Cantor manifolds and compact spaces

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Abstract. Properties of Cantor manifolds that could arise in the quantum gravitational path integral are investigated. The spin geometry is compared to the intersection of Cantor sets. Given a network consisting of unions of Riemann surfaces in a superstring theory, the probable value of the number of dimensions is initially ten, and after compactification of six coordinates, the space-time would be expected to be four-dimensional.

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1 Introduction

The classical theories of gravity and elementary particle interactions are formulated on continuous space-times. When the energy-momentum tensor of the matter distribution satisfies the dominant energy condition, caustics in geodesic congruences and curvature singularities are formed. A gravitational theory without singularities may be developed through desingularization or quantization.

The quantization of gauge theories can be extended to the metric and the manifold, and supergravity and superstring theories are described by actions with metric and matter fields. Amongst the discrete spaces that may be considered in adapting the quantum formalism to the manifold is the Cantor set. When the Cantor set is defined by subdivisions of the unit interval, which may be mapped conformally to ∂D, where D is the unit disk, it would be equivalent to the boundary of a Riemann surface.

The scattering of closed strings is described by a summation over worldsheets which are may be analytically continued to Riemann surfaces of finite genus in the Euclidean formalism. The sum over surfaces can be extended to noncompact infinite-genus surfaces. Since the series expansion must be retractible to closed finite-genus surfaces, the domain of string perturbation theory may include, in addition, only effectively closed infinite-genus surfaces. By the classification theory of Riemann surfaces, this set would be included in O_G, which is characterized by the vanishing of the harmonic measure of the ideal boundary [26].

Upon the calculation of the string coupling, it may be established that a factor of 2^g is required to match the unified gauge coupling in a minimal supersymmetric standard model [10, 29]. At finite genus, this factor can be derived either from

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the number of compositions of the genus g, occurring in the degeneration limits of the surfaces [8], or the cardinality of finite subdivisions of the unit interval. When $g \to \infty$, the sum of the harmonic measures over the set of thin ends comprising the ideal boundary must equal zero or the bifurcations should cause the vanishing of the size of the ends within this class of surfaces.

Nonperturbative effects are conventionally related to the coupling with open strings through the addition of Dirichlet boundaries to the surface [14]. There also could arise physical states described by nontrivial boundaries of infinite-genus surfaces. The distinction between the category of effectively closed surfaces and infinite-genus surfaces generating string states is evident from the first theorem, where it is demonstrated that the allowed values of the numbers of subdivisions of the intervals in the Cantor sets representing surfaces with zero harmonic measure and non-zero linear measure have a null intersection.

The path integral of quantum gravity is defined over a set of four-manifolds, which can be chosen to be initially compact by the no-boundary condition, relevant at early times in cosmological models [15], and matched to spaces with other characteristics at later times. Although Minkowski space-time is noncompact, there is a conformal compactification such that the topology of the boundary is $S^2 \times \mathbb{R}^1$. The Euclidean form could be included in the path integrals if a spherical boundary is added when the three-sections are $\mathbb{R}^3$. Hypersurfaces in compact four-manifolds would be bounded three-geometries. The basic three-geometries in the geometrization conjecture are necessary for a representation by a compact three-manifold [32].

The embedding of the string world sheet into a target space-time described by a σ-model would also provide an embedding of a Cantor set of ends of a Riemann surface into a continuous manifold after analytic continuation to a Euclidean signature. The uniformization group of the effectively closed infinite-genus surface would be an infinitely-generated group of Schottky type, and it is proven in the second theorem that upper limit for the Hausdorff dimension of the limit set is 2 when isometric circles form a bounded geometry. The ideal boundary, defined by the complement of the limit set, will be represented by a Cantor set of dimension between 0 and 1 that cannot separate an embedding space with a dimension greater than or equal to 3. The union of Cantor sets also tends to yield a space with non-integer dimension. Nevertheless, a transformation, defined by the composition of a dimension-raising mapping and a homeomorphism from finitely divided subset of the unit interval T_n, to any compact metric space, exists. The conditions for the mapping are not satisfied by a continuous metric in the limit $n \to \infty$. When the ideal boundary consists of border arcs of nonzero linear measure, which might be interpreted to be physical states, there is a mapping to compact three-geometries that are hypersurfaces of the four-manifolds in the quantum gravitational path integral. Having determined the effect of the embedding of Riemann surfaces in a continuous space, the most probable dimension may be given by a grand partition function where the weighting factor is given by e^{-I_E}, being the Euclidean action determined by the mapping of the surfaces into higher-dimensional manifolds. The known dimension of space-time is found, therefore, from the relative weightings in this grand partition function, given the renormalizability a theory based on the summation over the space of Riemann surfaces.

2 The capacity of the ideal boundary and the limit set of the uniformizing group

The capacity of a boundary β of a Riemann surface is $cap(\beta) = e^{-r(\beta)}$, where the Robin constant $r(\beta)$ is

$$r(\beta) = k_\beta = lim_{t\to 0}(g + \ln|t|) \qquad (2.1)$$
$$g(z;\zeta) = k_\beta(\zeta) + p_\beta(z;\zeta),$$

p is a principal function satisfying

$$L^*(p-s) = p - s \qquad (2.2)$$
$$L_0(p) = p$$
$$L_1(p) = p$$

in the region U_γ between β and γ, with γ being the inner boundary of an end, and L is a normal operator such that $minf \le Lf \le maxf$, $\int *d(Lf) = 0$, while $s = 0$ on U, $s = \ln|t|$ on $N - \{\zeta\}$, where N is the neighbourhood of the pole of the Green function, and $s = -\frac{2\pi}{\left(\int_{\partial U_\gamma} *du_\gamma\right)} \times u_\gamma$ with u_γ being a harmonic function on U_γ.

The Robin constants of Cantor sets $E(p_1...p_\nu)$ satisfy the inequalities [27]

$$\frac{1}{2}r_{n\nu} + \frac{1}{2}\ln\frac{2}{1-\frac{1}{p_{\nu-1}}} \le \frac{1}{2}r_{n,\nu-1} \le \frac{1}{2}r_{n\nu} + \frac{1}{2}\ln\frac{2}{1-\frac{1}{p_{\nu-1}}} + \frac{1}{2}\ln p_{\nu-1}. \qquad (2.3)$$

Summing these inequalities for $\nu = 1, ..., \nu+1$ gives

$$\begin{aligned} &\frac{1}{2}r_{n2} + \frac{1}{2}\ln\frac{2}{1-\frac{1}{p_1}} + \frac{1}{2}\left[\frac{1}{2}r_{n3} + \frac{1}{2}\ln\frac{2}{1-\frac{1}{p_n}}\right] \\ &\quad + \frac{1}{4}\left[\frac{1}{2}r_{n4} + \frac{1}{2}\ln\frac{2}{1-\frac{1}{p_3}}\right] + ... + \frac{1}{2^{n-1}}\left[\frac{1}{2}r_{n,n+1} + \frac{1}{2}\ln\frac{2}{1-\frac{1}{p_n}}\right] \\ &\quad \le r_{n1} + \frac{1}{2}r_{n2} + \frac{1}{4}r_{n3} + ... + \frac{1}{2^{n-1}}r_{nn} \\ &\quad \le \frac{1}{2}r_{n2} + \frac{1}{2}\ln\frac{2}{1-\frac{1}{p_1}} + \frac{1}{2}\ln p_1 + \frac{1}{2}\left[\frac{1}{2}r_{n3} + \frac{1}{2}\ln\frac{2}{1-\frac{1}{p_2}} + \frac{1}{2}\ln p_2\right] \\ &\quad + ... + \frac{1}{2^{n-1}}\left[\frac{1}{2}r_{n,n+1} + \frac{1}{2}\ln\frac{2}{1-\frac{1}{p_n}} + \frac{1}{2}\ln p_n\right] \end{aligned} \qquad (2.4)$$

and

$$\begin{aligned} &\ln\left(2^{\sum_{\nu=1}^{n} 2^{-\nu}}\right) - \sum_{\nu=1}^{n}\ln\left(1-\frac{1}{p_\nu}\right)^{2^{-\nu}} + \frac{1}{2^n}r_{n,n+1} \le r_{n1} \\ &\quad \le \ln\left(2^{\sum_{\nu=1}^{n} 2^{-\nu}}\right) - \sum_{\nu=1}^{\infty}\ln\left(1-\frac{1}{p_\nu}\right)^{2^{-\nu}} + \frac{1}{2}r_{n,n+1} + \sum_{\nu=1}^{n}\ln p_\nu^{2^{-\nu}}. \end{aligned} \qquad (2.5)$$

Since $r_{n1} = r_n$ and $r_{n,n+1} = r_0 = \ln 4$,

$$(2.6)\quad \left(1 - \frac{1}{2^n}\right)\ln 2 + \ln\left(\frac{1}{\prod_{\nu=1}^{n}\left(1-\frac{1}{p_\nu}\right)^{2^{-\nu}}}\right) + \frac{\ln 4}{2^n} \leq r_n$$

$$\leq \left(1 - \frac{1}{2^n}\right)\ln 2 + \ln\left(\frac{1}{\prod_{\nu=1}^{n}\left(1-\frac{1}{p_\nu}\right)^{2^{-\nu}}}\right) + \frac{\ln 4}{2^n} + \sum_{\nu=1}^{n}\frac{\ln p_\nu}{2^\nu}.$$

Two conditions suffice for an effectively closed surface with a boundary which is defined by a Cantor set $E(p_1 p_2 ...)$

$$(2.7)\quad \prod_{\nu=1}^{\infty}\left(1 - \frac{1}{p_\nu}\right)^{2^\nu} \to 0$$

$$\sum_{\nu=1}^{\infty}\frac{\ln p_\nu}{2^\nu} = \infty$$

From the first condition, all closed finite-genus surfaces can be included. It follows that $p_\nu \geq e^{\frac{2^\nu}{\nu(\ln\nu)^\alpha}}$. The dependence of p_ν on ν such that the surface belongs to O_G is related to a property of the upper bound for the Robin constant being saturated or infinite.

It may be verified that $p_\nu \sim e^{\frac{2^\nu}{\nu^k}}$, $k > 1$, is sufficient for the surface not to belong to O_G because $\sum_{\nu=1}^{\infty}\frac{\ln p_\nu}{2^\nu} < \infty$. The dimension of a boundary determined by a Cantor set $E(p_1 p_2 ...)$ can be evaluated.

The relation between partition functions when central charges that are fractional or equal to one [25] is indicative of a physical equivalence between string theories formulated such that the boundaries are Cantor sets of non-zero dimension and the circle S^1.

The capacity of the ideal boundary vanishes for closed finite-genus surfaces and parabolic infinite-genus surfaces. In the unit disk, the ideal boundary is represented by $(\partial D - \Lambda(G))/G$. When the discrete group acts on S^3, one-dimensional limit sets belong to a set of homeomorphism classes [16]. If the harmonic measure of the ideal boundary is zero, the linear measure vanishes for $g \geq 2$ [22].

Theorem 2.1. *The intersection of the category of Cantor sets representing O_G surfaces and the Cantor sets with non-zero linear measure is null.*

Proof. The factor introduced through division into p subintervals and deletion of the central subinterval is $\frac{p-1}{p}$. The linear measure of the set $E(p_1 p_2 ... p_n)$ is $\prod_{m=1}^{n}\left(1 - \frac{1}{p_m}\right)$. For finite n, it is non-zero and

$$(2.8)\quad lim_{n\to\infty}\prod_{m=1}^{n}\left(1 - \frac{1}{p_m}\right) \leq lim_{n\to\infty}\prod_{\nu=1}^{n}\left(1 - \frac{1}{p_{min}}\right) = lim_{n\to\infty}\left(1 - \frac{1}{p_{min}}\right)^n$$

which vanishes when $1 + \epsilon \leq p_{min} \leq p_m$ for all m, where ϵ is bounded away from zero. The linear measure of the Cantor set $E(p_1 p_2 ...)$ equals zero if p_ν is larger than one for all ν.

By the inequality,

$$\prod_{\nu=1}^{\infty}\left(1-\frac{1}{p_\nu}\right) > exp\left(-\sum_{\nu=1}^{\infty}\frac{1}{p_\nu}\right), \tag{2.9}$$

there is a lower bound for the linear measure when the sum $\sum_{\nu=1}^{\infty}\frac{1}{p_\nu}$ converges. Let $p_\nu \sim \nu^\alpha$, $\alpha > 1$. Since $\sum_{\nu=1}^{\infty}\frac{1}{\nu^\alpha} < 1 + \int_1^\infty \frac{dx}{x^\alpha} = 1 + \frac{1}{\alpha-1} = \frac{\alpha}{\alpha-1}$, it can be set equal to $e^{-\frac{\alpha}{\alpha-1}}$.

The class of surfaces derived from the two conditions in Eq.(2.7) yield either $p_\nu \geq e^{\frac{2^\nu}{\nu(\ln\nu)^\alpha}}$, $\alpha > 1$ for $\nu \geq 2$ or $1 \leq p_\nu < \frac{1}{1-e^{-2^\nu}}$ for sufficiently large ν. It has been demonstrated that the surfaces with a Cantor set of ends satisfying the first inequality do not belong to O_G because the capacity is non-zero [5].

Then, the allowed values of p_ν are bounded above by $\frac{1}{1-e^{-2^\nu}}$ for large ν if the surface is parabolic. The intersection of this set with $\{p_\nu | p_\nu \sim \nu^\alpha, \alpha > 1\}$ is null. □

When the limit set is a Cantor set characterized by the inequality $p_\nu < \nu^\alpha, \alpha < 1$, the Lebesgue measure will be zero and the Hausdorff dimension is less than one. The uniformizing groups are geometrically finite and isomorphic to Schottky-type groups [16].

The embedding of Teichmüller space in the space of quasiconformal homeomorphisms of the unit disk implies that the union of the spaces of complex structures on Riemann surfaces can be described by the diffeomorphisms on the boundary of an infinite-genus surface or a surface with an open disk deleted. Since the ideal boundary may be defined by a Cantor set, it would be necessary to consider homeomorphisms on this set, yielding a space different from $Diff(S^1)/S^1$. Indeed, beginning with a boundary consisting of a finite number of points, the increase in the dimension of the space of mappings from finite to infinite may be verified.

3 The embedding of Cantor sets in continuous manifolds

A compact metric space M such that $\dim M = n$ is an n-dimensional Cantor manifold if no closed subset L, with $\dim L \geq n-2$, separates the space M, if for every such set, the complement M/L is connected. By the Cantor manifold theorem, every compact metric space with $\dim M = n$ contains an n-dimensional Cantor manifold [11].

The difference between the dimension of a three-geometry in the quantum gravitational path integral and the boundary of a Riemann surface is greater than equal to 2. Therefore, when the three-geometries are three-dimensional Cantor manifolds, the boundaries of a Riemann surface can be mapped homeomorphically into these spaces without any separating the neighbouring region.

There is a continuous mapping from a Cantor set into any compact metric space [1]. From a theorem on dimension-raising mappings $f : M \to M'$ is a closed mapping between separable metric space M and M', there exists an integer $k \geq 1$ such that $|f^{-1}(y)| \geq k$ [11]. Although a metric cannot be defined on the Cantor set T_∞, because the Lebesgue measure is zero, it does exist on the ternary set T_n for finite n. There would exist a transformation, through a composition of a dimension-raising mapping with a homeomorphism, from T_n to a three-dimensional compact space that can be evolved into a four-dimensional space-time.

The Cantor set can represent the ends of a Riemann surface that have boundaries which are isomorphic to circles. There would be a multivalued mapping, continuous on each branch, from this ideal boundary to a compact metric space. Further, if the fundamental domain of the uniformizing group of the Riemann surface has a border arc of nonzero linear measure in ∂D, again, there would be a mapping from the boundary to a compact manifold. In both instances, the boundaries may be interpreted as quantum states that may generate a continuous geometry.

A manifold M of dimension n is $(n-2)$-connected when each Cantor set is contained in an open n-ball. The homeomorphism between a compact manifold and the n-sphere, if every Cantor set is contained in an open n-ball, is proven only for $n \geq 5$ [33].

A Cantor set K embedded in S^3 is homeomorphic to a subset of a real line if and only if $\pi_1(S^3 - (K \cup C))$ is finitely generated for all unknotted simple loops C [30]. For this embedding, $S^3 - K$ is a simply connected noncompact manifold. The Heegard splitting resembles a three-dimensional version of the thickened trivalent graph with a Cantor set of ends. By contrast, there are Cantor sets, which have non-simply connected complements in S^n, that are approximations to codimension two submanifolds [9]. An approximation to the Riemann surfaces of string theory by these Cantor sets is provided in four dimensions.

Theorem 3.1. *The upper bound for the Hausdorff dimension of the limit set of the extension of the classical Schottky group, acting on S^3, to an infinite number of generators with isometric circles belonging to a bounded geometry equals* 2.

Proof. The Hausdorff dimension of the limit set of a classical Schottky group on S^3 is less than 2 and $S^3 - \Lambda(G)$ would be simply connected [23, 31]. The limit sets of certain infinitely-generated classical Schottky groups have dimension less than n, where the discrete group acts on S^n [17]. In three dimensions, the dimension of the limit set of the infintely generated group is less than 3. This upper bound is derived from the union $\Lambda(G) = Q \cup (\Lambda(G) - Q)$, where $Q = \cap_m Q_m$, with $Q_1 = P_1$ being the ball representing the initial domain in a sequence of packings of isometric balls P such that $P_m < P_{m+1}$, P_{m+1} is a subdivision of P_m and $Q_{m+1} = \cup_{d \in Q_m} B^{int.}(d)$, where $B^{int.}(d)$ is the chain of maximal intermediate balls of the packing d, contained within crescent regions between Q_m and Q_{m+1}, which do not have the larger size under the action of group elements.

It is proven that $\dim(\Lambda(G) - Q) \leq 2$, since $\Lambda - Q_{m+1} = \Lambda - Q_m \cup \cup_{b \in Q}(\Lambda_b - B^{int.}(d))$, where $\{b\}$ consists of objects large enough to intersect the limit set of G, with $\Lambda_b = \Lambda \cap b$ [17]. By scaling arguments, $\dim\ Q \leq np = 3p$, and $\dim(\Lambda(G)) = \dim((\Lambda(G) - Q) \cup Q) \leq \max(3p, 2)$, where $p \in \left[\frac{3}{2}, 1\right]$ [28]. The limits of this range are based on the inequality

$$const. \sum_{k>N} k^{(n-1)-2np} \leq \epsilon \tag{3.1}$$

for some bounded value ϵ. However, the range $\left[\frac{3}{4}, 1\right]$ may be expanded to $\left[\frac{2}{3}, 1\right]$, since

$$\begin{aligned} \text{(3.2)} \qquad const. \sum_{k>N} k^{(n-1)-\frac{4n}{3}} &= const. \cdot k^{-1} \sum_{k>N} k^{-\frac{n}{3}} \\ &= const. \cdot \left[\zeta\left(\frac{n}{3}+1\right) - \zeta\left(\frac{n}{3}+1, N+1\right)\right] \\ &\le \epsilon. \end{aligned}$$

With the lower limit,

$$\text{(3.3)} \qquad \dim(\Lambda(G)) \le max\left(\frac{2}{3}n, n-1\right).$$

When $n = 3$,

$$\text{(3.4)} \qquad \dim(\Lambda(G)) \le 2.$$

Furthermore, the sum $\sum_{k>N} k^{(n-1)-2np}$ converges if $\frac{1}{2} < p < \frac{2}{3}$. Therefore,

$$\text{(3.5)} \qquad \begin{aligned} \dim(\Lambda(G)) &\le max\left(\left(\tfrac{2}{3}-\delta\right)n, n-1\right) = 2 \\ 0 &< \delta < \tfrac{1}{6}, \end{aligned}$$

in three dimensions. □

The limit sets of finitely-generated classical Schottky groups acting on S^3 with disjoint isometric spheres have simply connected complements, although this result is not valid necessarily for groups with other configurations of isometric spheres [16].

Given a representation of empty space by S^{-1}, with the isomorphism of reduced homology groups $\tilde{H}_{-1}(S^{-1}) = \tilde{H}_0(S^0) \simeq \mathbb{Z}$, a quantization procedure, defined by the designation of a creation operator at one point and an annihilation operator at the antipodal point on the real line, followed by an iteration of this representation for all points on $\mathbb{R}$ and multiplication of the operators by exponentials e^{-inz} and e^{inz} respectively, would describe the quantum creation of space and matter. Vacuum amplitudes in string theory would generate quantum states through the boundaries of Riemann surfaces, which can be the source of an interaction with the embedding space-time.

The dynamics of an abstract Cantor set C may be embedded into a homeomorphism R of the continuous manifold $\mathcal{M}$, by considering the product with a Cantor set of positive measure $K \subset \mathcal{M}$, such that there is a return to K after p iterations, with $R^p : \{x\} \to \{R^p x\}$. If there is such a mapping $f : \mathcal{M} \to \mathcal{M}$, then $f^p : \{x\} \times C \to \{R^p\} \times C$, which can be projected to $h_C^p : C \to C$ [2].

Let K be a compact set in $\mathbb{R}^q$, $q > 4$, and X be a compact metric space that satisfy the following conditions:

$$\text{(3.6)} \qquad \begin{aligned} \dim K + \dim X &\le q \\ \dim(K \times X) &< q \\ \dim K &\le q-3. \end{aligned}$$

When $q = 10$, K is a Cantor set of dimension less than or equal to 1 and $\dim X \leq 9$, the algebra of functions $C(X\backslash K, \mathbb{R}^{10})$ is dense in $C(X, \mathbb{R}^{10})$ [19].

A similar result holds for embeddings of Riemann surfaces in $\mathbb{R}^4$. If K is a Cantor set, then $C(X\backslash K, \mathbb{R}^4)$ is dense in $C(X, \mathbb{R}^4)$, since K is a locally homotopically one-co-connected set and X is a metric space with $\dim X \geq 2$. It follows that the embedding of Riemann surfaces in $\mathbb{R}^4$ would be the limit of a convergent sequence of embeddings of surfaces in $\mathbb{R}^4\backslash\varphi_n(K)$, where $\varphi_n : K \to \mathbb{R}^4$ and K is a Cantor set. The integrals representing amplitudes defined by these immersions would be equivalent in this limit. The existence of point particles in the space-time could be viewed as distinct from the embedding manifolds, and the new manifold would be defined with the removal of this point set. Given the representation of point particles as distributions on manifolds, this choice is the complement of the support of these distributions.

4 Spin geometry and the union of Cantor sets

A previous study of the force induced by the electromagnetic coupling of particle-antiparticle pairs indicated that two types of amplitudes could occur. In the string theory equivalent of vacuum polarization, the diagrams would be virtual or represent real processes [6] describing an interaction mediated by an extended string.

The splitting and joining of the strings in the worldsheet define separate space-time events, and the union of all such processes would generate a spin network, which approximates the space-time continuum. The spin network therefore resembles the intersection of Cantor sets that arise in fractal space-times. The connection will be considered further with regard to the estimate of the number of dimensions of the physical space-time.

The complex dimension of a self-similar fractal set can be found from the geometric zeta function [17]. For a self-similar system with scaling ratios $\{r_j\}_{i=1}^N$, the Hausdorff dimension of the attractor set [12, 21] is a solution to the equation

$$\sum_{j=1}^{N} r_j^\sigma = 1. \tag{4.1}$$

When $N = 2$ and $r_1 = \frac{x_1}{z_1}$, $r_2 = \frac{x_2}{z_2}$,

$$\begin{gathered} \left(\frac{x_1}{z_1}\right)^\sigma + \left(\frac{x_2}{z_2}\right)^\sigma = 1 \\ (x_1 z_2)^\sigma + (x_2 z_1)^\sigma = (z_1 z_2)^\sigma \end{gathered} \tag{4.2}$$

which has a nontrivial integer solutions for integer dimension only if $\sigma = 2$. For more general N and rational scalar ratios, the Hausdorff dimension of the attractor set is integer if

$$\begin{gathered} \left(\frac{x_1}{z_1}\right)^\sigma + ... + \left(\frac{x_N}{z_N}\right)^\sigma = 1 \\ (x_1 z_2 ... z_N)^\sigma + ... + (x_N z_1 ... z_{N-1})^\sigma = (z_1 ... z_N)^\sigma, \end{gathered} \tag{4.3}$$

and generically, a lower bound for N is determined by $g(\sigma)$ by the solution to Waring's problem.

The scaling ratio determines the form a Weyl transformation on the worldsheets in the string perturbation expansion. The corresponding diagrams would have nested sets of handles with a constant ratio between the sizes of consecutive handles. An endless iteration of the scaling transformation yields an infinite-genus surface.

It may be recalled that the Poincare series for the Schottky group uniformizing an infinite-genus surface converges if the radii r_n of the isometric circles decrease more rapidly than $\frac{cc'}{\sqrt{2}\zeta(2q)}n^{-q}$, with c being the lower bound for $\left|\frac{\frac{\alpha_m}{\gamma_m}-\xi_{1n}}{\xi_{2n}-\xi_{1n}}\right|$ and c' equal to the upper bound for $|\xi_{1n}-\xi_{2n}|$ and $q > \frac{1}{2}$ [4]. If $r_n \sim \rho^n$, $\rho < 1$, then it will satisfy the inequality for the definition of Green functions with two sources through an automorphic series.

It is known that a union of Cantor sets yields an expected Hausdorff dimension which is different from four [18]. A more general form of the Cantor set is defined to be

$$C(\lambda) = \sum_{r=0}^{\infty} \alpha_n \lambda^n, \qquad \alpha_n \in \mathcal{A}(\lambda) = \{0, 1-\lambda\}, \tag{4.4}$$

where λ is a real number in the interval $(0, \frac{1}{2})$, and the Hausdorff dimension is $d_H(C(\lambda)) = \frac{\ln 2}{\ln\left(\frac{1}{\lambda}\right)}$.

The arithmetic sum of two Cantor sets may yield an interval, a Cantorval, which contains intervals with endpoints that are limit points of sequences of gaps and intervals, or a disconnected Cantor set [20]. The Hausdorff dimension of the sum of the Cantor sets $C(\lambda)$ and $C(\lambda^\theta)$, for irrational θ [24], equals

$$d_H(C(\lambda) + C(\lambda^\theta)) = min\left(d_H(C(\lambda)) + d_H(C(\lambda^\theta), 1\right). \tag{4.5}$$

Since

$$d_H(C(\lambda)) + d_H(C(\lambda^\theta)) = \frac{\ln 2}{\ln\left(\frac{1}{\lambda}\right)}\left(1 + \frac{1}{\theta}\right), \tag{4.6}$$

$C(\lambda) + C(\lambda^\theta)$ has zero Lebesgue measure when

$$\frac{\ln 2}{\ln\left(\frac{1}{\lambda}\right)}\left(1 + \frac{1}{\theta}\right) < 1 \tag{4.7}$$

$$\lambda < 2^{-\left(1+\frac{1}{\theta}\right)}.$$

For each $\theta \geq 1$, let $\lambda_1(\theta)$ be the curve defined by $\lambda^\theta = 1 - 2\lambda$ and $\lambda_2(\theta)$ be the solution to $\frac{\lambda}{1-\lambda} + \frac{\lambda^\theta}{1-\lambda^\theta} = 1$. There exists a value $\lambda(\theta) \in [\lambda_1(\theta), \lambda_2(\theta)]$ such that $C(\lambda) + C(\lambda^\theta) = [0, 2]$ [27]. For (λ, θ) below the curve $\lambda_2(\theta)$ or above $\lambda_1(\theta)$, the sum of the two Cantor sets contains a disconnected component which would be a subset of a Cantor set.

The relative proportion of sums of two Cantor sets yielding one-dimensional sets and sets with disconnected components then would be less than the ratio of the areas

$$\frac{A_{interval}}{A_{Cantorset}} = \frac{\int_1^{\theta_B}(\lambda_1(\theta) - \lambda_2(\theta))d\theta}{\int_1^{\theta_B}\lambda_2(\theta)d\theta + \int_1^{\theta_B}\left(\frac{1}{2} - \lambda_1(\theta)\right)d\theta} \tag{4.8}$$

while the absolute frequency is less than

$$\frac{A_{interval}}{A_{entire}} = \frac{\int_1^{\theta_B}(\lambda_1(\theta) - \lambda_2(\theta))d\theta}{\frac{1}{2}(\theta_B - 1)} \tag{4.9}$$

in the λ, θ graph, where θ_B is chosen to be a sufficiently large number.

The frequency of sums of two higher-dimensional Cantor sets that are manifolds of integer dimension can be estimated from the n^{th} power of the ratio in one dimension. Sums of more than two Cantor sets with ratios of dissection greater than or equal to λ have been considered, and a minimum value $N(\lambda)$ follows from the inequality

$$(N-1)\frac{\lambda^2}{(1-\lambda)^3} + \frac{\lambda}{1-\lambda} \geq 1. \tag{4.10}$$

for $0 < \lambda \leq \frac{1}{3}$ [3].

This inequality follows from a relation between the sum of the length of the left subinterval and the gap of the interval for the two words w and wv, with an identification between the subdivisions and elements in the set of words,

$$x[wv]\frac{\lambda}{(1-\lambda)^2} \leq x[w]. \tag{4.11}$$

If the lengths of the words v and w are $|v| = j > 0$ and $|w| = k$ respectively, it may be shown [3] that

$$x[wv] = x[w]\left(\frac{1}{1-r_w}\right) r_{wv(k+1)} ... r_{wv(k+j)}(1 - r_{wv(k+j+1)}), \tag{4.12}$$

and from the bounds on the ratios of dissection,

$$\begin{aligned} 1 - r_{w1} &\geq \lambda \\ r_{wv(k+1)} ... r_{wv(k+j)} &\leq (1-\lambda)^j \\ 1 - r_{wv(k+j)1} &\leq 1 - \lambda, \end{aligned} \tag{4.13}$$

with the index 1 representing the choice of subinterval,

$$x[wv]\frac{1 - r_{w1}}{r_{wv(k+1)} ... r_{wv(k+j)}(1 - r_{wv(k+j)1})} \geq x[wv]\frac{\lambda}{(1-\lambda)^{j+1}}. \tag{4.14}$$

With $|v| \geq 1$, the inequality (4.11) can be made tighter for certain words.

It can be verified that $N\left(\frac{1}{3}\right) \geq \frac{7}{3}$ by the condition on the number of Cantor sets. However, it is known that the sum of the two Cantor sets $C\left(\frac{1}{3}\right) + C\left(\frac{1}{3}\right)$, where $C\left(\frac{1}{3}\right) = T_\infty$, is the interval $[0, 2]$. The coefficient of N in the inequality (4.10) may be increased to allow for additional sums of Cantor sets, including the value $N = 2$ for $\lambda = \frac{1}{3}$.

An estimate of the dimension of space-time, based on a union of Riemann surfaces, also may be given. In this network, there can be voids located between the handles of the surfaces, and the manifolds would be an approximation to a simply connected space-time.

Theorem 4.1. *The probability distribution with respect to the dimension of the embedding space is extremized first at $d = 10$ before a compactification of the extra dimensions to $d = 4$ is favored.*

Proof. The action for superstring theory may be interpreted to be a variational principle for the embedding of the string superworldsheet in the space-time background. It may be recalled that the bosonic string action is that of a σ-model

$$\frac{1}{4\pi\alpha'}\int_\Sigma d^2\xi h^{ab}\sqrt{h}g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu, \tag{4.15}$$

where X^μ is the coordinate field, is classically equivalent to the Nambu-Goto action, which is proportional to the area of the string worldsheet. The supersymmetric generalization of this integral would contain fermion fields ψ^α that are worldsheet spinors [13].

If it is embedded in a noncritical dimension, the theory is nonrenormalizable and will develop infinities. The counterterms that are added to the action will cause it to be infinite unless there is a cancellation with diagrams in the perturbative expansion of the S-matrix. The action with counterterms

(4.16)

$$I_{Eren} = \frac{1}{2\pi\alpha'}\left[\frac{1}{2}\int_\Sigma d^2\xi h^{ab}\sqrt{h}\Big[Z_{1i_{\mu\nu}}g_{\mu\nu}\partial_a X^\mu \partial X^\nu - \frac{1}{2}Z_{2i_{\mu\nu}}g_{\mu\nu}i\bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi^\mu \right.$$
$$\left. + Z_{3i_{\mu\nu}}\bar{\chi}_a\rho^\alpha\rho^a g_{\mu\nu}\psi^\mu\partial_\alpha X^\nu + \frac{1}{4}Z_{4i_{\mu\nu}}g_{\mu\nu}\bar{\psi}^\mu\psi^\nu\bar{\chi}_a\rho^a\rho^b\chi_b + ...\Big]\right]$$

would be infinite if the divergences do not cancel, since new types of terms are generated at each order of perturbation theory in the renormalization procedure. If an infinite number of terms is present and there are cancellations over a local region in field space, the fall-off conditions may be imposed on the bosons and fermions such that the divergences in the theory are removed and the action with counterterms would be finite. However, a contradiction arises with nonrenormalizability of the supersymmetric σ-model in a noncritical dimension, and there must be no such cancelation between an infinite number of terms in the action with counterterms. Since the bosonic string action is proportional to the area of the string worldsheet, and the supersymmetric σ-model is a generalization which can be reduced to the former action after integration of the fermionic coordinates, positivity would be evident. The weighting factor $e^{-I_{Eren}}$ then would vanish when the exponent tends to an infinite value.

In a grand partition function that includes a sum over dimension of the weighting factor $e^{-I_E[g_{\mu\nu},X^\mu,\psi^\nu]}$, the probability will be maximized in the preferred dimension 10. The four-dimensional nature of space-time would be predicted from the compactification of a ten-dimensional background, which has been established when the six-dimensional space, $G_2/SU(3)$, arises as the compact space in a solution to the effective field equations [7]. □

5 Conclusion

The physical effects of the boundaries of Riemann surfaces are known for gauge theories through the determination of the string coupling. The embedding in higher dimensions reveals a connection with the gravitational theory and the functional integral over the space of four-manifolds. The existence of a mapping from the boundaries of the Riemann surface to compact manifolds depends on the characteristics of the uniformizing group.

Both the capacity of the ideal boundary and the Hausdorff dimension of the limit set are considered with regard to the embedding space. It is found that, for Schottky groups with a finite number of generators and a class of infinitely-generated groups acting on S^3, the upper bound for the dimension of the limit set is 2. The generalization of the uniformizing groups of surfaces within the domain of string perturbation theory to three dimensions, and therefore will have limit sets and ideal boundaries that satisfy similar conditions. The approximation of higher-dimensional geometries by Cantor sets defined as a result of modeling boundaries of manifolds in this class does not occur, although there are approximations of codimension-two manifolds of this kind. The self-similarity of the voids resulting from the scaling in Cantor sets can be produced in Riemann surfaces. Considering a space consisting of a union of Riemann surfaces, the probabilistic value of the dimension of the noncompact component of the embedding manifold may be demonstrated to be initially ten, and then, four, after compactification.

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