A geodesic connection in Fréchet Geometry

Ali Suri

Abstract. In this paper first we propose a formula to lift a connection on M to its higher order tangent bundles T^rM, $r \in \mathbb{N}$. More precisely, for a given connection ∇ on T^rM, $r \in \mathbb{N} \cup \{0\}$, we construct the connection ∇^c on $T^{r+1}M$. Setting $\nabla^{c_i} = \nabla^{c_{i-1}c}$, we show that $\nabla^{c_\infty} = \varprojlim \nabla^{c_i}$ exists and it is a connection on the Fréchet manifold $T^\infty M = \varprojlim T^iM$ and the geodesics with respect to ∇^{c_∞} exist.

In the next step, we will consider a Riemannian manifold (M, g) with its Levi-Civita connection ∇. Under suitable conditions this procedure gives a sequence of Riemannian manifolds $\{(T^iM, g_i)\}_{i\in\mathbb{N}}$ equipped with a sequence of Riemannian connections $\{\nabla^{c_i}\}_{i\in\mathbb{N}}$. Then we show that ∇^{c_∞} produces the curves which are the (local) length minimizer of $T^\infty M$.

M.S.C. 2010: 58A05, 58B20.
Key words: Complete lift; spray; geodesic; Fréchet manifolds; Banach manifold; connection.

1 Introduction

In the first section we remind the bijective correspondence between linear connections and homogeneous sprays. Then using the results of [6] for complete lift of sprays, we propose a formula to lift a connection on M to its higher order tangent bundles T^rM, $r \in \mathbb{N}$. More precisely, for a given connection ∇ on T^rM, $r \in \mathbb{N} \cup \{0\}$, we construct its associated spray S and then we lift it to a homogeneous spray S^c on $T^{r+1}M$ [6]. Then, using the bijective correspondence between connections and sprays, we derive the connection ∇^c on $T^{r+1}M$ from S^c. Setting $\nabla^{c_i} = \nabla^{c_{i-1}c}$, we show that $\nabla^{c_\infty} = \varprojlim \nabla^{c_i}$ exists and it is a connection on the Fréchet manifold $T^\infty M = \varprojlim T^iM$ and the geodesics with respect to ∇^{c_∞} exist.

In the next step, we will consider a Riemannian manifold (M, g) with its Levi-Civita connection ∇. Using the results of the previous section, we construct the connection $\nabla^{c_1} := \nabla^c$ and the Sasaki Metric g_1 on $T^1M = TM$. It is known that if $\nabla^{c_0} := \nabla$ is a flat connection then, ∇^{c_1} is a flat connection too and ∇^{c_1} is the Levi-Civita connection of the Riemannian metric g_1 on T^1M.

Balkan Journal of Geometry and Its Applications, Vol.23, No.2, 2018, pp. 67-75.

Repeating this procedure gives a sequence of Riemannian manifolds $\{(T^iM, g_i)\}_{i\in\mathbb{N}}$ equipped with a sequence of Riemannian connections $\{\nabla^{c_i}\}_{i\in\mathbb{N}}$. Note that the limit connection $\nabla^{c_\infty} = \varprojlim \nabla^{c_i}$ is a generalized linear connection on the non-Banach manifold $T^\infty M$. The advantage of using this procedure lies in the fact that ∇^{c_∞} produces the curves which are the (local) length of $T^\infty M$ and hence we call it a geodesic connection.

In this paper we assume that all the maps and manifolds are smooth and when the partition of unity is necessary we suppose that our manifolds admit partition of unity.

2 Preliminaries

Let M be a smooth manifold modeled on the Banach space $\mathbb{E}$ and $\pi_0 : TM \longrightarrow M$ be its tangent bundle. We remind that $TM = \bigcup_{x\in M} T_xM$ such that T_xM consists of all equivalence classes of the form $[c, x]$ where

$$c \in C_x = \{c : (\epsilon, \epsilon) \longrightarrow M;\ \epsilon > 0,\ c \text{ is smooth and } c(0) = x\},$$

under the equivalence relation

$$c_1 \sim_x c_2 \Longleftrightarrow c_1'(0) = c_2'(0)$$

for $c_1,\ c_2 \in C_x$. The projection map $\pi_0 : TM \longrightarrow M$ maps $[f, x]$ onto x. If $\mathcal{A}_0 = \{(\phi_{\alpha_0} := \phi_\alpha, U_{\alpha_0} := U_\alpha);\ \alpha \in I\}$ is an atlas for M, then we have the canonical atlas $\mathcal{A}_1 = \{\big(\phi_{\alpha_1} := D\phi_{\alpha_0}, U_{\alpha_1} := \pi_0^{-1}(U_\alpha)\big);\ \alpha \in I\}$ for TM where

$$\begin{array}{rcl} \phi_{\alpha_1} : \pi_0^{-1}(U_{\alpha_0}) \longrightarrow & \longrightarrow & U_{\alpha_0} \times \mathbb{E} \\ [c, x] & \longmapsto & ((\phi_{\alpha_0} \circ c)(0), (\phi_{\alpha_0} \circ c)'(0)). \end{array}$$

Inductively one can define an atlas for $T^rM := T(T^{r-1}M)$, $r \in \mathbb{N}$, by

$$\mathcal{A}_r := \{\big(\phi_{\alpha_r} := D\phi_{a_{r-1}}, U_{\alpha_r} := \pi_{r-1}^{-1}(U_{\alpha_{r-1}})\big);\ \alpha \in I\}$$

for which $\pi_{r-1} : T^rM \longrightarrow T^{r-1}M$ is the natural projection. The model spaces for TM and T^rM are $\mathbb{E}_1 := \mathbb{E}^2$ and $\mathbb{E}_r := \mathbb{E}^{2^r}$ respectively. We add here to the convention that $T^0M = M$.

Set $\kappa_1 := id_{TM}$ and for $r \geq 2$ consider the canonical involution $\kappa_r : T^rM \longrightarrow T^rM$ which satisfies $\partial_t\partial_s f(t, s) = \kappa_r\partial_s\partial_t f(t, s)$, for any smooth map $f : (-\epsilon, \epsilon)^2 \longrightarrow T^{r-2}M$. If we consider $T^{r-2}M$ as a smooth manifold modeled on $\mathbb{E}_{r-2}$, then the charts of $T^{r-1}M$, T^rM and $T^{r+1}M$ take their values in $\mathbb{E}_{r-1} = \mathbb{E}_{r-2}^2$, $\mathbb{E}_r = \mathbb{E}_{r-2}^4$ and $\mathbb{E}_{r+1} = \mathbb{E}_{r-2}^8$ respectively. It is easy to check that the local representation of κ_r is given by

$$\begin{array}{rcl} \kappa_{r\alpha} := \phi_{\alpha_r} \circ \kappa_r \circ \phi_{\alpha_r}^{-1} : U_{\alpha_{r-2}} \times \mathbb{E}_{r-2}^3 & \longrightarrow & U_{\alpha_{r-2}} \times \mathbb{E}_{r-2}^3 \\ (x, y, X, Y) & \longmapsto & (x, X, y, Y). \end{array}$$

For $r \geq 1$ a semispray on $T^{r-1}M$ is a vector field $S : T^rM \longrightarrow T^{r+1}M$ with the additional property $\kappa_{r+1} \circ S = S$ (or equivalently $D\pi_{r-1} \circ S = id_{T^rM}$) ([1]).

Considering the atlas $\mathcal{A}_{r-1}$, we observe that locally on the chart $\big(\phi_{a_{r-1}}, U_{\alpha_{r-1}}\big)$ the local representation of the vector field S is $S_\alpha := \phi_{\alpha_{r+1}} \circ S \circ \phi_{\alpha_r}^{-1}$ and

$$S_\alpha : U_{\alpha_{r-1}} \times \mathbb{E}_{r-1} \longrightarrow U_{\alpha_{r-1}} \times \mathbb{E}^3_{r-1}\ ; \quad (x,y) \longmapsto \big(x, y; y, -2G_\alpha(x,y)\big).$$

where $G_\alpha : U_{\alpha_{r-1}} \times \mathbb{E}_{r-1} \longrightarrow \mathbb{E}_{r-1}$ locally represents S (see e.g. [1, 6]).

Definition 2.1. A semispray S on T^rM is called a (2-homogeneous) spray if for any $\alpha \in I$, $\lambda \in \mathbb{R}$ and $(x,y) \in U_{\alpha_r} \times \mathbb{E}_r$, $G_\alpha(x, \lambda y) = \lambda^2 G_\alpha(x,y)$.

Definition 2.2. The complete lift of the semispray (spray) S is defined by

$$S^c = D\kappa_{r+2} \circ \kappa_{r+3} \circ DS \circ \kappa_{r+2}. \tag{2.1}$$

According to propositions 3.5 and 4.13 of [6], the complete lift of a semispray (spray) is a semispray (respectively spray).

3 Complete lift of connections

In this section we compute the iterated complete lifts of a linear connection ∇ on M to a linear connection ∇^{c_i} on T^iM for $i \in \mathbb{N} \cup \{\infty\}$. To this end, first we establish a bijective correspondence between sprays and linear connections. Then, using the results of [6] we will lift a connection to its higher order tangent bundles. The first result is known due to Lang [4]. However, in order to make our exposition as much as possible self contained, we state a proof adopted to our terminology. First, we recall the definition of a connection from [7].

Definition 3.1. A connection on M is a vector bundle morphism

$$\nabla : TTM \longrightarrow TM$$

with the family of local components (Christoffel symbols) $\{\Gamma_\alpha\}_{\alpha \in I}$ where;

$$\Gamma_\alpha : \phi(U_\alpha) \times \mathbb{E} \to \mathcal{L}(\mathbb{E}, \mathbb{E})\,; \quad \alpha \in I,$$

and the local expression of ∇, i.e. $\nabla_\alpha = \phi_{\alpha_1} \circ \nabla \circ \phi_{\alpha_2}^{-1}$ is

$$\begin{array}{rcl} \nabla_\alpha : \phi_\alpha(U_\alpha) \times \mathbb{E}^3 & \longrightarrow & \phi_\alpha(U_\alpha) \times \mathbb{E} \\ (x, y; X, Y) & \longmapsto & (x, Y + \Gamma_\alpha(x,y)X). \end{array}$$

If the local components $\{\Gamma_\alpha\}_{\alpha \in I}$ are linear with respect to their second variable (and symmetric i.e. $\Gamma_\alpha(x)(y, X) = \Gamma_\alpha(x)(X, y)$ for all $(y, X) \in \mathbb{E}^2$), then the connection is called a linear (and symmetric) connection.

Let $\gamma : (-\epsilon, \epsilon) \longrightarrow TM$ be a curve. Then γ is a called a geodesic of ∇ if $\nabla_{\gamma'(t)}\gamma'(t) := \nabla \circ \gamma''(t) = 0$. The local representation of $\nabla_{\gamma'(t)}\gamma'(t) = 0$ is

$$\gamma''_\alpha(t) + \Gamma_\alpha(\gamma_\alpha(t))\big(\gamma'_\alpha(t), \gamma'_\alpha(t)\big) = 0;\ t \in (-\epsilon, \epsilon),\ \alpha \in I. \tag{3.1}$$

Proposition 3.1. i. *Any spray S on M gives rise to a linear and symmetric connection ∇ on M and vice versa.*
ii. *The curve $\gamma : (-\epsilon, \epsilon) \longrightarrow M$ is a geodesic of the spray S if and only if γ is a geodesic of ∇.*

Proof. Suppose that S be a spray on M with the local expression

$$\begin{array}{rcl} S_\alpha := \phi_{\alpha_2} \circ S \circ {\phi_{\alpha_1}}^{-1} : TM & \longrightarrow & TTM \\ (x,y) & \longmapsto & \big(x,y,y,-2G_\alpha(x,y)\big) \end{array}$$

where $G_\alpha, \alpha \in I$, is a 2-homogeneous function with respect to its second variable. Set $B_\alpha(x) = \dfrac{1}{2}\partial_2^2 G_\alpha(x,0)$. Clearly $B(x) \in \mathcal{L}^2_{sym}(\mathbb{E},\mathbb{E})$. According to [4] (chapter I, section 3) $G_\alpha(x,y) = \frac{1}{2}\partial_2^2 G_\alpha(x,0)(y,y)$ for any $(x,y) \in U_\alpha \times \mathbb{E}$.

Define the local components of the connection ∇ on M by $\Gamma_\alpha(x)(y,y) = 2B_\alpha(x)(y,y) = 2G_\alpha(x,y)$. Then for $(y,z) \in \mathbb{E} \times \mathbb{E}$ we have

$$\Gamma_\alpha(x)(y,z) = \frac{1}{2}\{\Gamma_\alpha(x)(y+z,y+z) - \Gamma_\alpha(x)(y,y) - \Gamma_\alpha(x)(z,z)\}.$$

Clearly $\Gamma_\alpha : U_\alpha \subset M \longrightarrow \mathcal{L}^2_{sym}(\mathbb{E},\mathbb{E})$, $\alpha \in I$, is smooth. Then the connection map, again denoted by ∇, is define by

$$\begin{array}{rcl} \nabla_\alpha : TTM|_{U_\alpha} & \longrightarrow & TM|_{U_\alpha} \\ (x,y,X,Y) & \longmapsto & \big(x, Y + \Gamma_\alpha(x)(y,X)\big). \end{array}$$

We will show that Γ_α and Γ_β satisfy the known compatibility condition

$$\begin{array}{rcl} d\phi_{\alpha\beta}\{\Gamma_\beta(x)(y,z)\} & = & d^2\phi_{\alpha\beta}(x)(y,z) \\ & & +\Gamma_\alpha\big(\phi_{\alpha\beta}(x)\big)[d\phi_{\alpha\beta}(x)y, d\phi_{\alpha\beta}(x)z]. \end{array} \tag{3.2}$$

(see e.g. [5]). Since S is a spray on M then for $\alpha,\beta \in I$ with $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$ we have $\phi_{\alpha_2} \circ {\phi_{\beta_2}}^{-1} \circ S_\beta = S_\alpha \circ \phi_{\alpha_1} \circ {\phi_{\beta_1}}^{-1}$ or equivalently

$$2d\phi_{\alpha\beta}(x)(G_\beta(x,y)) = d^2\phi_{\alpha\beta}(x)(y,y) + 2G_\alpha\big(\phi_{\alpha\beta}(x), d\phi_{\alpha\beta}(x)y\big). \tag{3.3}$$

where $\phi_{\alpha\beta} := \phi_{\alpha_1} \circ \phi_{\beta_1}^{-1}$.

As a consequence, for any $(x,y) \in U_{\alpha\beta} \times \mathbb{E}$ we have

$$\begin{array}{rcl} d\phi_{\alpha\beta}\{\Gamma_\beta(x)(y,y)\} & = & d\phi_{\alpha\beta}(x)\{2G_\beta(x,y)\} \\ & = & d^2\phi_{\alpha\beta}(x)(y,y) + 2G_\alpha(\phi_{\alpha\beta}(x), d\phi_{\alpha\beta}(x)y) \\ & = & d^2\phi_{\alpha\beta}(x)(y,y) + \Gamma_\alpha(\phi_{\alpha\beta}(x))\big(d\phi_{\alpha\beta}(x)y, d\phi_{\alpha\beta}(x)y\big). \end{array}$$

Moreover for any $(x,y,z) \in U_{\alpha\beta} \cap \mathbb{E} \times \mathbb{E}$ we have

$$\begin{array}{rcl} d\phi_{\alpha\beta}\{\Gamma_\beta(x)(y,z)\} & = & \dfrac{1}{2} d\phi_{\alpha\beta}(x)\{\Gamma_\beta(x)(y+z,y+z) - \Gamma_\beta(x)(y,y) \\ & & -\Gamma_\beta(x)(z,z)\} \end{array}$$

Using the compatibility condition for Γ_α and Γ_β on the diagonal elements we get

$$\begin{aligned} d\phi_{\alpha\beta}\{\Gamma_\beta(x)(y,z)\} &= \frac{1}{2}d^2\phi_{\alpha\beta}(x)(y+z,y+z) \\ &+\frac{1}{2}\Gamma_\alpha\big(\phi_{\alpha\beta}(x)\big)\big(d\phi_{\alpha\beta}(x)(y+z),d\phi_{\alpha\beta}(x)(y+z)\big) \\ &-\frac{1}{2}d^2\phi_{\alpha\beta}(x)(y,y)-\frac{1}{2}\Gamma_\alpha\big(\phi_{\alpha\beta}(x)\big)\big(d\phi_{\alpha\beta}(x)y,d\phi_{\alpha\beta}(x)y\big) \\ &-\frac{1}{2}d^2\phi_{\alpha\beta}(x)(z,z)-\frac{1}{2}\Gamma_\alpha\big(\phi_{\alpha\beta}(x)\big)\big(d\phi_{\alpha\beta}(x)z,d\phi_{\alpha\beta}(x)z\big) \\ &= d^2\phi_{\alpha\beta}(x)(y,z)+\Gamma_\alpha\big(\phi_{\alpha\beta}(x)\big)\big(d\phi_{\alpha\beta}(x)y,d\phi_{\alpha\beta}(x)z\big). \end{aligned}$$

Now, using partition of unity we can build a global map ∇ which is a linear and symmetric connection ([7, 5]).

For the converse suppose that ∇ be a linear and symmetric connection on M with the local components $\{\Gamma_\alpha\}_{\alpha\in I}$. Since ∇ is a global map then, the compatibility condition (3.2) is satisfied. For $\alpha\in I$, define $G_\alpha(x,y)=\frac{1}{2}\Gamma_\alpha(x)(y,y)$ and set

$$\begin{aligned} S_\alpha : TM|_{U_\alpha} &\longrightarrow TTM|_{U_\alpha} \\ (x,y) &\longmapsto (x,y,y,-2G_\alpha(x,y)) \end{aligned}$$

However,

$$\begin{aligned} 2d\phi_{\alpha\beta}(x)\big(G_\beta(x,y)\big) &= 2d\phi_{\alpha\beta}(x)\big(\frac{1}{2}\Gamma_\beta(x)(y,y)\big) \\ &= d\phi_{\alpha\beta}(x)\big(\Gamma_\beta(x)(y,y)\big) \\ &= d^2\phi_{\alpha\beta}(x)(y,y)+\Gamma_\alpha\big(\phi_{\alpha\beta}(x)\big)\big(d\phi_{\alpha\beta}(x)y,d\phi_{\alpha\beta}(x)y\big) \\ &= d^2\phi_{\alpha\beta}(x)(y,y)+2G_\alpha\big(\phi_{\alpha\beta}(x),d\phi_{\alpha\beta}(x)y\big) \end{aligned}$$

that is, the spray S can be defined.

ii. Now, suppose that $\gamma:(-\epsilon,\epsilon)\longrightarrow M$ is a geodesic of the spray S. Then, according to [6], definition 3.6, we have

$$\gamma_\alpha''(t)+2G_\alpha(\gamma_\alpha(t),\gamma_\alpha'(t))=0 \quad ;\alpha\in I.$$

Setting $\Gamma_\alpha(x)(y,y)=2G_\alpha(x,y)$ we observe that

$$\gamma_\alpha''(t)+\Gamma_\alpha(\gamma_\alpha(t))\big(\gamma_\alpha'(t),\gamma_\alpha'(t)\big)=0;\ \alpha\in I.$$

as desired (See also equation (3.1)). The converse can be proved similarly.

□

Motivation for writing proposition 3.1 was twofold. First we will use this correspondence to lift a connection form T^rM, $r\in\mathbb{N}\cup\{0\}$, to $T^{r+1}M$ and consequently to $T^\infty M$. Then, using theorem 4.11 of [6], we will show that the geodesics on $T^\infty M$ exist.

Now, suppose that ∇ be a connection on T^rM. Then, according to proposition 3.1, ∇ induces the spray S on T^rM. We compute S^c as proposed in [6] and then by using proposition 3.1, we derive its associated connection ∇^c.

Proposition 3.2. *∇^c is a connection on $T^{r+1}M$ with the local components*

$$\Gamma^c_\alpha : \phi^{-1}_{\alpha_{r+1}}(U_\alpha) \subseteq T^{r+1}M \longrightarrow \mathcal{L}^2_{sym}(\mathbb{E}_{r+1} \times \mathbb{E}_{r+1}, \mathbb{E}_{r+1}) \ ; \quad \alpha \in I,$$

which maps $\big((x,y),(X,Y),(u,v)\big) \in \phi^{-1}_{\alpha_{r+1}}(U_\alpha) \times \mathbb{E}^2_{r+1}$ to

$$\big(\Gamma_\alpha(x)(X,u), \partial_1\Gamma_\alpha(x)(X,u)y + \Gamma_\alpha(x)(Y,u) + \Gamma_\alpha(x)(X,v)\big).$$

Proof. Let ∇ be a connection on T^rM. According to proposition 3.1 we can construct its associated spray S. Using definition 2.2 (and also definition 3.4 of [6]) we have a lifted spray which is locally given by

$$S^c_\alpha(x,y,X,Y) = \big(x,y,X,Y,X,Y,-2G_\alpha(x,X), -2dG_\alpha(x,X)(y,Y)\big).$$

Set

$$\begin{array}{rcl} G^c_\alpha : \phi^{-1}_{\alpha_{r+2}}(U_\alpha) & \longrightarrow & \mathbb{E}_{r+1} = \mathbb{E}^2_r \\ \big((x,y),(X,Y)\big) & \longmapsto & \big(G_\alpha(x,X), dG_\alpha(x,X)(y,Y)\big). \end{array}$$

Using proposition 3.1 we define

$$\Gamma^c_\alpha(x,y)\big((X,Y),(X,Y)\big) := 2G^c_\alpha\big((x,y),(X,Y)\big).$$

that is

$$\begin{array}{rcl} \Gamma^c_\alpha(x,y)\big((X,Y),(X,Y)\big) & = & \Big(\Gamma_\alpha(x)(X,X), \partial_1\Gamma_\alpha(x)(X,X)y \\ & & +\Gamma_\alpha(x)(X,Y) + \Gamma_\alpha(x)(Y,X)\Big). \end{array}$$

Then Γ^c_α on non-diagonal elements $\big((X,Y),(u,v)\big) \in \mathbb{E}^2_{r+1}$ is

$$\begin{aligned} &\Gamma^c_\alpha(x,y)\big((X,Y),(u,v)\big) = \frac{1}{2}\{\Gamma^c_\alpha(x,y)\big((X,Y)+(u,v),(X,Y)+(u,v)\big) \\ &-\Gamma^c_\alpha(x,y)\big((X,Y),(X,Y)\big) - \Gamma^c_\alpha(x,y)\big((u,v),(u,v)\big)\} \\ &= \big(\Gamma_\alpha(x)(X,u), \partial_1\Gamma_\alpha(x)(X,u)y + \Gamma_\alpha(x)(Y,u) + \Gamma_\alpha(x)(X,v)\big) \end{aligned}$$

as we promised. □

Remark 3.2. For $r = 0$, the complete lift of connections in the above proposition coincides with theorem 2 of [7], that is

$$\nabla^c = \kappa_2 \circ D\nabla \circ \kappa_3 \circ D\kappa_2.$$

Definition 3.3. Let ∇ be the connection map of a linear and symmetric connection on T^rM, $r \in \mathbb{N}\cup\{0\}$. The complete lift of ∇ denoted by ∇^c, is a linear and symmetric connection on $T^{r+1}M$ with the connection map

$$\nabla^c := \kappa_{r+2} \circ D\nabla \circ \kappa_{r+3} \circ D\kappa_{r+2} \tag{3.4}$$

and its local components (Christoffel symbols) are given by proposition 3.2

Corollary 3.3. *The geodesics of ∇^c are Jacobi fields of ∇.*

Proof. Since the concept of lift in our framework coincides with that of [7], according to [7], the geodesics of ∇^c are Jacobi fields of ∇. $\square$

Theorem 3.4. i. *Let ∇ be a connection on M. If $\nabla^{c_r} = (\nabla^{c_{r-1}})^c$, $r \in \mathbb{N}$, then the projective limit, $\nabla^{c_\infty} = \varprojlim \nabla^{c_i}$ exists and is a connection on $T^\infty M$.*
ii. *For the given initial conditions, there exists a unique geodesic of ∇^{c_∞} on $T^\infty M$.*

Proof. **i.** Keeping the formalism of [6], first we show that the following diagram is commutative.

$$\begin{array}{ccc} T^2(T^{r+1}M) & \xrightarrow{\nabla^{c_{r+1}}} & T(T^{r+1}M) \\ D^2\pi_r \downarrow & & \downarrow D\pi_r \\ T^2(T^rM) & \xrightarrow{\nabla^{c_r}} & T(T^rM) \end{array}$$

Using equation (7), $D\pi_r = \pi_{r+1} \circ \kappa_{r+2}$ we see that

$$\begin{aligned} D\pi_r \circ \nabla^{c_{r+1}} &= D\pi_r \circ (\kappa_{r+2} \circ D\nabla^{c_r} \circ \kappa_{r+3} \circ D\kappa_{r+2}) \\ &= (\pi_{r+1} \circ \kappa_{r+2}) \circ (\kappa_{r+2} \circ D\nabla^{c_r} \circ \kappa_{r+3} \circ D\kappa_{r+2}) \\ &= \pi_{r+1} \circ D\nabla^{c_r} \circ \kappa_{r+3} \circ D\kappa_{r+2} \\ &= \nabla^{c_r} \circ \pi_{r+2} \circ \kappa_{r+3} \circ D\kappa_{r+2} \\ &= \nabla^{c_r} \circ D\pi_{r+1} \circ D\kappa_{r+2} \\ &= \nabla^{c_r} \circ D^2\pi_r. \end{aligned}$$

i.e. $\{\nabla^{c_r}\}_{r\in\mathbb{N}\cup\{0\}}$ form a projective system of connections on $\{T^rM\}_{r\in\mathbb{N}\cup\{0\}}$. Set $\nabla^{c_\infty} = \varprojlim \nabla^{c_i}$. Theorem 3.1 of [2] guarantees that ∇^{c_∞} is a connection on the PLB manifold $T^\infty M = \varprojlim T^rM$.
ii. For any $r \in \mathbb{N} \cup \{0\}$, let S^{c_r} be the corresponding spray induced by proposition 3.1 and the connection ∇^{c_r}. Then, according to [6], theorem 4.11, for any $\xi = (\xi_i)_{i\in\mathbb{N}} \in T_xT^\infty M$, there exists a unique geodesic $\gamma = \varprojlim \gamma_r : (-\epsilon, \epsilon) \longrightarrow T^\infty M$ of $S^{c_\infty} = \varprojlim S^{c_r}$ such that $\gamma(0) = x$ and $\gamma'(0) = \xi$. Moreover for any $r \in \mathbb{N} \cup \{0\}$, γ_r is a geodesic of S^{c_r} and consequently ∇^{c_r} with $\gamma_r(0) = x_r$ and $\gamma'(0) = \xi_r$. But

$$\begin{aligned} \nabla^{c_\infty}_{\gamma'(t)}\gamma'(t) = \nabla^{c_\infty} \circ \gamma''(t) &= (\varprojlim \nabla^{c_r}) \circ (\varprojlim \gamma''_r)(t) \\ &= \varprojlim(\nabla^{c_r} \circ \gamma''_r)(t) = 0. \end{aligned}$$

as desired. $\square$

4 A geodesic connection on $T^\infty M$

Let (M, g) be a Riemannian manifold. Moreover suppose that $\nabla^{c_0} := \nabla$ be the Levi-civita connection of the metric g. Then the Sasaki metric g_1 on $T^1M := TM$ is defined by

$$g_1(x, y)(X, Y) = g(x)(d\pi_0X, d\pi_0Y) + g(x)(\nabla^{c_0}X, \nabla^{c_0}Y) \tag{4.1}$$

for any $(x, y) \in TM$ and $X, Y \in T_{(x,y)}TM$. It is known that the Levi-Civita connection of the metric g_1 and ∇^{c_1} coincide if and only if curvature of ∇^{c_0} vanishes (see e.g. p. 238, [7]).

Moreover we have the following theorem from [3].

Theorem 4.1. *Let R_i be the curvature tensor of the connection ∇^{c_i}, $i = 0, 1$. Then $R_1 = 0$ if and only if $R = 0$.*

Now, suppose that (M, g) be a flat Riemannian manifold. Then (T^1M, g_1) is a flat Riemannian manifold with the Levi-Civita connection ∇^{c_1}. Repeating the above procedure gives a sequence of Riemannian manifolds with the associated metric connections $\{\nabla^{c_i}\}_{i\in\mathbb{N}}$ respectively. Then, theorem 3.4 guarantees that $\nabla^{c_\infty} = \varprojlim \nabla^{c_i}$ exists and it is a generalized connection on $T^\infty M = \varprojlim T^r M$.

Moreover for any $x \in T^\infty M$ and $X \in T_x T^\infty M$, there exists a unique geodesic (of ∇^{c_∞}) $\gamma(-\epsilon, \epsilon) \longrightarrow T^\infty M$ such that $\gamma(0) = x$ and $\gamma'(0) = X$.

Now, suppose that $t_0 < t_1$ be two points in $(-\epsilon, \epsilon)$, $\gamma(t_0) = x$ and $\gamma(t_1) = y$. Then, for any $i \in \mathbb{N}$, $\gamma_i(t_0) = x_i$ and $\gamma_i(t_1) = y_i$ where $x_i = p_i(x)$, $y_i = p_i(y)$ and $p_i : T^\infty M \longrightarrow T^i M$ is projection to the i'th factor. However, for any $i \in \mathbb{N}$, γ_i is the shortest path which connects x_i to y_i in $T^i M$ that is

$$l(\gamma_i) := \int_{t_0}^{t_1} g_i(\gamma_i(t))\big(\dot{\gamma}_i(t), \dot{\gamma}_i(t)\big)dt = d_i(x_i, y_i) \tag{4.2}$$

where

$$d_i(x_i, y_i) = \inf\{l(c);\ c \text{ is a curve joinig } x_i \text{ to } y_i \ \text{ on } \ T^i M\}.$$

As a consequence, γ is the curve which

$$L(\gamma) = d(x, y) \tag{4.3}$$

where

$$d(x, y) := \sum_{i=0}^{\infty} \frac{d_i(x_i, y_i)}{2^i(1 + d_i(x_i, y_i))}. \tag{4.4}$$

and

$$L(\gamma) := \sum_{i=0}^{\infty} \frac{l(\gamma_i)}{2^i(1 + l(\gamma_i))}. \tag{4.5}$$

Consider the family $\mathcal{M} = \{M_i,\ \varphi_{ji}\}_{i,j\in\mathbb{N}}$ where M_i, $i \in \mathbb{N}$, is a manifold modeled on the Hilbert space $\mathbb{E}_i$ and $\varphi_{ji} : M_j \longrightarrow M_i$, $j \geq i$, is a differentiable map. Moreover we need to

i) the model spaces $\{\mathbb{E}_i, \rho_{ji}\}_{i,j\in\mathbb{N}}$ form a projective system of vector spaces,

ii) for any $x = (x_i) \in M = \varprojlim M_i$ there exists a projective family of charts $\{(\phi_i, U_i)\}$ such that $x_i \in U_i \subseteq M_i$ and for $j \geq i$, $\rho_{ji} \circ \phi_j = \phi_i \circ \varphi_{ji}$.

Moreover suppose that g_i be a Riemannian metric on M_i, $i \in \mathbb{N}$. Then for any $x_i \in M_i$, we have the canonical linear isomorphism

$$\begin{array}{rcl} g_i(x_i)^\flat : T_{x_i}M & \longrightarrow & T^*_{x_i}M \\ v_i & \longmapsto & g_i(x_i)^\flat(v_i) = g_i(x_i)(v_i, .) \end{array}$$

with the inverse $g_i(x_i)^\sharp$.

For $j \geq i$, consider the induced bounded linear morphisms $\psi_{ji}(x_j) : T^*_{x_j}M_j \longrightarrow T^*_{x_i}M_i$ given by

$$\psi_{ji}(x_j) := g_i(x_i) \circ T_{x_j}\varphi_{ji}(x_j) \circ g_j(x_j)^\sharp.$$

It is easily seen that $\{T^*_{x_j}M_j, \psi_{ji}(x_j)\}$ forms a projective system of Hilbert spaces. For any $x \in M$ set $T^*_x M = \varprojlim T^*_{x_i}M_i$ which is endowed with the projective topology.

Definition 4.1. The 2-tensor g on $M = \varprojlim M_i$ is called compatible with the family of Riemannain metrics $\{g_i\}_{i\in\mathbb{N}}$ if for any $x = (x_i)_{i\in\mathbb{N}}$, $g(x) : T_x M \longrightarrow T^*_x M$ is equal to $\varprojlim g_i(x_i)$.

The above definition is compatible with our construction and section 4 proposes an example for this structure.

References

[1] I. Bucataru, M.F. Dahl, *A complete lift for semisprays*, Int. J. Geom. Methods Mod. Phys. 7, 2 (2010), 267-287.

[2] G.N. Galanis, *Differential and geometric structure for the tangent bundle of a projective limit manifold*, Rendiconti del Seminario Matematico di Padova 112 (2004), 1-12.

[3] O. Kowalski, *Curvature of the induced Riemannian metric on the tangent bundle of a Riemannian manifold*, J. Reine Angew. Math. 250 (1971), 124-129.

[4] S. Lang, *Fundamentals of differential geometry*, Graduate Texts in Mathematics, Vol. 191, Springer-Verlag, New York 1999.

[5] A. Suri, *Geometry of the double tangent bundles of Banach manifolds*, J. Geom. Phys. 74 (2013), 91-100.

[6] A. Suri and S. Rastegarzadeh, *Complete lift of vector fields and sprays to $T^\infty M$*, Int. J. Geom. Methods Mod. Phys. 12, 10 (2015), 1550113.

[7] J. Vilms, *Connections on tangent bundles*, J. Diff. Geom. 1 (1967), 235-243.

Author's address:

Ali Suri
Department of Mathematics, Faculty of Sciences,
Bu-Ali Sina University, Hamedan, Iran.
E-mail: a.suri@basu.ac.ir & ali.suri@gmail.com