi_2)y := \tfrac{1}{2}\{\partial_1\,\overset{1}{M}_\alpha\,(x,\xi_1)(y,\xi_1)+\,\overset{1}{M}_\alpha\,(x,\xi_1)[\overset{1}{M}_\alpha\,(x,\xi_1)y]\}+\,\overset{1}{M}_\alpha\,(x,\xi_2)y$$

where $\{\Gamma_\alpha\}_{\alpha\in I}$ are the local components associated to the linear connection ∇. Clearly the introduced local maps $\overset{1}{M}$ and $\overset{2}{M}$ depend only on the connection ∇. To prove that the pairs $\{\overset{1}{M}_\alpha, \overset{2}{M}_\alpha\}_{\alpha\in I}$ are the local components of a connection map, it suffices to show that they satisfy the compatibility conditions (3.3) and (3.4). The relation (3.3) is a direct consequence of the compatibility condition for the local forms of the connection ∇ ([7])

$$d\psi_{\alpha\beta}(x)\Gamma_\beta(x)[\xi, y] = d^2\psi_{\alpha\beta}(x)(y,\xi) + \Gamma_\alpha(\psi_{\alpha\beta}(x))[d\psi_{\alpha\beta}(x)\xi, d\psi_{\alpha\beta}(x)y].$$

For more details we refer the reader to [7], [9] or [10]. The second equality holds due to the fact that

$$\begin{aligned}
&d\psi_{\alpha\beta}(x)\partial_1 \overset{1}{M}_\beta (x,\xi_1)(y,\xi_1) = d^3\psi_{\alpha\beta}(x)(\xi_1,\xi_1,y) - d^2\psi_{\alpha\beta}(x)\big(\xi_1, \overset{1}{M}_\beta (x,\xi_1)y\big)\\
&+ \overset{1}{M}_\alpha \Big(\psi_{\alpha\beta}(x), d\psi_{\alpha\beta}(x)\xi_1\Big)d^2\psi_{\alpha\beta}(x)(\xi_1,y)\\
&+ \overset{1}{M}_\alpha \Big(\psi_{\alpha\beta}(x), d^2\psi_{\alpha\beta}(x)(\xi_1,\xi_1)\Big)d\psi_{\alpha\beta}(x)y\\
&+\partial_1 \overset{1}{M}_\alpha \Big(\psi_{\alpha\beta}(x), d\psi_{\alpha\beta}(x)\xi_1\Big)\big[d\psi_{\alpha\beta}(x)y, d\psi_{\alpha\beta}(x)\xi_1\big]
\end{aligned}$$

and

$$\begin{aligned}
&d\psi_{\alpha\beta}(x) \overset{1}{M}_\alpha (x)\Big(\xi_1, \overset{1}{M}_\alpha (x,\xi_1,y)\Big) = \overset{1}{M}_\alpha \Big(\psi_{\alpha\beta}(x), d\psi_{\alpha\beta}(x)\xi_1\Big)d^2\psi_{\alpha\beta}(x)(\xi_1,y)\\
&+ \overset{1}{M}_\alpha \Big(\psi_{\alpha\beta}(x), d\psi_{\alpha\beta}(x)\xi_1\Big) \overset{1}{M}_\alpha \Big(\psi_{\alpha\beta}(x), d\psi_{\alpha\beta}(x)\xi_1\Big)d\psi_{\alpha\beta}(x)y\\
&+d^2\psi_{\alpha\beta}(x)\Big(\xi_1, \overset{1}{M}_\beta (x,\xi_1)y\Big).
\end{aligned}$$

As a consequence, we get

$$\begin{aligned}
&d\psi_{\alpha\beta}(x) \overset{2}{M}_\beta (x,\xi_1,\xi_2)y = \frac{1}{2}\{d\psi_{\alpha\beta}(x)\partial_1 \overset{1}{M}_\beta (x,\xi_1)(y,\xi_1)\\
&+d\psi_{\alpha\beta}(x) \overset{1}{M}_\beta (x,\xi_1)[\overset{1}{M}_\beta (x,\xi_1)y]\} + d\psi_{\alpha\beta}(x) \overset{1}{M}_\beta (x,\xi_2)y\\
= \;&\frac{1}{2}d^3\psi_{\alpha\beta}(x)\big(\xi_1,\xi_1,y\big) + d^2\psi_{\alpha\beta}(x)(\xi_2,y) + \overset{1}{M}_\alpha \Big(\psi_{\alpha\beta}(x), d\psi_{\alpha\beta}(x)\xi_1\Big)d^2\psi_{\alpha\beta}(x)\\
&(\xi_1,y) + \overset{2}{M}_\alpha \Big(\psi_{\alpha\beta}(x), d\psi_{\alpha\beta}(x)\xi_1, d\psi_{\alpha\beta}(x)\xi_2 + d^2\psi_{\alpha\beta}(x)(\xi_1,\xi_1)\Big)d\psi_{\alpha\beta}(x)y
\end{aligned}$$

which completes the proof. □

References

[1] I. Bucataru, *Characterisations of the nonlinear connection in the higher order geometry*, BJGA 2, 2 (1997), 13-22.

[2] I. Bucataru, O. Constantinescu and M. Dahl, *A geometric setting for systems of ordinary differential equations*, Int. J. Geom. Methods in Mod. Phys. 8, 6 (2011), 1291-1327.

[3] C. T. J. Dodson and G. N. Galanis, *Second order tangent bundles of infinite dimensional manifolds*, J. Geom. Phys. 52 (2004), 127-136.

[4] C. T. J. Dodson and M. S. Radivoiovici, *Tangent and frame bundles of order two*, Analele Şt. Univ. "Al. I. Cuza" 28 (1982), 63-71.

[5] S. Lang, *Fundumentals of Differential Geometry*, Graduate Texts in Mathematics 191, Springer, 1999.

[6] R. Miron, *The geometry of higher order Lagrange spaces. Applications to Mechanics and Physics*, Kluwer Academic Publishers 1997.

[7] A. Suri, *Geometry of the double tangent bundles of Banach manifodls*, J. Geom. Phys. 74 (2013), 91-100.

[8] A. Suri, *Higher order tangent bundles*, http://arxiv.org/abs/1403.3111v1.

[9] A. Suri and M. Aghasi, *Connections and second order differential equations on infinite dimensional manifolds*, Int. Elec. J. Geom. 6, 2 (2013), 45-56.

[10] J. Vilms, *Connections on tangent bundles*, J. Diff. Geom. 1 (1967), 235-243.

Author's address:

Ali Suri and Hossein Abedi
Group of Mathematics, Faculty of Sciences,
Bu Ali Sina University, Hamedan, Iran.
E-mail addresses: ali.suri@gmail.com , a.suri@basu.ac.ir , h.abedi@basu.ac.ir