Harmonic maps on generalized metric manifolds

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Abstract. In this paper, we study harmonic map, pluriharmonicity and harmonic morphisms on trans-S-manifolds. Different results are discussed for different cases of trans-S-manifolds as trans-S-manifolds are the genralization of C-manifolds, f-Kenmotsu and S-manifolds.

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1 Introduction

Harmonic maps on Riemannian manifolds have been studied for many years started with the paper of J. Eells and J. H. Sampson [4]. Because of analytic and geometric properties, harmonic maps have become an important and attractive field of research.

The study of harmonic maps on Riemannian manifolds endowed with some structures has its origin in a paper of Lichnerowicz's [10], in which he proved that a holomorphic map between Kähler manifolds is not only a harmonic map but also attains the minimum of the energy in its homotopy class. More over, Rawnsley [11] discussed structure preserving harmonic maps between f-manifolds by using twistorial methods. Harmonic maps on $\mathcal{C}$-manifolds were studied by Gherghe and Kenmotsu in [9]. On contact geometry, similar results as Lichnerowicz's, were attained by Ianuş, Gherghe, Pastore, Chinea [8, 3]. In [12], [13], N. A. Rehman obtained the results related to harmonic maps on Kenmotsu manifolds and on S-Manifolds. Now recently, Pablo Alegre, Luis M. Fernandez and Alicia Prieto-Martin have introduced a new class of metric S-manifolds, called trans-S-manifols [1], so it is natural to find the results of harmonicity on trans-S-manifols which are the genaralization of C-manifolds, S-manifolds and f-Kenmotsu manifolds.

The aim of this paper is to find some results concerning harmonic maps and harmonic morphisms on trans-S-manifolds. After we recall some well known facts about harmonic maps, trans-S-manifolds (Section 2), we prove some results concerning harmoic maps on trans S-manifolds (Section 3).

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2 Preliminaries

As a generalization of both almost complex (in even dimension) and almost contact (in odd dimension) structures, Yano introduced in [14] the notion of f-structure on a smooth manifold of dimension $2n+s$, i.e., a tensor field of type $(1,1)$ and rank $2n$ satisfying $f^3+f=0$. The existence of such a structure is equivalent to a reduction of the structural group of the tangent bundle to $U(n)\times O(s)$. Let N be a $(2n+s)$-dimensional manifold with f-structure of rank $2n$. If there exist s global vector fields $\xi_1,\xi_2,\dots,\xi_s$ on N such that:

$$f\xi_\alpha=0,\qquad \eta_\alpha\circ f=0,\qquad f^2=-I+\sum\xi_\alpha\otimes\eta_\alpha, \tag{2.1}$$

where η_α are the dual 1-forms of ξ_α, we say that the f-structure has complemented frames. For such a manifold there exists a Riemannian metric g such that

$$g(X,Y)=g(fX,fY)+\sum\eta_\alpha(X)\eta_\alpha(Y)$$

for any vector fields X and Y on N. See [2], then F is metric f-structure on manifold.

An f-structure f is normal, if it has complemented frames and

$$[f,f]+2\sum\xi_\alpha\otimes d\eta_\alpha=0,$$

where $[f,f]$ is Nijenhuis torsion of f.

Let Ω be the fundamental 2-form defined by $\Omega(X,Y)=g(X,fY)$, X, $Y\in T(N)$. A metric f-manifold is said to be a K-manifold if it is normal and $d\Omega=0$. In a K-manifold M, the structure vector fields are Killing vector fields. A K-manifold is called an S-manifold if $\Omega=d\eta$, for any i, and a C-manifold if $d\eta=0$, for any i. Note that, for $s=0$, a K-manifold is a Kählerian manifold and, for $s=1$, a K-manifold is a quasi-Sasakian manifold, an S-manifold is a Sasakian manifold and a C-manifold is a cosymplectic manifold. When $s\geq 2$, non-trivial examples can be found in [1]. Moreover, a K-manifold M is an S-manifold if and only if

$$\nabla_X\xi=-fX,\quad X\in\chi(M)\quad i=1,\dots,s. \tag{2.2}$$

and it is a C-manifold if and only if:

$$\nabla_X\xi=0,\quad X\in\chi(M)\quad i=1,\dots,s. \tag{2.3}$$

It is easy to show that in an S-manifold,

$$(\nabla_Xf)Y=\sum_1^s\{g(fx,fy)\xi_i+\eta_i(Y)f^2X\},\quad X\in\chi(M)\quad i=1,\dots,s. \tag{2.4}$$

for any $X,Y\in\chi(M)$ and in a C-manifold,

$$\nabla F=0. \tag{2.5}$$

A $(2n+s)-$dimensional metric f-manifold M is said to be a nearly trans-S-manifold if it satisfies (see [1])

$$(\nabla_Xf)Y=\sum_{i=1}^s[\alpha_i\{g(fX,fY)\xi_i+\eta_i(Y)f^2X\}+\beta_i\{g(fX,Y)\xi_i-\eta_i(Y)fX\}], \tag{2.6}$$

for certain smooth functions α, β, i = 1....s, on M and any $X, Y \in \chi(M)$. If, moreover, M is normal, then it is said to be a trans-S-manifold.
A nearly trans-S-manifold M is a trans-S-manifold if and only if

$$\nabla_X \xi_i = -\alpha_i f X - \beta_i f^2 X$$

holds for any $X \in X(M)$ and any $i = 1, \dots, s$.

Now we recall some well known facts about harmonic maps on Riemannian manifolds(for details see [4]).

Let $F : (M, g) \longrightarrow (N, h)$ be a smooth map between two Riemannian manifolds of dimensions m and n respectively. The energy density of F is a smooth function $e(F) : M \longrightarrow [0, \infty)$ given by

$$e(F)_p = \frac{1}{2} Tr_g (F^* h)(p) = \frac{1}{2} \sum_{i=1}^{m} h(F_{*p} u_i, F_{*p} u_i),$$

for any $p \in M$ and any orthonormal basis $\{u_1, \dots, u_m\}$ of $T_p M$. If M is a compact Riemannian manifold, the energy $E(F)$ of F is the integral of its energy density:

$$E(F) = \int_M e(F) \upsilon_g,$$

where υ_g is the volume measure associated with the metric g on M. A map $F \in C^\infty(M, N)$ is said to be harmonic if it is a critical point of the energy functional E on the set of all maps between (M, g) and (N, h). Now, let (M, g) be a compact Riemannian manifold. If we look at the Euler-Lagrange equation for the corresponding variational problem, a map $F : M \longrightarrow N$ is a harmonic if and only if $\tau(F) \equiv 0$, where $\tau(F)$ is tension field which is defined by

$$\tau(F) = Tr_g \tilde{\nabla} dF,$$

where $\tilde{\nabla}$ is the connection induced by the Levi-Civita connection on M, the F-pull back connection of the Levi Civita connection on N and

$$\tilde{\nabla} dF(X, Y) = \nabla^F_X dF(Y) - dF(\nabla^M_X Y). \tag{2.7}$$

A map $F : M \to M'$ is called pluriharmonic if the second fundamental form α of the map satisfies

$$\alpha(X, Y) + \alpha(fX, fY) = 0, \quad for \quad X, Y \in \Gamma TM,$$

and is called $f-$pluriharmonic if $X, Y \in D$.

3 Main Results

A smooth map $F : M \to N$ from a trans-S-manifold $M^{2m+s}(f, \xi_i, \eta_k, g, \alpha_k, \beta_k)$: $k = 1, \dots, s$ to a trans-S-manifold $N^{2n+s'}(f', \xi'_j, \eta'_j, g', \alpha'_j, \beta'_j)$: $j = 1, \dots, s'$ is defined as (f, f')-holomorphic if its differential interwines the structures, that is $dF \circ f = f' \circ dF$. In the following theorem we discuss the harmonicity of such maps.

Theorem 3.1. *Let $M^{2m+s}(f, \xi_k, \eta_k, g, \alpha_k, \beta_k)$: $k = 1, \ldots, s$ and $N^{2n+s'}(f', \xi'_j, \eta'_j, h, \alpha'_j, \beta'_j)$: $j = 1, \ldots, s'$ be two trans-S-manifolds. Let $F : M \to N$ be (f, f')-holomorphic map such that $\{\xi'_1, \ldots, \xi'_{s'}\} \subset (ImgdF)^{\perp}$, then*

1. *F is D-pluriharmonic iff $|dFX|^2 = 0$ or provided $\beta'_j = 0$, $\forall\, j \in \{1, \ldots, n\}$*

2. *F is harmonic iff F is constant or provided $\beta'_j = 0$, $\forall\, j \in \{1, \ldots, n\}$*

Proof. By definition of a trans-S-manifold, let trans-S-manifold has decomposition

$$TM = D_1 + < \xi_1, \ldots, \xi_s >, where D_1 = \{X \in M : g(x, \xi_i) = 0, \forall i = 1, \ldots, s\}.$$

The distribution D_1 is invariant decomposition with respect to structure f. Since F is (f, f')-holomorphic map, so $kerdF$ and $(kerdF)^{\perp}$ are invariant with respect to structure f and,

$$\begin{aligned} f\xi_i &= 0 \qquad \forall i = 1, \ldots, s \\ \therefore f'(dF\xi_i) &= 0 \Rightarrow \xi_i \in kerdF, \qquad \forall i = 1, \ldots, s. \end{aligned}$$

(1) Using equation (2.7),

$$\tilde{\nabla} dF(X, fY) = \nabla^F_X dF(fY) - dF(\nabla^M_X fY),$$

from definintion of trans-S-manifold, using equation (2.6),

$$\begin{aligned} \tilde{\nabla} dF(X, fY) &= \sum_{j=1}^{s'} \alpha'_j \left\{ h(dFX, dFY)\xi'_j - \eta'_j(dFY)dFX \right\} + \\ &+ \beta'_j \left\{ h(f'dFX, dFY)\xi'_j - \eta'_j(dFY)f'dFX \right\} - \\ &- dF \sum_{k=1}^{s} \alpha_k \{g(X, Y)\xi_k - \eta_k(Y)X\} + \beta_k \{g(fX, Y)\xi_k - \eta_k(Y)fX\} + \\ &+ f'\tilde{\nabla} dF(X, Y), \end{aligned}$$

and

$$\begin{aligned} \tilde{\nabla} dF(X, fY) &- \tilde{\nabla} dF(Y, fX) = \sum_{j=1}^{s'} \alpha'_j \left\{ -\eta'_j(dFY)dFX + \eta'_j(dFX)dFY \right\} + \\ &+ \beta'_j \left\{ -2h(dFX, f'dFY)\xi'_j - \eta'_j(dFY)f'dFX + \eta'_j(dFX)f'dFY \right\} + \\ &+ \sum_{k=1}^{s} \alpha_k \{\eta_k(Y)dFX - \eta_k(X)dFY\} + \beta_k \left\{ \eta_k(Y)f'dFX - \eta_k(X)f'dFY \right\}, \end{aligned}$$

taking $Y = fX$

$$\begin{aligned} \tilde{\nabla} dF(X, X) &+ \tilde{\nabla} dF(fX, fX) = \sum_{j=1}^{s'} \alpha'_j \left\{ -\eta'_j(dFX)f'dFX \right\} + \\ &+ \beta'_j \left\{ -2h(dFX, dFX)\xi'_j + (\eta'_j(dFX))^2\xi'_j + \eta'_j(dFX)dFX \right\} + \\ &+ \sum_{k=1}^{s} \alpha_k \eta_k(X)f'dFX + \beta_k \{-\eta_k(X)dFX + \eta_k(X)\eta_k(dFX)\xi_k\}, \end{aligned}$$

for $X \in D$

$$\begin{aligned}\tilde{\nabla} dF(X,X) &+ \tilde{\nabla} dF(fX,fX) = \sum_{j=1}^{s'} \alpha_j' \left\{-\eta_j'(dFX)f'dFX\right\} + \\ &+ \beta_j' \left\{-2h(dFX,dFX)\xi_j' + (\eta_j'(dFX))^2\xi_j' + \eta_j'(dFX)dFX\right\}\end{aligned}$$

for $X \in D$ and $\xi_2 \in (ImgdF)^{\perp}$

$$\tilde{\nabla} dF(X,X) + \tilde{\nabla} dF(fX,fX) = -2h(dFX,dFX)\sum_{j=1}^{s'} \beta_j'\xi_j',$$

therefore F is D-pluriharmonic iff $h(dFX,dFX) = 0$ or $\sum_{j=1}^{s'} \beta_j'\xi_j' = 0$ i. e. given all beta $\beta_j : j = 1, \ldots, s'$ are zero.
(2) To see harmonicity, let $\{e_1, \ldots, e_m, fe_1, \ldots, fe_m, \xi_1, \ldots, \xi_s\}$ be orthonormal frame on TM, then

$$\begin{aligned}\tau(F) &= \sum_{i=1}^{m} \tilde{\nabla} dF(fe_i, fe_i) + \sum_{i=1}^{m} \tilde{\nabla} dF(fe_i, fe_i) + \sum_{i=1}^{s} \tilde{\nabla} dF(fe_i, fe_i) \\ &= \sum_{i=1}^{m}\left(\sum_{j=1}^{s'} -2\beta_i'\xi_i'\right) h(dFe_i, dFe_i),\end{aligned}$$

F is harmonic map iff
$\sum_{i=1}^{m} h(dFe_i, dFe_i) = 0$ or $\sum_{j=1}^{s'} -2\beta_i'\xi_i' = 0$,
this implies F is harmonic map iff F is a constant map or $\sum_{j=1}^{s'} \beta_j'\xi_j' = 0$ i. e. given all beta $\beta_j : j = 1, \ldots, s'$ are zero.

Corollary 3.2. *A (f, f')-holomorphic map from trans-S-manifold to a C-manifold is plurihamonic and hence harmonic.*

Proof. If $\alpha_j' = 0$, $\beta_j' = 0$ for all $j = 1, \ldots, s'$, in Theorem 3.1 then we get required result.

Remark 3.1.
- If $s = 1$, $s' = 1$ for any α, β we have the results on trans-Sasakian manifold as proved in [7].

Theorem 3.3. *Let $M^{2m+s}(f, \xi_i, \eta_i, g, \alpha_i, \beta_i) : \quad i = 1, \ldots, s$ be a trans-S-manifold, $N^{2n}(J,h)$ be Kähler manifold and let $F : M \to N$ be a (f, J)-holomorphic map, then F is a harmonic map.*

Proof. We have

$$\begin{aligned}\tilde{\nabla} dF(X, fY) &= \nabla_X^F JdF(Y) - dF(\nabla_X^M fY), \\ &= (\nabla_X^N J)dF(Y) - dF(\nabla_X^M f)Y + J\tilde{\nabla} dF(X,Y),\end{aligned}$$

since N is a Käler manifold, first term on right hand side becomes zero and in second term on right hand side using the definition of trans-S-manifold from equation (2.7), we have

$$\begin{aligned}\tilde{\nabla} dF(X, fY) &= -dF\left(\sum_{i=1}^{s} \alpha_i \left\{g(X,Y)\xi_i - \eta_i(Y)X\right\} + \beta_i \left\{g(fX,Y)\xi_i - \eta_i(Y)fX\right\}\right) + \\ &+ J\tilde{\nabla} dF(X,Y),\end{aligned}$$

then

$$\begin{aligned}\tilde{\nabla} dF(X, fY) - \tilde{\nabla} dF(Y, fX) &= \sum_{i=1}^{s} \alpha_i \left\{\eta_i(Y)dFX - \eta_i(X)dFY\right\} + \\ &+ \beta_i \left\{\eta_i(Y)dF(fX) - \eta_i(X)dF(fY)\right\},\end{aligned}$$

take $Y = fX$ and $X \in D$, we have

$$\tilde{\nabla} dF(X,X) + \tilde{\nabla} dF(fX, fX) = 0, \tag{3.1}$$

let $\{e_1, \ldots, e_m, fe_1, \ldots, fe_m, \xi_1, \ldots, \xi_s\}$ be orthonormal frame on TM, then

$$\tau(F) = \sum_{i=1}^{m} \tilde{\nabla} dF(fe_i, fe_i) + \sum_{i=1}^{m} \tilde{\nabla} dF(fe_i, fe_i) + \sum_{i=1}^{s} \tilde{\nabla} dF(\xi_i, \xi_i).$$

Sum of first two terms vanishes From (3.1) and third term is also equal to zero by the definition of trans-S-manifold and using the fact that $\xi_i \in (kerdF)$, $\forall\ i = 1, \ldots, s$. Therefore

$$\tau(F) = 0.$$

Harmonic morphism are maps which pull back germs of real valued harmonic functions on the target manifold to germs of harmonic functions on the domain, that is, a smooth map $F : (M, g) \to (N, h)$ is a harmonic morphism if for any harmonic function $f : U \to \mathbb{R}$, defined on an open subset U of N such that $\pi^{-1}(U)$ is non-empty, the composition $f \circ F : \pi^{-1}(U) \to \mathbb{R}$ is a harmonic function. The following characterization of harmonic morphisms is due to Fuglede and Ishihara: A smooth map F is a harmonic morphism if and only if F is a horizontally conformal harmonic map (see [5] and [6]). Now we discuss a result for harmonic morphisms defined on trans-S-manifolds.

Theorem 3.4. *Let $M^{2m+s}(f, \xi_i, \eta_i, g, \alpha_i, \beta_i) : \quad i = 1, \ldots, s$ be a trans-S-manifold, $N^{2n}(J, h)$ be semi Kähler manifold and let $F : M \to N$ be a horizontally conformal (f, J)-holomorphic map, then F is a harmonic morphism.*

Proof. It can be similarly proved as in ([8]), that for a horizontally conformal (f, J)-holomorphic map F from an S-manifold $M(f, \xi_\alpha, \eta_\alpha, g)$ to an almost Hermitian manifold $N(J, h)$, any two of the following conditions imply the third: (i) $divJ = 0$ (ii) $dF(divf) = 0$ (iii) F is harmonic and so is harmonic morphism.

Let $\{e_1, ..., e_m, fe_1, ..., fe_m, \xi_1, \ldots, \xi_s\}$ be a local orthonormal f-adapted basis on TM, Then we have

$$
\begin{aligned}
divf &= \sum_{i=1}^{2m+s}(\nabla_{e_i}f)e_i = \sum_{i=1}^{m}(\nabla_{e_i}f)e_i + \sum_{i=1}^{m}(\nabla_{fe_i}f)fe_i + \sum_{i=1}^{s}(\nabla_{\xi_i}f)\xi_i \\
&= \sum_{i=1}^{m}\sum_{j=1}^{s}\alpha_j\{g(fe_i, fe_i)\xi_j + \eta_j(e_i)f^2e_i\} + \beta_j\{g(fe_i, e_i)\xi_j + \eta_j(e_i)fe_i\} + \\
&+ \sum_{i=1}^{m}\sum_{j=1}^{s}\alpha_j\{g(f^2e_i, f^2e_i)\xi_j + \eta_j(fe_i)f^2fe_i\} + \beta_j\{g(f^2e_i, fe_i)\xi_j + \eta_j(fe_i)f^2e_i\} + \\
&+ \sum_{i=1}^{s}\sum_{j=1}^{s}\alpha_j\{g(f\xi_i, f\xi_i)\xi_j + \eta_j(\xi_i)f^2\xi_i\} + \beta_j\{g(f\xi_i, \xi_i)\xi_j + \eta_j(\xi_i)f\xi_i\}.
\end{aligned}
$$

Therefore $divf = \sum_{i=1}^{m}\sum_{j=1}^{s} 2g(fe_i, fe_i)\xi_j$ and $dF(divf) = 0$, as F is (f, J)-holomorphic $F_*(\xi_j) = 0$, for any $j = 1, \ldots, s$. Because F is a horizontally conformal (f, J)-holomorphic map, it follows that F is a harmonic morphism if and only if $divJ = 0$, i.e., N is semi-Kähler.

From Theorems 3.3, 3.4, we notice that:

Remark 3.2.

- If all $\alpha_i = 1 : i = 1, \ldots, s$ and all $\beta_i = 0 : i = 1, \ldots, s$, then we have the result from S-manifolds to Kähler manifolds, as proved in [13].
- If all $\alpha_i = 0 : i = 1, \ldots, s$ and all $\beta_i = 1 : i = 1, \ldots, s$, then we have the result from Kenmotsu manifolds to Kähler manifolds, as proved in [12].

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