Homogeneous pseudo-Riemannian structures of class $\mathcal{T}_2$ on low-dimensional generalized symmetric spaces

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Abstract. We give a contribution about the study of homogeneous pseudo-Riemannian structures on a type of homogeneous spaces, namely, the *pseudo-Riemannian generalized symmetric spaces* of low dimensions. It is well known that generalized symmetric Riemannian spaces admit homogeneous structures of the class $\mathcal{T}_2 \oplus \mathcal{T}_3$. The same property holds in the pseudo-Riemannian context. The aim of the present paper is to improve this result, showing that in the pseudo-Riemannian case three- and four-dimensional generalized symmetric spaces, not symmetric, admit homogeneous structures of class $\mathcal{T}_2$.

M.S.C. 2010: 53C20, 53C30, 53C50
Key words: Riemannian and pseudo-Riemannian metrics; homogeneous structures; generalized symmetric spaces.

1 Introduction

A (connected) pseudo-Riemannian manifold (M, g) is *homogeneous* if there exists a connected Lie group G of isometries acting transitively and effectively on it. Denoted by H the isotropy group at a fixed point $o \in M$ (the *origin*), then (M, g) can be identified with $(G/H, g)$. In general, there exists more than one group $G \subset I(M)$ (= full group of isometries of (M, g)); for any fixed choice $M = G/H$, the Lie group G acts transitively and effectively on G/H from the left. The pseudo-Riemmanian metric g of M can be considered as a G-invariant metric on G/H. The pair $(G/H, g)$ is called *homogeneous pseudo-Riemannian space.* We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H, respectively and by $\mathfrak{m}$ a complement of $\mathfrak{h}$ in $\mathfrak{g}$. If $\mathfrak{m}$ is stable under the action of $\mathfrak{h}$, then $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ is called a *reductive split*, and $(\mathfrak{g}, \mathfrak{h})$ a *reductive pair.* Contrary to the Riemannian case, for a homogeneous pseudo-Riemannian space $(M = G/H, g)$ the Lie algebra $\mathfrak{g}$ of G needs not to admit a reductive decomposition.

A special class of homogeneous spaces are symmetric spaces [18]. Roughly speaking, these are (pseudo-)Riemannian manifolds for which the geodesic symmetries are isometries. Locally, a symmetric space has the covariant derivative of its Riemann-Christoffel curvature tensor that vanishes at each point. A larger class of homogeneous

Balkan Journal of Geometry and Its Applications, Vol.19, No.2, 2014, pp. 73-85.

spaces, which includes the class of symmetric spaces, is provided by the *generalized symmetric spaces*. The theory of the generalized symmetric spaces was begun by P.J. Graham and A.J. Ledger in 1967; after, many authors contributed to this topic (A.S. Fedenko, A. Gray, V.G. Kac, K. Sekigawa, J. Wolf and so on). A systematic exposition appeared for the first time in the book by O. Kowalski in 1980 [22]. In low dimension, an explicit classification of such spaces is due to O. Kowalski in the Riemannian case [21] and to J. Černý and O. Kowalski in the pseudo-Riemannian case [6]. More recently, generalized symmetric spaces have been intensively studied under several different points of view. A basic property of these spaces is that all of them are reductive homogeneous. Taking into account their classification, a complete description about the set of homogeneous geodesics for each metric in dimension four was given in [8]. Other aspects of the geometry of four-dimensional generalized symmetric spaces have been investigated, as curvature properties [5], algebraic Ricci solitons [3], and harmonicity of vector fields [4]. S. Terzić classified generalized symmetric spaces defined as quotients of compact simple Lie groups, describing their real cohomology algebras explicitly [30] and calculating their real Pontryagin characteristic classes [31]. D. Kotschick and S. Terzić [20] proved that all generalized symmetric spaces are formal, that is, their rational homotopy type is determined by their rational cohomology algebra alone.

Strictly referred to homogeneity of a space is the concept of *homogeneous structure*. Homogeneous Riemannian structures were classified by F. Tricerri and L. Vanhecke [32], based on the W. Ambrose and I.M. Singer characterization of homogeneous Riemannian spaces [1]. The classification has three primitive classes, $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$. They correspond, under suitable topological conditions, to the real hyperbolic space, to the strictly cotorsionless (or strictly conaturally reductive) spaces, and to the naturally reductive spaces, respectively. The classification was extended to the pseudo-Riemannian case by P.M. Gadea and J.A. Oubiña in [16]. Naturally reductive spaces have been much studied for several reasons: one of these is that they are considered the simplest kind of homogeneous Riemannian spaces, after Riemannian symmetric spaces; moreover, the fact that the canonical connection corresponding to a given reductive decomposition has totally skew-symmetric torsion, has applications in Physics. Strictly cotorsionless spaces (sometimes called strictly conaturally reductive spaces) have been much less investigated. Actually, they are also of interest, as they are in some senses simpler than naturally reductive ones. For instance, they have a Dirac operator as that of a Riemannian symmetric space. Moreover, the easily seen relation with closed differential 2-forms, also makes them interesting in Physics. As a point of comparation, in the Kähler-like context, they are related to almost Kähler manifolds, as naturally reductive spaces are related to nearly Kähler manifolds.

Few simply connected homogeneous Riemannian spaces of class $\mathcal{T}_2$ are known. In [23], O. Kowalski and F. Tricerri classified Riemannian manifolds of dimensions three and four admitting a homogeneous structure of class $\mathcal{T}_2$. Few more examples are known. On the other hand, homogeneous structures of generalized symmetric Riemannian spaces were studied in [13]. In [32] and [34] F. Tricerri and L. Vanhecke proved that any simply-connected generalized symmetric Riemannian space in Kowalski's sense, admits a homogeneous Riemannian structure of class $\mathcal{T}_2 \oplus \mathcal{T}_3$. The same property holds in the pseudo-Riemannian context yet [9]. In the present paper the author classifies the spaces admitting a homogeneous pseudo-Riemannian structure of class $\mathcal{T}_2$, among three- and four-dimensional generalized symmetric pseudo-Riemannian spaces of any signature. Since the number of known spaces admitting a $\mathcal{T}_2$-homogeneous structure is rather scarce, as said above, this result wants to be an addition to the literature on the topic. On the other hand, this contribution is a easy consequence of the main theorem included into a wider study that the author is going to develop into a forthcoming paper, and concerning the classification of pseudo-Riemannian manifolds in dimension $n \leq 4$ which admit a non-trivial homogeneous structure of class $\mathcal{T}_2$.

In Section 2, we give some informations concerning pseudo-Riemannian generalized symmetric spaces and their classification in low dimensions. We also remind the definition and some properties of homogeneous structures.

In Section 3, we give a proof that each generalized symmetric space in dimension three and in dimension four of type A, B and D, admits a homogeneous structure of class $\mathcal{T}_2$.

2 Preliminaries

Let (M, g) a pseudo-Riemannian manifold of class $\mathcal{C}^\infty$. An *s-structure* on M is a family $\{s_x : x \in M\}$ of isometries of (M, g) (called *symmetries*), such that s_x has the point x as only isolated fixed point. An s-structure $\{s_x\}$ on (M, g) is said *regular* if:

1. the map $(x, y) \to s_x(y)$ of $M \times M$ into M is $\mathcal{C}^\infty$;
2. $s_x \circ s_y = s_z \circ s_x$, , where $z = s_x(y)$,

for all points x, y of M. If we define the tensor field S of type $(1, 1)$ by $S_x = (s_x)_{*x}$, for each $x \in M$, we can see that $\{s_x\}$ is regular if and only if the tensor field S is smooth and invariant with respect to all symmetries s_x. The tangent map $S_x = (s_x)_{*x}$ is a pseudo-orthogonal transformation on $T_x(M)$, admitting the null vector as the only fixed one.

An s-structure $\{s_x : x \in M\}$ is said of *order* k ($k \geq 2$) if $(s_x)^k = id_M$ and k is the least integer with this property. We say that an s-structure is of *infinity order* if such k does not exist. A *generalized symmetric pseudo-Riemannian space* is a connected pseudo-Riemannian manifold (M, g), admitting at least one regular s-structure. *Order* of a generalized symmetric pseudo-Riemannian space is the infimum of all integers $k \geq 2$ such that M admits a regular s-structure of order k (it may be that $k = \infty$). In particular, each symmetric pseudo-Riemannian space is generalized symmetric of order 2, and conversely.

Let (M, g) be a generalized symmetric pseudo-Riemannian space and $\{s_x\}$ a fixed regular s-structure on (M, g). Then, the triplet $(M, g, \{s_x\})$ is called a *regular s-manifold*. Let now ∇ denote the Levi-Civita connection and S the tangent tensor field of $\{s_x\}$. Following [14], it is possible to define a new connection $\widetilde{\nabla}$ of the s-manifold $(M, g, \{s_x\})$ by the formula

$$\widetilde{\nabla} = \nabla - T, \tag{2.1}$$

and the tensor field T is given by $T_X Y = (\nabla S)(S^{-1}Y, (I - S)^{-1}X) = (\nabla_{(I-S)^{-1}X} S)(S^{-1}Y)$, for all $X, Y \in \mathfrak{X}(M)$. The connection $\widetilde{\nabla}$ (called the *canonical connection*) is the unique linear connection on M which is invariant with respect to $\{s_x\}$ and such that $\widetilde{\nabla} S = 0$. Indeed, $\widetilde{\nabla}$ is complete and has parallel curvature $\widetilde{R}$ and parallel torsion $\widetilde{T}$, that is, $\widetilde{\nabla}\widetilde{R} = \widetilde{\nabla}\widetilde{T} = 0$.

As in the Riemannian case [22], the following results hold:

Proposition 2.1. *[6] The group G of all automorphisms of a regular s-manifold $(M, g, \{s_x\})$ is a transitive Lie transformation group of M. G is a Lie subgroup of the group of all affine transformations of $(M, \widetilde{\nabla})$. At each point $p \in M$, the factor space G/G_p with respect to the isotropy subgroup $G_p \subset G$, is a reductive homogeneous space, and the canonical connection of G/G_p coincides with $\widetilde{\nabla}$ if we identify G/G_p with M.*

Corollary 2.2. *[6] Any generalized symmetric pseudo-Riemannian space possesses at least one structure of a reductive homogeneous space with an invariant metric.*

J. Černý and O. Kowalski provided a similar classification concerning generalized symmetric pseudo-Riemannian spaces in dimension $n \leq 4$ [6]. More precisely, they proved the following theorems.

Theorem 2.3. *Any proper, simply connected generalized symmetric pseudo-Riemannian space (M, g) of dimension $n = 3$ is of order 4. It is indecomposable, and described (up to an isometry) as follows: the underlying homogeneous space G/H is the matrix group*

$$\begin{pmatrix} e^{-t} & 0 & x \\ 0 & e^t & y \\ 0 & 0 & 1 \end{pmatrix}.$$

(M, g) is the space $\mathbb{R}^3(x, y, t)$ with the pseudo-Riemannian metric

$$g = \pm(e^{2t}dx^2 + e^{-2t}dy^2) + \lambda dt^2,$$

where $\lambda \neq 0$ is a real constant. The possible signatures of g are (3,0) (0,3), (2,1), (1,2). The typical symmetry of order 4 at the initial point $(0, 0, 0)$ is the transformation

$$x' = -y, \quad y' = x, \quad t' = -t.$$

Theorem 2.4. *All proper, simply connected generalized symmetric pseudo-Riemannian spaces (M, g) of dimension $n = 4$ are of order 3 or 4, or infinity. All these spaces are indecomposable, and belong (up to an isometry) to the following four types:*

Type A: *The underlying homogeneous space is G/H, where*

$$G = \begin{pmatrix} a & b & u \\ c & d & v \\ 0 & 0 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with $ad - bc = 1$. (M, g) is the space $\mathbb{R}^4(x, y, u, v)$ with the pseudo-Riemannian metric

$$\begin{aligned} g = \; & \pm[(-x + \sqrt{1 + x^2 + y^2})du^2 + (x + \sqrt{1 + x^2 + y^2})dv^2 - 2y^2 dudv] + \\ & + \tfrac{\lambda}{(1+x^2+y^2)}[(1 + y^2)dx^2 + (1 + x^2)dy^2 - 2xydxdy], \end{aligned}$$

where $\lambda \neq 0$ is a real constant. The order is $k = 3$ and the possible signatures are (4,0), (0,4), (2,2). *The typical symmetry of order* 3 *at the initial point* $(0,0,0,0)$ *is the transformation*

$$u' = -(1/2)u - (\sqrt{3}/2)v, \quad v' = -(\sqrt{3}/2)u - (1/2)v,$$
$$x' = -(1/2)x + (\sqrt{3}/2)y, \quad y' = -(\sqrt{3}/2)x - (1/2).$$

Type B: *The underlying homogeneous space is G/H, where*

$$G = \begin{pmatrix} e^{-(x+y)} & 0 & 0 & a \\ 0 & e^x & 0 & b \\ 0 & 0 & e^y & c \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 & 0 & -w \\ 0 & 1 & 0 & -2w \\ 0 & 0 & 1 & 2w \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(M,g) *is the space* $\mathbb{R}^4(x,y,u,v)$ *with the pseudo-Riemannian metric*

$$g = \lambda(dx^2 + dy^2 + dxdy) + e^{-y}(2dx + dy)dv + e^{-x}(dx + 2dy)du,$$

where λ is a real constant. The order is $k = 3$ and the signature is always (2,2). *The typical symmetry of order* 3 *at the initial point* $(0,0,0,0)$ *is the transformation*

$$u' = -ue^{(y-x)} - v, \qquad v' = ue^{-(y+2x)} \qquad x' = y, \qquad y' = -(x+y).$$

Type C: *The underlying homogeneous space G/H is the matrix group*

$$\begin{pmatrix} e^{-t} & 0 & 0 & x \\ 0 & e^t & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(M,g) *is the space* $\mathbb{R}^4(x,z,u,t)$ *with the pseudo-Riemannian metric*

$$g = \pm(e^{2t}dx^2 + e^{-2t}dy^2) + dzdt.$$

The order is $k = 4$ and the possible signatures are (1,3), (3,1).

Type D: *The underlying homogeneous space is G/H, where*

$$G = \begin{pmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with $ad - bc = 1$. (M,g) is the space $\mathbb{R}^4(x,y,u,v)$ *with the pseudo-Riemannian metric*

$$\begin{aligned} g = \ & (\sinh(2u) - \cosh(2u)\sin(2v))dx^2 + (\sinh(2u) + \\ & + \cosh(2u)\sin(2v))dy^2 - 2\cosh(2u)\cos(2v)dxdy + \lambda(du^2 - \cosh^2(2u)dv^2), \end{aligned}$$

where $\lambda \neq 0$ is a real constant. The order is infinite and the signature is (2,2). *The typical symmetry at the initial point* $(0,0,0,0)$ *is induced by the automorphism of G of the form:*

$$a' = a, \ \ b' = (1/\alpha^2)b, \ \ c' = \alpha^2 c, \ \ d' = d, \ \ x' = (1/\alpha)x, \ \ y' = \alpha y,$$

where $\alpha \neq 0, \pm 1$.

Let $(M = G/H, g)$ be a (connected) homogeneous pseudo-Riemannian manifold. If the metric g is positive definite, then the homogeneous Riemannian space $(G/H, g)$ is always a *reductive homogeneous space*; this means that, denoted by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H respectively, there exists a subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ and $Ad(H)\mathfrak{m} \subset \mathfrak{m}$ (where $Ad : H \times \mathfrak{g} \to \mathfrak{g}$ is the adjoint representation of H in $\mathfrak{g}$). When extending Riemannian homogeneous spaces G/H to arbitrary signature, the problem arises that the group H can be noncompact, so the existence of an $Ad(H)$-invariant complement $\mathfrak{m} \subset \mathfrak{g}$ to the Lie algebra $\mathfrak{h}$ of H is not ensured, and the reductive decomposition of the Lie algebra $\mathfrak{g}$ may not exists (see an example in 4.4 [12]). It is important to stress that reductivity is not an intrinsic property of (M, g), but of the description of M as coset space G/H. In fact, the so-called *Kaigorodov space* is an example of homogeneous Lorentzian manifold which has two different coset descriptions, but only one of them is reductive [12].

For nonreductive homogeneous pseudo-Riemannian manifolds, see for instance Fels and Renner [11], Figueroa-O'Farrill, Meessen and Philip [12], and Dŭsek [10].

In the present paper, we consider homogeneous pseudo-Riemannian structures on homogeneous pseudo-Riemannian manifolds. In [16], the following definition is given:

Definition 2.1. A *homogeneous pseudo-Riemannian structure* on a pseudo- Riemannian manifold (M, g) is a tensor field T of type $(1, 2)$ on M, such that the connection $\tilde{\nabla} = \nabla - T$ satisfies $\tilde{\nabla} g = 0$, $\tilde{\nabla} R = 0$, $\tilde{\nabla} T = 0$, where ∇ denotes the Levi-Civita connection.

The geometric meaning of the existence of a homogeneous pseudo-Riemannian structure is explained by the following

Theorem 2.5. *[16] A connected, simply connected and complete pseudo-Riemannian manifold (M, g) admits a homogeneous pseudo-Riemannian structure if and only if it is a reductive homogeneous pseudo-Riemannian manifold.*

Following [17], we consider a real vector space V endowed with an inner product $\langle , \rangle$ of signature $(k, n-k)$. Let $\mathcal{T}(V)$ be the vector space of the tensor of type $(0, 3)$ on $(V, \langle , \rangle)$, such that $\mathcal{T}(V) = \{T \in \otimes^3 V^* : T_{XYZ} + T_{XZY} = 0, X, Y, Z \in V\}$, where $T_{XYZ} = \langle T_X Y, Z \rangle$. Let $c_{12} : \mathcal{T}(V) \to V^*$ be the map defined by

$$c_{12}(T)(Z) = \sum_{i=1}^{n} \varepsilon_i T_{e_i e_i Z}$$

for all $Z \in V$, where $\{e_i\}$ is a pseudo-orthonormal basis of V, $\varepsilon_i = \langle e_i, e_i \rangle$, $\varepsilon_i = -1$ if $1 \leq i \leq k$, $\varepsilon_i = 1$, if $k + 1 \leq i \leq n$.

The following theorem holds (see [17] for details).

Theorem 2.6. *[17] If dim $V \geq 3$, then $\mathcal{T}(V)$ decomposes into the orthogonal direct sum of subspaces which are invariant and irreducible under the action of the pseudo-orthogonal group $O(k, n-k)$, $\mathcal{T}(V) = \mathcal{T}_1(V) \oplus \mathcal{T}_2(V) \oplus \mathcal{T}_3(V)$, where*

$$\begin{aligned}
\mathcal{T}_1(V) &= \{T \in \mathcal{T}(V) : T_{XYZ} = \langle X, Y \rangle \varphi(Z) - \langle X, Z \rangle \varphi(Y), \varphi \in V^*\} \\
\mathcal{T}_2(V) &= \{T \in \mathcal{T}(V) : \mathfrak{S}_{XYZ} T_{XYZ} = 0, c_{12}(T) = 0\} \\
\mathcal{T}_3(V) &= \{T \in \mathcal{T}(V) : T_{XYZ} + T_{YXZ} = 0\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}_1(V)\oplus\mathcal{T}_2(V) &= \{T\in\mathcal{T}(V): \mathfrak{S}_{XYZ}T_{XYZ}=0\}\\
\mathcal{T}_1(V)\oplus\mathcal{T}_3(V) &= \{T\in\mathcal{T}(V): T_{XYZ}+T_{YXZ}=2\langle X,Y\rangle\varphi(Z)\\
&\quad -\langle X,Z\rangle\varphi(Y)-\langle Y,Z\rangle\varphi(X), \varphi\in V^*\}\\
\mathcal{T}_2(V)\oplus\mathcal{T}_3(V) &= \{T\in\mathcal{T}(V): c_{12}(T)=0\}.
\end{aligned}$$

Let (M,g) be a reductive homogeneous pseudo-Riemannian manifold and $(V,\langle,\rangle)$ the model for the tangent space T_xM, for each $x\in M$. A homogeneous pseudo-Riemannian structure T is of type $\{0\}$, $\mathcal{T}_i$ $(i=1,2,3)$, $\mathcal{T}_i\oplus\mathcal{T}_j$ $(1\le i<j\le 3)$, or $\mathcal{T}$ if, for each point $x\in M$, $T(x)\in\mathcal{T}(T_xM)$ belongs to $\{0\}$, $\mathcal{T}_i(T_xM)$ $(i=1,2,3)$, $\mathcal{T}_i(T_xM)\oplus\mathcal{T}_j(T_xM)$ $(1\le i<j\le 3)$, or $\mathcal{T}(T_xM)$, respectively.

Clearly, homogeneous spaces of type $\{0\}$ are just symmetric ones and it is worth knowing that the homogeneous spaces with a $\mathcal{T}_3$ structure are *naturally reductive spaces*. In [7], the author reports the classification of four-dimensional naturally reductive pseudo-Riemannian manifolds, similar to the one given for Riemannian case by O. Kowalski and L. Vanhecke [24]. In [17], P.M. Gadea and J.A. Oubiña provided a characterization of each of the primitive classes $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, and of $\mathcal{T}_1\oplus\mathcal{T}_2$. In [2], homogeneous pseudo-Riemannian structures of *linear type* are reviewed and studied. In the Riemannian case, they furnish characterizations of the real, complex and quaternionic hyperbolic spaces. In the Lorentzian case, a related class gives characterizations of singular homogeneous plane waves. P. Meessen proved in [25] that a time-dependent singular homogeneous plane wave admits a degenerate homogeneous pseudo-Riemannian structure in the class $\mathcal{T}_1\oplus\mathcal{T}_3$, and the singular homogeneous plane waves exhaust the degenerate case in the $\mathcal{T}_1\oplus\mathcal{T}_3$ class. Moreover he proved that a connected homogeneous Lorentzian space admitting a non-degenerate homogeneous pseudo-Riemannian structure in the class $\mathcal{T}_1\oplus\mathcal{T}_3$ is a locally symmetric space. Actually, a space of constant curvature.

3 Homogeneous pseudo-Riemannian structures on generalized symmetric spaces

In [9], the authors presented a slight modification of the proof, due to O. Kowalski, of a result valid in the Riemannian case (see [34]): homogeneous Riemannian structures of type $\mathcal{T}_2\oplus\mathcal{T}_3$ occur on a whole class of Riemannian manifolds, namely the pseudo-Riemannian generalized symmetric spaces. We are now going to adapt to the pseudo-Riemannian case a simpler proof of the previous result due to F. Tricerri and L. Vanhecke [33].

Theorem 3.1. *Each generalized symmetric pseudo-Riemannian space, which is not locally symmetric, admits a non-vanishing homogeneous pseudo-Riemannian structure of the class $\mathcal{T}_2\oplus\mathcal{T}_3$.*

Proof. It holds that T is S-invariant, i.e. $T_{SX}SY=ST_XY$. Fixed a point $P\in M$, let $\{E_1,\dots,E_n\}$ be an arbitrary pseudo-orthonormal basis of $T_P(M)$. Then, since T and g are S-invariant, we get

$$\Sigma_{i=1}^n\varepsilon_iT_{E_i}E_i=\Sigma_{i=1}^n\varepsilon_iT_{SE_i}SE_i=S(\Sigma_{i=1}^n\varepsilon_iT_{E_i}E_i),$$

$\varepsilon_i = \pm 1$. Hence,

$$(I - S)\Sigma_{i=1}^{n} \varepsilon_i T_{E_i} E_i = 0$$

and since $(I - S)$ is non-singular, we have

$$\Sigma_{i=1}^{n} \varepsilon_i T_{E_i} E_i = 0,$$

that is, $T \in \mathcal{T}_2 \oplus \mathcal{T}_3$. □

Reminding that a g.s. space admits a homogeneous structure of type $\mathcal{T}_2 \oplus \mathcal{T}_3$, then condition $c_{12}(T) = 0$ is always verified. Thus, in order to prove that a homogeneous structure belongs to the class $\mathcal{T}_2$, it is enough to prove the other condition:

$$\mathfrak{S}_{XYZ} T_{XYZ} = 0. \tag{3.1}$$

But, if (M, g) is n-dimensional generalized symmetric space, with $n \leq 4$, endowed with a pseudo-Riemannian metric, then for each type of metric of the classification made in [6], the difference tensor T of (2.1), is given by the formula

$$2g(T_X Y, Z) = g(\widetilde{T}_X Y, Z) + g(\widetilde{T}_X Z, Y) + g(\widetilde{T}_Y Z, X),$$

where $\widetilde{T}$ is the torsion of the canonical connection $\widetilde{\nabla}$. Thus, condition (3.1) is equivalent to condition

$$\mathfrak{S}_{XYZ} \widetilde{T}_{XYZ} = 0. \tag{3.2}$$

3.1 *Three-dimensional case*

According to Theorem (2.3), any proper, simply connected generalized symmetric pseudo-Riemannian space (M, g) of dimension $n = 3$ is isometric, as homogeneous space, to the group G of matrices of the form

$$\begin{pmatrix} e^{-t} & 0 & x \\ 0 & e^{t} & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Following [6] and [22], the Lie bracket [,] of the Lie algebra $\mathfrak{g}$ of G is defined by:

$$[X', Y'] = 0, \quad [X', Z'] = -X', \quad [Y', Z'] = Y',$$

where $\{X', Y', Z'\}$ is an orthogonal basis with respect to a fixed scalar product $\langle , \rangle$ on $\mathfrak{g}$.
Let $\langle X', X' \rangle = \langle Y', Y' \rangle = a$ and $\langle Z', Z' \rangle = b$; then we shall suppose $a > 0$ and $b < 0$. Thus, the scalar product $\langle , \rangle$ induces a Lorentzian metric on the matrix group G.
Put $X = X'/\sqrt{a}$, $Y = Y'/\sqrt{a}$, $Z = Z'/\sqrt{-b}$, then $\{X, Y, Z\}$ is a Lorentzian orthonormal basis such that

$$[X, Y] = 0, \quad [X, Z] = -\rho X, \quad [Y, Z] = \rho Y,$$

where $\rho = 1/\sqrt{-b}$.

Three-dimensional unimodular Lie groups endowed with a left invariant Lorentzian metric, were classified by S. Rahmani [29], who obtained a result corresponding to one made by Milnor [26] in the Riemannian case. In particular, S. Rahmani proved that: *if G is a three-dimensional connected unimodular Lie group, then there exists a pseudo-orthonormal frame field $\{e_1, e_2, e_3\}$, with e_3 time-like, such that the Lie algebra of G is one of the following:*

$$(\mathfrak{g}_1): \begin{array}{l} [e_1,e_2] = \alpha e_1 - \beta e_3 \\ [e_1,e_3] = -\alpha e_1 - \beta e_2 \\ [e_2,e_3] = \beta e_1 + \alpha e_2 + \alpha e_3, \quad \alpha \neq 0, \end{array} \qquad (\mathfrak{g}_2): \begin{array}{l} [e_1,e_2] = -\gamma e_2 - \beta e_3 \\ [e_1,e_3] = -\beta e_2 + \gamma e_3, \quad \gamma \neq 0 \\ [e_2,e_3] = \alpha e_1, \end{array}$$

$$(\mathfrak{g}_3): \begin{array}{l} [e_1,e_2] = -\gamma e_3 \\ [e_1,e_3] = -\beta e_2 \\ [e_2,e_3] = \alpha e_1, \end{array} \qquad (\mathfrak{g}_4): \begin{array}{l} [e_1,e_2] = -e_2 + (2\varepsilon - \beta) e_3, \quad \varepsilon = \pm 1 \\ [e_1,e_3] = -\beta e_2 + e_3 \\ [e_2,e_3] = \alpha e_1. \end{array}$$

Taking into account the above classification, we can notice that 3-dimensional Lorentzian generalized symmetric spaces belong to the algebra of type $\mathfrak{g}_2$, with $\alpha = 0 = \beta, \gamma = -1$. Thus, the relation (3.2) is always verified, since it is equivalent to the condition $\mathfrak{S}_{X,Y,Z}\, g([X,Y],Z) = 0$. Moreover, we get that $T \in \mathcal{T}_3$ if and only if $T_{XXZ} = 0$ [32], which is a not satisfied condition by these spaces; indeed, we know that they are not naturally reductive.

3.2 *Four-dimensional case*

3.2.1 Spaces of type A: signature $(+,+,+,+)$

The Lie algebra $\mathfrak{g}$ of the Lie group G has a reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ and $\{X_1, Y_1, X_2, Y_2, B\}$ is a basis of $\mathfrak{g}$ with $\{X_1, Y_1, X_2, Y_2\}$ basis of $\mathfrak{m}$ and $\{B\}$ basis of $\mathfrak{h}$; the Lie bracket $[\ ,\ ]$ on $\mathfrak{g}$ is given by the following table:

$[\ ,\ ]$	X_1	Y_1	X_2	Y_2	B
X_1	0	0	$-X_1$	Y_1	Y_1
Y_1	0	0	Y_1	X_1	$-X_1$
X_2	X_1	$-Y_1$	0	$-2B$	$-2Y_2$
Y_2	$-Y_1$	$-X_1$	$2B$	0	$2X_2$
B	$-Y_1$	X_1	$2Y_2$	$-2X_2$	0

and the scalar product of $\mathfrak{m}$ is:

$$\langle X_1, X_1 \rangle = \langle Y_1, Y_1 \rangle = 1, \qquad \langle X_2, X_2 \rangle = \langle Y_2, Y_2 \rangle = 2/\rho^2.$$

We put $\lambda = 2/\rho^2$ [8],[6]. With respect to the basis $\{X_1, Y_1, X_2, Y_2\}$ of $\mathfrak{m}$, the torsion tensor field $\widetilde{T}$ has components

$$\widetilde{T}(X_1, X_2) = X_1, \qquad \widetilde{T}(X_1, Y_2) = -Y_1, \qquad \widetilde{T}(Y_1, X_2) = -Y_1, \qquad \widetilde{T}(Y_1, Y_2) = -X_1.$$

Thus, with a simple computation we see that relation (3.2) is always verified.

3.2.2 Spaces of type A: signature $(+,+,-,-)$

Following [8], [6] the Lie algebra $\mathfrak{g}$ of the Lie group G has a reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ and $\{X_1, Y_1, X_2, Y_2, B\}$ is a basis of $\mathfrak{g}$ with $\{X_1, Y_1, X_2, Y_2\}$ basis of $\mathfrak{m}$ and

$\{B\}$ basis of $\mathfrak{h}$ such that the Lie bracket $[\ ,\ ]$ on $\mathfrak{g}$ is given by the following table:

$[\ ,\ ]$	X_1	Y_1	X_2	Y_2	B
X_1	0	0	$-\delta X_1$	δY_1	Y_1
Y_1	0	0	δY_1	δX_1	$-X_1$
X_2	δX_1	$-\delta Y_1$	0	$-2\delta^2 B$	$-2Y_2$
Y_2	$-\delta Y_1$	$-\delta X_1$	$2\delta^2 B$	0	$2X_2$
B	$-Y_1$	X_1	$2Y_2$	$-2X_2$	0

and the scalar product of $\mathfrak{m}$ is:

$$\langle X_1, X_1\rangle = \langle Y_1, Y_1\rangle = 1, \qquad \langle X_2, X_2\rangle = \langle Y_2, Y_2\rangle = -2,$$

where $\delta > 0$ is a real constant. With respect to the basis $\{X_1, Y_1, X_2, Y_2\}$ of $\mathfrak{m}$, the torsion tensor field $\widetilde{T}$ has non-vanishing components

$$\widetilde{T}(X_1, X_2) = \delta X_1, \quad \widetilde{T}(X_1, Y_2) = -\delta Y_1, \quad \widetilde{T}(Y_1, X_2) = -\delta Y_1, \quad \widetilde{T}(Y_1, Y_2) = -\delta X_1.$$

Thus, with a simple computation we see that relation (3.2) is always verified.

3.2.3 Spaces of type B

The Lie algebra $\mathfrak{g}$ of the Lie group G has a reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ and $\{X_1, Y_1, X_2, Y_2, A\}$ is a basis of $\mathfrak{g}$ with $\{X_1, Y_1, X_2, Y_2\}$ basis of $\mathfrak{m}$ and $\{A\}$ basis of $\mathfrak{h}$ such that the Lie bracket $[\ ,\ ]$ on $\mathfrak{g}$ is given by the following table:

$[\ ,\ ]$	X_1	Y_1	X_2	Y_2	A
X_1	0	0	$-X_1$	$\pm A + Y_1$	0
Y_1	0	0	$\mp A + Y_1$	X_1	0
X_2	X_1	$\pm A - Y_1$	0	0	$2Y_1$
Y_2	$\mp A - Y_1$	$-X_1$	0	0	$-2X_1$
A	0	0	$-2Y_1$	$2X_1$	0

and the scalar product $\langle , \rangle$ on $\mathfrak{m}$ is:

$$\langle X_1, X_2\rangle = \langle Y_1, Y_2\rangle = -1, \qquad \langle X_2, X_2\rangle = \langle Y_2, Y_2\rangle = 2\lambda.$$

With respect to the basis $\{X_1, Y_1, X_2, Y_2\}$ of $\mathfrak{m}$, the torsion tensor field $\widetilde{T}$ has non-vanishing components

$$\widetilde{T}(X_1, X_2) = X_1, \quad \widetilde{T}(X_1, Y_2) = -Y_1, \quad \widetilde{T}(Y_1, X_2) = -Y_1, \quad \widetilde{T}(Y_1, Y_2) = -X_1.$$

Thus, with a simple computation we see that relation (3.2) is always verified.

3.2.4 Spaces of type D

The Lie algebra $\mathfrak{g}$ of the Lie group G has a reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$; there exists $\{U_1, U_2, U_3, U_4, A\}$, basis of $\mathfrak{g}$ with $\{U_1, U_2, U_3, U_4\}$ basis of $\mathfrak{m}$ and $\{A\}$ basis of

$\mathfrak{h}$, such that the Lie bracket $[\ ,\ ]$ on $\mathfrak{g}$ is given by the following table:

$[\ ,\ ]$	U_1	U_2	U_3	U_4	A
U_1	0	0	0	$-U_2$	U_1
U_2	0	0	$-U_1$	0	$-U_2$
U_3	0	U_1	0	$-A$	$2U_3$
U_4	U_2	0	A	0	$-2U_4$
A	$-U_1$	U_2	$-2U_3$	$2U_4$	0

and the scalar product $\langle , \rangle$ on $\mathfrak{m}$ is:

$$\langle U_1, U_2 \rangle = 1, \qquad \langle U_3, U_4 \rangle = \lambda,$$

with λ real constant, $\lambda \neq 0$. With respect to the basis $\{U_1, U_2, U_3, U_4\}$ of $\mathfrak{m}$, the torsion tensor field $\widetilde{T}$ has non-vanishing components:

$$\widetilde{T}(U_1, U_4) = U_2, \qquad \widetilde{T}(U_2, U_3) = U_1.$$

Thus, with a simple computation we see that relation (3.2) is always verified.

Summing up the above results, we can state the following theorems:

Theorem 3.2. *Let (M, g) be a three-dimensional generalized symmetric space endowed with the metric described in theorem* (2.3)*. Then M always admits a Lorentzian homogeneous structure of class $\mathcal{T}_2$, but not of class $\mathcal{T}_3$ (since M is not a naturally reductive space).*

Theorem 3.3. *Let (M, g) be a four-dimensional generalized symmetric space endowed with one of the metrics described in theorem* (2.4)*, with exclusion of the case C. Then M always admits a pseudo-Riemannian homogeneous structure of class $\mathcal{T}_2$, but not of class $\mathcal{T}_3$ (since M is not a naturally reductive space).*

Acknowledgements. The author wishes to express her sincere gratitude toward Prof. R.A. Marinosci for her valuable comments during the preparation of this paper. The author also wishes to thank the referees for their useful remarks and suggestions.

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