Affine and conformal submersions with horizontal distribution and statistical manifolds

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Abstract. Conformal submersion with horizontal distribution is defined in this paper, which is a generalization of the affine submersion with horizontal distribution. Then, proved a necessary and sufficient condition for a semi-Riemannian manifold to become a statistical manifold in the case of a conformal submersion with horizontal distribution. Also, obtained a necessary and sufficient condition for the tangent bundle to become a statistical manifold with respect to the Sasaki lift metric and the complete lift connection.

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1 Introduction

Statistical manifolds play a central role in the discipline of information geometry which has got application in various fields like statistical mechanics, machine learning, neural networks, statistics, neuroscience, etc.,[2]. Statistical manifold was originally introduced by S.L Lauritzen [9], later Kurose [8] reformulated this from the viewpoint of the affine differential geometry. Riemannian submersions from a statistical viewpoint were first mentioned by Barndroff-Neilsen and Jupp [4]. O'Neill [12] defined a Riemannian submersion and obtained the fundamental equations of Riemannian submersions for Riemannian manifolds. Also in [13], O'Neill defined a semi-Riemannian submersion. Abe and Hasegawa [1] defined an affine submersion with horizontal distribution and obtained the fundamental equations. For the semi-Riemannian submersion $\pi : (M, g_M) \to (B, g_B)$, Abe and Hasegawa [1] obtained a necessary and sufficient condition for (M, ∇, g_M) to become a statistical manifold with respect to the affine submersion with horizontal distribution $\pi : (M, \nabla) \to (B, \nabla^*)$. Conformal submersion and the fundamental equations of conformal submersion were also studied by many researchers, see [14], [6] for example. Harmonic morphisms between Riemannian manifolds of arbitrary dimensions and horizontally conformal submersions were introduced by Fuglede [5] and Ishihara [7] independently. Harmonic morphisms are nothing but harmonic and horizontally conformal maps. Their

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study focuses on the conformality relation between the metrics on Riemannian manifolds and the Levi-Civita connections. Our interest is on the conformal submersion between Riemannian manifolds $\mathbf{M}$ and $\mathbf{B}$ and the conformality relation between any two affine connections ∇ and ∇^* (not necessarily be Levi-Civita connections) on $\mathbf{M}$ and $\mathbf{B}$, respectively. In this paper, the concept of the conformal submersion with horizontal distribution is defined, which is a generalization of the affine submersion with horizontal distribution. Then, we study the statistical manifold structure for affine and conformal submersions with horizontal distribution.

The projection from the tangent bundle $T\mathbf{M}$ to the manifold $\mathbf{M}$ can be considered as a submersion. Matsuzoe and Inoguchi [10] obtained a necessary and sufficient condition for the tangent bundle $T\mathbf{M}$ to become a statistical manifold with respect to the Sasaki lift metric and the horizontal lift connection and also with respect to the horizontal lift metric and the horizontal lift connection. In [3], V. Balan et al studied statistical structures on the tangent bundle of a statistical manifold with the Sasaki metric. We have shown that the submersion $\pi : (T\mathbf{M}, \nabla^c) \longrightarrow (\mathbf{M}, \nabla)$ is an affine submersion with horizontal distribution and $\pi : (T\mathbf{M}, g^s) \longrightarrow (\mathbf{M}, g)$ is a semi-Riemannian submersion. Also, obtained a necessary and sufficient condition for $T\mathbf{M}$ to become a statistical manifold with respect to the Sasaki lift metric and the complete lift connection for an affine submersion with horizontal distribution.

In section 2, a statistical structure is obtained on the manifold $\mathbf{B}$ induced by the affine submersion $\pi : \mathbf{M} \longrightarrow \mathbf{B}$ with the horizontal distribution $\mathcal{H}(\mathbf{M}) = \mathcal{V}^{\perp}(\mathbf{M})$. In section 3, we introduced the concept of conformal submersion with horizontal distribution, which is a generalization of the affine submersion with horizontal distribution. For a conformal submersion of semi-Riemannian manifolds $\pi : (\mathbf{M}, g_M) \to (\mathbf{B}, g_B)$, we proved that $\pi : (\mathbf{M}, \nabla) \longrightarrow (\mathbf{B}, \nabla^*)$ is a conformal submersion with horizontal distribution if and only if $\pi : (\mathbf{M}, \overline{\nabla}) \longrightarrow (\mathbf{B}, \overline{\nabla^*})$ is a conformal submersion with horizontal distribution. Then, proved a necessary and sufficient condition for $(\mathbf{M}, \nabla, g_M)$ to become a statistical manifold for a conformal submersion with horizontal distribution. This is a generalization of the theorem for an affine submersion with horizontal distribution proved by Abe and Hasegawa [1]. In section 4, we proved that the submersion $\pi : (T\mathbf{M}, \nabla^c) \to (\mathbf{M}, \nabla)$ is an affine submersion with horizontal distribution and the submersion $\pi : (T\mathbf{M}, g^s) \to (\mathbf{M}, g)$ is a semi-Riemannian submersion, where ∇^c is the complete lift of affine connection ∇ on $\mathbf{M}$ and g^s is the Sasaki lift metric. Then, we obtained a necessary and sufficient condition for $(T\mathbf{M}, \nabla^c, g^s)$ to become a statistical manifold. Throughout this paper, all the objects are assumed to be smooth.

2 Statistical manifolds and semi-Riemannian submersions

In this section, we show that the geometric structure $(\nabla', \tilde{g})$ induced by an affine submersion of a statistical manifold $\pi : \mathbf{M} \longrightarrow \mathbf{B}$ with the horizontal distribution $\mathcal{H}(\mathbf{M}) = \mathcal{V}^{\perp}(\mathbf{M})$ is a statistical manifold structure.

A semi-Riemannian manifold $(\mathbf{M}, g)$ with a torsion-free affine connection ∇ is called a statistical manifold if ∇g is symmetric. For a statistical manifold $(\mathbf{M}, \nabla, g)$ the dual connection $\overline{\nabla}$ is defined by $Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \overline{\nabla}_X Z)$, for X, Y and Z in $\mathcal{X}(\mathbf{M})$, where $\mathcal{X}(\mathbf{M})$ denotes the set of all vector fields on $\mathbf{M}$. If (∇, g) is a statistical structure on $\mathbf{M}$ so is $(\overline{\nabla}, g)$. Then $(\mathbf{M}, \overline{\nabla}, g)$ becomes a statistical manifold, called the dual statistical manifold of $(\mathbf{M}, \nabla, g)$. Let R^{∇} and $R^{\overline{\nabla}}$ be the curvature tensors of ∇ and $\overline{\nabla}$, respectively. Using the above relation of ∇ and $\overline{\nabla}$ one can show that $g(R^{\nabla}(X,Y)Z,W) = -g(Z, R^{\overline{\nabla}}(X,Y)W)$ for X, Y, Z and W in $\mathcal{X}(\mathbf{M})$.

Let $\mathbf{M}$ be an n dimensional manifold and $\mathbf{B}$ be an m dimensional manifold $(n > m)$. An onto map $\pi : \mathbf{M} \longrightarrow \mathbf{B}$ is called a submersion if $\pi_{*p} : T_p\mathbf{M} \longrightarrow T_{\pi(p)}\mathbf{B}$ is onto for all $p \in \mathbf{M}$. For a submersion $\pi : \mathbf{M} \longrightarrow \mathbf{B}$, $\pi^{-1}(b)$ is a submanifold of $\mathbf{M}$ of dimension $(n-m)$ for each $b \in \mathbf{B}$. These submanifolds $\pi^{-1}(b)$ are called the fibers. Set $\mathcal{V}(\mathbf{M})_p = Ker(\pi_{*p})$ for each $p \in \mathbf{M}$.

Definition 2.1. A submersion $\pi : \mathbf{M} \longrightarrow \mathbf{B}$ is called a submersion with horizontal distribution if there is a smooth distribution $p \longrightarrow \mathcal{H}(M)_p$ such that $T_p\mathbf{M} = \mathcal{V}(\mathbf{M})_p \bigoplus \mathcal{H}(M)_p$.

We call $\mathcal{V}(\mathbf{M})_p$ $(\mathcal{H}(M)_p)$ the vertical (horizontal) subspace of $T_p\mathbf{M}$. $\mathcal{H}$ and $\mathcal{V}$ denote the projection of the tangent space of $\mathbf{M}$ onto the horizontal and vertical subspaces, respectively.

Note 2.1. *Let $\pi : \mathbf{M} \to \mathbf{B}$ be a submersion with horizontal distribution $\mathcal{H}(M)$. Then, $\pi_* |_{\mathcal{H}(M)_p} : \mathcal{H}(M)_p \to T_{\pi(p)}\mathbf{B}$ is an isomorphism for each $p \in \mathbf{M}$.*

Definition 2.2. Let $(\mathbf{M}, g_M)$, $(\mathbf{B}, g_B)$ be semi-Riemannian manifolds of dimensions n, m respectively $(n > m)$. A submersion $\pi : \mathbf{M} \longrightarrow \mathbf{B}$ is called a semi-Riemannian submersion if all the fibers are semi-Riemannian submanifolds of $\mathbf{M}$ and π_* preserves the length of horizontal vectors.

Note 2.2. *A vector field Y on $\mathbf{M}$ is said to be projectable if there exists a vector field Y_* on $\mathbf{B}$ such that $\pi_*(Y_p) = Y_{*\pi(p)}$ for each $p \in \mathbf{M}$, that is, Y and Y_* are π- related. A vector field X on $\mathbf{M}$ is said to be basic if it is projectable and horizontal. Every vector field X on $\mathbf{B}$ has a unique smooth horizontal lift, denoted by $\tilde{X}$, to $\mathbf{M}$.*

Definition 2.3. Let ∇ and ∇^* be affine connections on $\mathbf{M}$ and $\mathbf{B}$, respectively. $\pi : (\mathbf{M}, \nabla) \longrightarrow (\mathbf{B}, \nabla^*)$ is said to be an affine submersion with horizontal distribution if $\pi : \mathbf{M} \longrightarrow \mathbf{B}$ is a submersion with horizontal distribution and satisfies $\mathcal{H}(\nabla_{\tilde{X}}\tilde{Y}) = (\nabla^*_X Y)\tilde{}$, for $X, Y \in \mathcal{X}(\mathbf{B})$.

Note 2.3. *Abe and Hasegawa [1] proved that the connection ∇ on $\mathbf{M}$ induces a connection $\nabla^{'}$ on $\mathbf{B}$ when $\pi : \mathbf{M} \longrightarrow \mathbf{B}$ is a submersion with horizontal distribution and $\mathcal{H}(\nabla_{\tilde{X}}\tilde{Y})$ is projectable for all the vector fields X and Y on $\mathbf{B}$.*

A connection $\mathcal{V}\nabla\mathcal{V}$ on the subbundle $\mathcal{V}(\mathbf{M})$ is defined by $(\mathcal{V}\nabla\mathcal{V})_E V = \mathcal{V}(\nabla_E V)$ for any vertical vector field V and for any vector field E on $\mathbf{M}$. For each $b \in \mathbf{B}$, $\mathcal{V}\nabla\mathcal{V}$ induces a unique connection $\hat{\nabla}^b$ on the fiber $\pi^{-1}(b)$. Abe and Hasegawa [1] proved that, if ∇ is torsion-free then $\hat{\nabla}^b$ and $\nabla^{'}$ are also torsion-free.

Definition 2.4. Let $\pi : (\mathbf{M}, \nabla) \longrightarrow (\mathbf{B}, \nabla^*)$ be an affine submersion with horizontal distribution $\mathcal{V}^{\perp}(\mathbf{M})$, g be a semi-Riemannian metric on $\mathbf{M}$ and $\mathcal{H}(\nabla_{\tilde{X}}\tilde{Y})$ be projectable. Define the induced semi-Riemannian metric $\tilde{g}$ and the induced connection $\nabla^{'}$ on $\mathbf{B}$ as $\tilde{g}(X,Y) = g(\tilde{X}, \tilde{Y})$ and $\nabla^{'}_X Y = \pi_*(\nabla_{\tilde{X}}\tilde{Y})$ where X, Y are vector fields on $\mathbf{B}$.

Now, we show that $(\mathbf{B}, \nabla^{'}, \tilde{g})$ is a statistical manifold.

Theorem 2.1. *Let $(\mathbf{M}, \nabla, g)$ be a statistical manifold and $\pi : \mathbf{M} \longrightarrow \mathbf{B}$ be an affine submersion with horizontal distribution $\mathcal{H}(\mathbf{M}) = \mathcal{V}^{\perp}(\mathbf{M})$ and $\mathcal{H}(\nabla_{\tilde{X}}\tilde{Y})$ be projectable. Then, $(\mathbf{B}, \nabla^{'}, \tilde{g})$ is a statistical manifold.*

Proof. Let X, Y, Z be vector fields on $\mathbf{B}$, we have

$$\begin{aligned}(\nabla^{'}_X \tilde{g})(Y, Z) &= X\tilde{g}(Y, Z) - \tilde{g}(\nabla^{'}_X Y, Z) - \tilde{g}(Y, \nabla^{'}_X Z) \\ &= \tilde{X} g(\tilde{Y}, \tilde{Z}) - g(\nabla_{\tilde{X}}\tilde{Y}, \tilde{Z}) - g(\tilde{Y}, \nabla_{\tilde{X}}\tilde{Z}) \\ &= (\nabla_{\tilde{X}} g)(\tilde{Y}, \tilde{Z}).\end{aligned}$$

Since $(\mathbf{M}, \nabla, g)$ is a statistical manifold, $(\mathbf{B}, \nabla^{'}, \tilde{g})$ is also a statistical manifold. □

Definition 2.5. Let $\pi : (\mathbf{M}, \nabla, g_M) \longrightarrow (\mathbf{B}, \nabla^*, g_B)$ be an affine submersion with horizontal distribution $\mathcal{H}(M)$. The fundamental tensors T and A are defined as

$$T_E F = \mathcal{H}(\nabla_{\mathcal{V}E}\mathcal{V}F) + \mathcal{V}(\nabla_{\mathcal{V}E}\mathcal{H}F), \tag{2.1}$$

$$A_E F = \mathcal{V}(\nabla_{\mathcal{H}E}\mathcal{H}F) + \mathcal{H}(\nabla_{\mathcal{H}E}\mathcal{V}F), \tag{2.2}$$

for vector fields E and F on $\mathbf{M}$. Also, we denote the fundamental tensors correspond to the dual connection $\overline{\nabla}$ of ∇ by $\overline{T}$ and $\overline{A}$.

Note that, T and A are $(1, 2)$-tensors. These tensors can be defined in a general situation, namely, it is enough that a manifold $\mathbf{M}$ has a splitting $T\mathbf{M} = \mathcal{V}(\mathbf{M}) \bigoplus \mathcal{H}(\mathbf{M})$. Also note that, T_E and A_E reverses the horizontal and vertical subspaces and $T_E = T_{\mathcal{V}E}$, $A_E = A_{\mathcal{H}E}$.

The inclusion map $(\pi^{-1}(b), \hat{\nabla}^b) \longrightarrow (\mathbf{M}, \nabla)$ is an affine immersion in the sense of [11] and the corresponding Guass and Weingarten formulae follow.

3 Conformal submersion with horizontal distribution

In this section, we generalize the concept of an affine submersion with horizontal distribution. Then, prove that $\pi : (\mathbf{M}, \nabla) \longrightarrow (\mathbf{B}, \nabla^*)$ is a conformal submersion with horizontal distribution if and only if $\pi : (\mathbf{M}, \overline{\nabla}) \longrightarrow (\mathbf{B}, \overline{\nabla^*})$ is a conformal submersion with horizontal distribution. Also, a necessary and sufficient condition for $(\mathbf{M}, \nabla, g_M)$ to become a statistical manifold for a conformal submersion with horizontal distribution is obtained.

Definition 3.1. Let $(\mathbf{M}, g_M)$ and $(\mathbf{B}, g_B)$ be Riemannian manifolds. A submersion $\pi : (\mathbf{M}, g_M) \longrightarrow (\mathbf{B}, g_B)$ is called a conformal submersion if there exists a $\phi \in C^{\infty}(\mathbf{M})$ such that $g_M(X, Y) = e^{2\phi} g_B(\pi_* X, \pi_* Y)$, for horizontal vector fields $X, Y \in \mathcal{X}(\mathbf{M})$.

For $\pi : (\mathbf{M}, \nabla) \longrightarrow (\mathbf{B}, \nabla^*)$ an affine submersion with horizontal distribution, $\pi_*(\nabla_{\tilde{X}}\tilde{Y}) = \nabla^*_X Y$, for $X, Y \in \mathcal{X}(\mathbf{B})$. In the case of a conformal submersion we prove the following theorem, which is the motivation for us to generalize the concept of an affine submersion with horizontal distribution.

Theorem 3.1. *Let $\pi : (\mathbf{M}, g_M) \longrightarrow (\mathbf{B}, g_B)$ be a conformal submersion. If ∇ on $\mathbf{M}$ and ∇^* on B are the Levi-Civita connections, then*

$$\begin{aligned} g_B(\pi_*(\nabla_{\tilde{X}}\tilde{Y}), Z) &= g_B(\nabla^*_X Y, Z) - d\phi(\tilde{Z})g_B(X,Y) \\ &+ \{d\phi(\tilde{X})g_B(Y,Z) + d\phi(\tilde{Y})g_B(Z,X)\}, \end{aligned}$$

where $X, Y, Z \in \mathcal{X}(\mathbf{B})$ and $\tilde{X}, \tilde{Y}, \tilde{Z}$ denote the unique horizontal lifts on $\mathbf{M}$.

Proof. We have the Koszul formula for the Levi-Civita connection,

$$\begin{aligned} 2g_M(\nabla_{\tilde{X}}\tilde{Y}, \tilde{Z}) &= \tilde{X}g_M(\tilde{Y}, \tilde{Z}) + \tilde{Y}g_M(\tilde{Z}, \tilde{X}) - \tilde{Z}g_M(\tilde{X}, \tilde{Y}) \\ &- g_M(\tilde{X}, [\tilde{Y}, \tilde{Z}]) + g_M(\tilde{Y}, [\tilde{Z}, \tilde{X}]) + g_M(\tilde{Z}, [\tilde{X}, \tilde{Y}]). \end{aligned} \tag{3.1}$$

Now consider

$$\begin{aligned} \tilde{X}g_M(\tilde{Y}, \tilde{Z}) &= \tilde{X}(e^{2\phi}g_B(Y,Z)) \\ &= \tilde{X}(e^{2\phi})g_B(Y,Z) + e^{2\phi}\tilde{X}(g_B(Y,Z)) \\ &= 2e^{2\phi}d\phi(\tilde{X})g_B(Y,Z) + e^{2\phi}Xg_B(Y,Z). \end{aligned}$$

Similarly we have,

$$\begin{aligned} \tilde{Y}g_M(\tilde{X}, \tilde{Z}) &= 2e^{2\phi}d\phi(\tilde{Y})g_B(X,Z) + e^{2\phi}Yg_B(X,Z) \\ \tilde{Z}g_M(\tilde{X}, \tilde{Y}) &= 2e^{2\phi}d\phi(\tilde{Z})g_B(X,Y) + e^{2\phi}Zg_B(X,Y). \end{aligned}$$

Also, $g_M(\tilde{X}, [\tilde{Y}, \tilde{Z}]) = e^{2\phi}g_B(X, [Y,Z])$, $g_M(\tilde{Y}, [\tilde{Z}, \tilde{X}]) = e^{2\phi}g_B(Y, [Z,X])$ and $g_M(\tilde{Z}, [\tilde{X}, \tilde{Y}]) = e^{2\phi}g_B(Z, [X,Y])$. Then, from the equation (3.1) and the above equations

$$\begin{aligned} 2g_M(\nabla_{\tilde{X}}\tilde{Y}, \tilde{Z}) &= 2d\phi(\tilde{X})e^{2\phi}g_B(Y,Z) + 2d\phi(\tilde{Y})e^{2\phi}g_B(X,Z) \\ &- 2d\phi(\tilde{Z})e^{2\phi}g_B(X,Y) + 2e^{2\phi}g_B(\nabla^*_X Y, Z). \end{aligned}$$

This implies

$$\begin{aligned} g_B(\pi_*(\nabla_{\tilde{X}}\tilde{Y}), Z) &= g_B(\nabla^*_X Y, Z) - d\phi(\tilde{Z})g_B(X,Y) \\ &+ \{d\phi(\tilde{X})g_B(Y,Z) + d\phi(\tilde{Y})g_B(Z,X)\}. \end{aligned}$$

□

Now, we generalize the concept of an affine submersion with horizontal distribution as follows:

Definition 3.2. Let $\pi : (\mathbf{M}, g_M) \longrightarrow (\mathbf{B}, g_B)$ be a conformal submersion and let ∇ and ∇^* be affine connections on $\mathbf{M}$ and $\mathbf{B}$, respectively. Then, $\pi : (\mathbf{M}, \nabla) \longrightarrow (\mathbf{B}, \nabla^*)$ is said to be a conformal submersion with horizontal distribution $\mathcal{H}(\mathbf{M}) = \mathcal{V}(\mathbf{M})^{\perp}$ if

$$\begin{aligned} g_B(\pi_*(\nabla_{\tilde{X}}\tilde{Y}), Z) &= g_B(\nabla^*_X Y, Z) - d\phi(\tilde{Z})g_B(X,Y) \\ &\quad +\{d\phi(\tilde{X})g_B(Y,Z) + d\phi(\tilde{Y})g_B(Z,X)\}, \end{aligned}$$

for some $\phi \in C^\infty(\mathbf{M})$ and for all $X, Y, Z \in \mathcal{X}(\mathbf{B})$.

Note 3.1. *If ϕ is constant, it turns out to be an affine submersion with horizontal distribution.*

Example 3.3. Let $H^n = \{(x_1, ..., x_n) \in \mathbf{R}^n : x_n > 0\}$ and $\tilde{g} = \frac{1}{x_n^2} g$ be a Riemannian metric on H^n, where g is the Euclidean metric on $\mathbf{R}^n$. Let $\pi : H^n \longrightarrow \mathbf{R}^{n-1}$ be defined by $\pi(x_1, ..., x_n) = (x_1, ..., x_{n-1})$. Let $\phi : H^n \longrightarrow \mathbf{R}$ be defined by $\phi(x_1, ..., x_n) = \log(\frac{1}{x_n^2})$. Then, we have

$$\tilde{g}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = e^{\phi} g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$$

hence, $\pi : (H^n, \tilde{g}) \longrightarrow (\mathbf{R}^{n-1}, g)$ is a conformal submersion. Then, by the theorem (3.1), $\pi : (H^n, \nabla) \longrightarrow (\mathbf{R}^{n-1}, \nabla^*)$ is a conformal submersion with horizontal distribution, where ∇ and ∇^* are Levi-Civita connections on H^n and $\mathbf{R}^{n-1}$ respectively.

Now, for semi-Riemannian manifolds $(\mathbf{M}, g_M)$, $(\mathbf{B}, g_B)$ with affine connections ∇ and ∇^* and the dual connections $\overline{\nabla}$ and $\overline{\nabla^*}$ respectively, we prove

Proposition 3.2. *Let $\pi : (\mathbf{M}, g_M) \longrightarrow (\mathbf{B}, g_B)$ be a conformal submersion. Then, $\pi : (\mathbf{M}, \nabla) \longrightarrow (\mathbf{B}, \nabla^*)$ is a conformal submersion with horizontal distribution if and only if $\pi : (\mathbf{M}, \overline{\nabla}) \longrightarrow (\mathbf{B}, \overline{\nabla^*})$ is a conformal submersion with horizontal distribution.*

Proof. Consider,

$$\begin{aligned} \tilde{X} g_M(\tilde{Y}, \tilde{Z}) &= 2e^{2\phi} d\phi(\tilde{X}) g_B(Y, Z) + e^{2\phi} X g_B(Y, Z) \\ &= 2e^{2\phi} d\phi(\tilde{X}) g_B(Y, Z) + e^{2\phi}\{g_B(\nabla^*_X Y, Z) + g_B(Y, \overline{\nabla^*}_X Z)\}. \end{aligned}$$

Now consider

$$\begin{aligned} \tilde{X} g_M(\tilde{Y}, \tilde{Z}) &= g_M(\nabla_{\tilde{X}} \tilde{Y}, \tilde{Z}) + g_M(\tilde{Y}, \overline{\nabla}_{\tilde{X}} \tilde{Z}) \\ &= e^{2\phi} g_B(\pi_*(\nabla_{\tilde{X}} \tilde{Y}), Z) + e^{2\phi} g_B(Y, \pi_*(\overline{\nabla}_{\tilde{X}} \tilde{Z})). \end{aligned} \tag{3.2}$$

Since,

$$\begin{aligned} g_B(\pi_*(\nabla_{\tilde{X}} \tilde{Y}), Z) &= g_B(\nabla^*_X Y, Z) - d\phi(\tilde{Z}) g_B(X, Y) \\ &+ \{d\phi(\tilde{X}) g_B(Y, Z) + d\phi(\tilde{Y}) g_B(Z, X)\} \end{aligned} \tag{3.3}$$

from (3.2) and (3.3) we get

$$\begin{aligned} g_B(\pi_*(\overline{\nabla}_{\tilde{X}} \tilde{Z}), Y) &= g_B(\overline{\nabla^*}_X Z, Y) - d\phi(\tilde{Y}) g_B(X, Z) \\ &+ \{d\phi(\tilde{X}) g_B(Y, Z) + d\phi(\tilde{Z}) g_B(X, Y)\}. \end{aligned}$$

Hence, $\pi : (\mathbf{M}, \overline{\nabla}) \longrightarrow (\mathbf{B}, \overline{\nabla^*})$ is a conformal submersion with horizontal distribution. Converse is obtained by interchanging ∇, ∇^* with $\overline{\nabla}$, $\overline{\nabla^*}$ in the above proof. □

Lemma 3.3. *Let $\pi : (\mathbf{M}, g_M) \longrightarrow (\mathbf{B}, g_B)$ be a conformal submersion and $\pi : (\mathbf{M}, \nabla) \longrightarrow (\mathbf{B}, \nabla^*)$ be a conformal submersion with horizontal distribution $\mathcal{V}(\mathbf{M})^\perp$,* *then*

$$(3.4)\qquad (\nabla_{\tilde{X}} g_M)(\tilde{X}_1, \tilde{X}_2) = e^{2\phi}(\nabla^*_X g_B)(X_1, X_2),$$
$$(3.5)\qquad (\nabla_V g_M)(X, Y) = -g_M(S_V X, Y),$$
$$(3.6)\qquad (\nabla_X g_M)(V, Y) = -g_M(A_X V, Y) + g_M(\overline{A}_X V, Y),$$
$$(3.7)\qquad (\nabla_X g_M)(V, W) = -g_M(S_X V, W),$$
$$(3.8)\qquad (\nabla_V g_M)(X, W) = -g_M(T_V X, W) + g_M(\overline{T}_V X, W),$$
$$(3.9)\qquad (\nabla_U g_M)(V, W) = (\hat{\nabla}_U \hat{g}_M)(V, W),$$

for horizontal vector fields X, Y on $\mathbf{M}$ and vertical vector fields U, V, W on $\mathbf{M}$. $\tilde{X}_i$ are the horizontal lifts of vector fields X_i on $\mathbf{B}$, $\hat{g}$ is the induced metric on the fibers and $S_V X = \nabla_V X - \overline{\nabla}_V X$.

Proof. Consider

$$\begin{aligned}(\nabla_{\tilde{X}} g_M)(\tilde{X}_1, \tilde{X}_2) &= \tilde{X} g_M(\tilde{X}_1, \tilde{X}_2) - g_M(\nabla_X \tilde{X}_1, \tilde{X}_2) - g_M(\tilde{X}_1, \nabla_X \tilde{X}_2)\\ &= \tilde{X} e^{2\phi} g_B(X_1, X_2) - e^{2\phi} g_B(\pi_*(\nabla_{\tilde{X}} \tilde{X}_1), X_2)\\ &- e^{2\phi} g_B(X_1, \pi_*(\nabla_{\tilde{X}} \tilde{X}_2))\\ &= 2e^{2\phi} d\phi(\tilde{X}) g_B(X_1, X_2) + e^{2\phi} X g_B(X_1, X_2)\\ &- e^{2\phi} g_B(\pi_*(\nabla_{\tilde{X}} \tilde{X}_1), X_2) - e^{2\phi} g_B(X_1, \pi_*(\nabla_{\tilde{X}} \tilde{X}_2)).\end{aligned}$$

Since

$$\begin{aligned}g_B(\pi_*(\nabla_{\tilde{X}} \tilde{X}_i), X_j) &= g_B(\nabla^*_X X_i, X_j) - d\phi(\tilde{X}_j) g_B(X, X_i)\\ &+ \{d\phi(\tilde{X}) g_B(X_i, X_j) + d\phi(\tilde{X}_i) g_B(X_j, X)\},\end{aligned}$$

where $i, j = 1, 2$ and $i \neq j$, we get

$$(\nabla_{\tilde{X}} g_M)(\tilde{X}_1, \tilde{X}_2) = e^{2\phi}(\nabla^*_X g_B)(X_1, X_2).$$

Similarly, we can prove the other equations. □

Now, we prove a necessary and sufficient condition for (M, ∇, g_M) to be a statistical manifold for a conformal submersion with horizontal distribution.

Theorem 3.4. *Let $\pi : (\mathbf{M}, g_M) \longrightarrow (\mathbf{B}, g_B)$ be a conformal submersion and $\pi : (\mathbf{M}, \nabla) \longrightarrow (\mathbf{B}, \nabla^*)$ be a conformal submersion with horizontal distribution $\mathcal{H}(\mathbf{M}) = \mathcal{V}^{\perp}(\mathbf{M})$ and ∇ be torsion-free. Then, $(\mathbf{M}, \nabla, g_M)$ is a statistical manifold if and only if*

1. *$\mathcal{H}(S_V X) = A_X V - \overline{A}_X V$.*
2. *$\mathcal{V}(S_X V) = T_V X - \overline{T}_V X$.*
3. *$(\pi^{-1}(b), \hat{\nabla}^b, \hat{g}^b_M)$ is a statistical manifold for each $b \in \mathbf{B}$.*
4. *$(\mathbf{B}, \nabla^*, g_B)$ is a statistical manifold.*

Proof. Suppose $(\mathbf{M}, \nabla, g_M)$ is a statistical manifold, then ∇g_M is symmetric. So $(\nabla_V g_M)(X, Y) = (\nabla_X g_M)(V, Y)$, where X, Y are horizontal vector fields and V is a vertical vector field. Then, from (3.5) and (3.6) of the above lemma $g_M(S_V X, Y) = g_M(A_X V, Y) - g_M(\overline{A}_X V, Y)$. This implies, $\mathcal{H}(S_V X) = A_X V - \overline{A}_X V$. Similarly from (3.7) and (3.8) of the above lemma, we have $\mathcal{V}(S_X V) = T_V X - \overline{T}_V X$.

Since ∇g_M is symmetric, from (3.9) of the above lemma, we get $\hat{\nabla}^b \hat{g}^b$ is symmetric, so $(\pi^{-1}(b), \hat{\nabla}^b, \hat{g}^b_M)$ is a statistical manifold. Also from (3.4) of the above lemma, we get $(\nabla_{\tilde{X}} g_M)(\tilde{X}_1, \tilde{X}_2) = e^{2\phi}(\nabla^*_X g_B)(X_1, X_2)$, where $\tilde{X}_i$ are the horizontal lift of the vector fields X_i on $\mathbf{B}$. Since, ∇g_M is symmetric $\nabla^* g_B$ is also symmetric. Hence, $(\mathbf{B}, \nabla^*, g_B)$ is a statistical manifold.

Conversely, if all the four conditions hold then from the above lemma $\nabla_E g_M(F, G) = \nabla_F g_M(E, G)$, for vector fields E, F and G on $\mathbf{M}$. That is, ∇g_M is symmetric on $\mathbf{M}$ and hence $(\mathbf{M}, \nabla, g_M)$ is a statistical manifold. □

4 Statistical structures on the tangent bundle

In this section, we show that the submersion $\pi : (T\mathbf{M}, \nabla^c) \to (\mathbf{M}, \nabla)$ is an affine submersion with horizontal distribution and $\pi : (T\mathbf{M}, g^s) \to (\mathbf{M}, g)$ is a semi-Riemannian submersion. Also, prove a necessary and sufficient condition for $T\mathbf{M}$ to become a statistical manifold with respect to the Sasaki lift metric and the complete lift connection for an affine submersion with horizontal distribution.

Let $\mathbf{M}$ be an n dimensional manifold and $T\mathbf{M} = \amalg_{x \in \mathbf{M}} T_x\mathbf{M}$ denote the tangent bundle on $\mathbf{M}$. Let $\pi : T\mathbf{M} \longrightarrow \mathbf{M}$ be the natural projection defined by $X_x \in T_x\mathbf{M} \longrightarrow x \in \mathbf{M}$. Let $(U; x^1, ..., x^n)$ be a local coordinate system on $\mathbf{M}$ and the induced co-ordinate system on $\pi^{-1}(U)$ be $(x^1, ..x^n; u^1, ..u^n)$. Let $(x; u)$ be a point on $T\mathbf{M}$, denote the kernel of $\pi_{*(x;u)}$ by $\mathcal{V}_{(x;u)}$ called the vertical subspace of $T_{(x;u)}(T\mathbf{M})$ at $(x; u)$. Note that the vertical subspace $\mathcal{V}_{(x;u)}$ is spanned by $\{\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}, \dots \frac{\partial}{\partial u^n}\}$. The two linear spaces $T_x\mathbf{M}$ and $\mathcal{V}_{(x;u)}$ have the same dimension, so there is a canonical linear isomorphism $V : T_x\mathbf{M} \longrightarrow \mathcal{V}_{(x;u)}$ called the vertical lift.

Let $f : \mathbf{M} \longrightarrow \mathbf{R}$ be a smooth function on $\mathbf{M}$. The vertical lift f^v of f is defined by $f^v = f \circ \pi$. For a vector field $X = X^i \frac{\partial}{\partial x^i}$ on $\mathbf{M}$, the vertical lift X^v is defined by $X^v = (X^i)^v \frac{\partial}{\partial u^i}$. Note that $[X^v, Y^v] = 0$, for any two vector fields X, Y on $\mathbf{M}$. The vertical lift of df is defined by $(df)^v = d(f^v)$, in particular, $(dx^i)^v = d(x^i)^v$ for local co-ordinate functions x^i. The vertical lift of 1-form $\omega = \omega_i dx^i$ is defined as $\omega^v = (\omega_i)^v d(x^i)^v$, the vertical lift operation extends on the full tensor algebra $\mathcal{T}(\mathbf{M})$ by the rule $(P \otimes Q)^v = P^v \otimes Q^v$, for tensor fields P and Q on $\mathbf{M}$.

Let $f : \mathbf{M} \longrightarrow \mathbf{R}$ be a smooth map, the complete lift f^c of f on $T\mathbf{M}$ is defined as $f^c = i(df) = u^i \frac{\partial f}{\partial x^i}$. The complete lift X^c on $T\mathbf{M}$ of the vector field X on $\mathbf{M}$ is characterized by the formula $X^c(f^c) = (Xf)^c$, for all $f \in C^\infty(\mathbf{M})$. In local co-ordinates, the complete lift X^c of $X = X^i \frac{\partial}{\partial x^i}$ has the local expression $X^c = (X^i)\frac{\partial}{\partial x^i} + u^j \frac{\partial X^i}{\partial x^j} \frac{\partial}{\partial u^i}$.

The complete lift to the 1−from ω is defined as $\omega^c(X^c) = (\omega(X))^c$. More generally, the complete lift to full tensor algebra $\mathcal{T}(\mathbf{M})$ is given by the rule $(P \otimes Q)^c = P^c \otimes Q^v + P^v \otimes Q^c$, for tensor fields P and Q on $\mathbf{M}$. Let ∇ be a linear connection on $\mathbf{M}$, then the complete lift ∇^c on $T\mathbf{M}$ is defined as $\nabla^c_{X^c} Y^c = (\nabla_X Y)^c$, for $X, Y \in \mathcal{X}(\mathbf{M})$.

Remark 4.1. Matsuzoe and Inoguchi [10] have proved that if $(\mathbf{M}, \nabla, g)$ is a statistical manifold, then $(T\mathbf{M}, \nabla^c, g^c)$ is a statistical manifold. Moreover, the conjugate connection of ∇^c is $\overline{(\nabla^c)} = (\overline{\nabla})^c$.

Now, we look at the horizontal lifts on the tangent bundle. Let $\mathbf{M}$ be a smooth n dimensional manifold and ∇ be a torsion-free linear connection on $\mathbf{M}$. The vertical subspace $\mathcal{V}_{(x;u)}$ of $T_{(x;u)}(T\mathbf{M})$ at $(x;u)$ defines a smooth distribution $\mathcal{V}$ on $T\mathbf{M}$ called the vertical distribution. Also, there exists a smooth distribution $x \longrightarrow \mathcal{H}(T\mathbf{M})_x$ depending on the linear connection ∇ such that $T_{(x;u)}(T\mathbf{M}) = \mathcal{H}(T\mathbf{M})_x \oplus \mathcal{V}_{(x;u)}$. This distribution is denoted by $\mathcal{H}_\nabla$, called the horizontal distribution. Let X be a vector field on $\mathbf{M}$, then the horizontal lift of X on $T\mathbf{M}$ is the unique vector field X^H on $T\mathbf{M}$ such that $\pi_*(X^H_{(x;u)}) = X_{\pi((x;u))}$ for all $(x;u) \in T\mathbf{M}$. In local co-ordinates, $X^H = X^i \frac{\partial}{\partial x^i} - X^j u^k \Gamma^i_{j,k} \frac{\partial}{\partial u_i}$ for $X = X^i \frac{\partial}{\partial x^i}$. Here $\Gamma^i_{j,k}$ are the connection coefficients of ∇.

Let g be a semi-Riemannian metric on $\mathbf{M}$, then the horizontal lift g^H on $\mathbf{M}$ is defined as $g^H(X^H, Y^H) = g^H(X^v, Y^v) = 0$ and $g^H(X^H, Y^v) = g(X, Y)$, for $X, Y \in \mathcal{X}(\mathbf{M})$. The horizontal lift ∇^H on $\mathbf{M}$ of linear connection ∇ on $\mathbf{M}$ is defined as $\nabla^H_{X^v} Y^v = 0$, $\nabla^H_{X^v} Y^H = 0$, $\nabla^H_{X^H} Y^v = (\nabla_X Y)^v$, $\nabla^H_{X^H} Y^H = (\nabla_X Y)^H$, for $X, Y \in \mathcal{X}(\mathbf{M})$. Note that even if ∇ is torsion-free, its horizontal lift ∇^H may have non-trivial torsion.

Let g be a semi-Riemannian metric on $(\mathbf{M}, \nabla)$. Define a semi-Riemannian metric g^s on $T\mathbf{M}$ as, $g^s_{(x;u)}(X^H, Y^H) = g_x(X, Y)$, $g^s_{(x;u)}(X^H, Y^v) = 0$, $g^s_{(x;u)}(X^v, Y^v) = g_x(X, Y)$. The metric g^s is called the Sasaki lift metric.

In [15], Yano and Ishihara introduced γ operator for defining the horizontal lift from the complete lift. Let X be a vector field on $\mathbf{M}$, with $X = X^i \frac{\partial}{\partial x^i}$, $\nabla X = X^i_j \frac{\partial}{\partial x_i} \otimes dx^j$, where $X^i_j = \frac{\partial X^i}{\partial x^j} + X^k \Gamma^i_{j,k}$. Define $\gamma(\nabla X) = u^j X^i_j \frac{\partial}{\partial u_i}$ with respect to the induced co-ordinate $(x^1, ..., x^n; u^1, ..., u^n)$. Then we can see that $X^H = X^c - \gamma(\nabla X)$, note that $\gamma(\nabla X)$ is the vertical part of X^c.

Remark 4.2. Matsuzoe and Inoguchi [10] proved that if $(\mathbf{M}, \nabla, g)$ is a statistical manifold, then $(T\mathbf{M}, \nabla^H, g^s)$ or $(T\mathbf{M}, \nabla^H, g^H)$ is a statistical manifold if and only if $\nabla g = 0$. Also, they obtained that for a statistical manifold $(\mathbf{M}, \nabla, g)$ both $(T\mathbf{M}, g^s, C^H)$ and $(T\mathbf{M}, g^H, C^H)$ are statistical manifolds, where C^H is the horizontal lift of the cubic form $C = \nabla g$.

A necessary and sufficient condition for $T\mathbf{M}$ to become a statistical manifold with respect to the Sasaki lift metric and the complete lift connection for an affine submersion with horizontal distribution is obtained in this section.

Consider the submersion $\pi : T\mathbf{M} \longrightarrow \mathbf{M}$. Let ∇ be an affine connection on $\mathbf{M}$. Then, there is a horizontal distribution $\mathcal{H}$ such that $T_{(x;u)}(T\mathbf{M}) = \mathcal{H}_{(x;u)}(T\mathbf{M}) + \mathcal{V}_u$ for every $(x;u) \in T\mathbf{M}$.

Now, we show that the submersion π of $T\mathbf{M}$ into $\mathbf{M}$ with the complete lift of the affine connection is an affine submersion with horizontal distribution.

Proposition 4.1. *The submersion* $\pi : (T\mathbf{M}, \nabla^c) \longrightarrow (\mathbf{M}, \nabla)$ *is an affine submersion with horizontal distribution.*

Proof. We need to show that $\mathcal{H}(\nabla^c_{X^H}Y^H) = (\nabla_X Y)^H$. Consider, $X^H = X^c - \gamma(\nabla X)$, then

$$\begin{aligned}\nabla^c_{X^H}Y^H &= \nabla^c_{X^c-\gamma(\nabla X)}Y^c - \gamma(\nabla Y)\\ &= \nabla^c_{X^c-\gamma(\nabla X)}Y^c - \nabla^c_{X^c-\gamma(\nabla X)}\gamma(\nabla Y)\\ &= \nabla^c_{X^c}Y^c - \nabla^c_{\gamma(\nabla X)}Y^c - \nabla^c_{X^c}\gamma(\nabla Y) + \nabla^c_{\gamma(\nabla X)}\gamma(\nabla Y).\end{aligned}$$

Using $\nabla^c_{X^v}Y^v = 0$ ([10]) we have

$$\nabla^c_{X^H}Y^H = (\nabla_X Y)^c - [\nabla^c_{\gamma(\nabla X)}Y^c + \nabla^c_{X^c}\gamma(\nabla Y)]. \tag{4.1}$$

By definition

$$(\nabla_X Y)^c = (\nabla_X Y)^H + \gamma(\nabla(\nabla_X Y)). \tag{4.2}$$

From (4.1) and (4.2), $\mathcal{H}(\nabla^c_{X^H}Y^H) = (\nabla_X Y)^H$. Hence the submersion $\pi : (T\mathbf{M}, \nabla^c) \longrightarrow (\mathbf{M}, \nabla)$ is an affine submersion with horizontal distribution. □

Proposition 4.2. *The submersion $\pi : (T\mathbf{M}, g^s) \longrightarrow (\mathbf{M}, g)$ is a semi Riemannian submersion.*

Proof. Clearly $\pi^{-1}(p) = T_p\mathbf{M}$, for $p \in \mathbf{M}$, is a semi-Riemannian submanifold of $T\mathbf{M}$ and by the definition of g^s we have $g^s(X^H, Y^H) = g(X, Y)$. Hence π is a semi-Riemannian submersion. □

Now, we give a necessary and sufficient condition for the tangent bundle to be a statistical manifold with the Sasaki lift metric and the complete lift connection.

Theorem 4.3. *$(T\mathbf{M}, \nabla^c, g^s)$ is a statistical manifold if and only if*

1. $\mathcal{H}(S_V X) = A_X V - \overline{A}_X V$.
2. $\mathcal{V}(S_X V) = T_V X - \overline{T}_V X$.
3. *$(T_p\mathbf{M}, \hat{\nabla}^c, \hat{g}^s)$ is a statistical manifold for each $p \in \mathbf{M}$.*
4. *$(\mathbf{M}, \nabla, g)$ is a statistical manifold.*

Note that, since $g^s(X^H, Y^v) = 0$, we can take $\mathcal{H}(\mathbf{M}) = \mathcal{V}(\mathbf{M})^\perp$.

Proof. From propositions (4.1) and (4.2), we get that $\pi : (T\mathbf{M}, \nabla^c, g^s) \longrightarrow (\mathbf{M}, \nabla, g)$ is an affine submersion with horizontal distribution. Since $g^s(X^H, Y^v) = 0$, we can take $\mathcal{H}(\mathbf{M}) = \mathcal{V}(\mathbf{M})^\perp$. First, we show that the following equations hold for horizontal vector fields X, Y and vertical vector fields U, V, W

$$(\nabla^c_V g^s)(X, Y) = -g^s(S_V X, Y), \tag{4.3}$$
$$(\nabla^c_X g^s)(V, Y) = -g^s(A_X V, Y) + g^s(\overline{A}_X V, Y), \tag{4.4}$$
$$(\nabla^c_X g^s)(V, W) = -g^s(S_X V, W), \tag{4.5}$$
$$(\nabla^c_V g^s)(X, W) = -g^s(T_V X, W) + g^s(\overline{T}_V X, W), \tag{4.6}$$
$$(\nabla^c_U g^s)(V, W) = (\hat{\nabla}^c_U \hat{g}^s)(V, W), \tag{4.7}$$
$$(\nabla^c_{\tilde{X}} g^s)(\tilde{X}_1, \tilde{X}_2) = (\nabla_X g)(X_1, X_2), \tag{4.8}$$

where $\tilde{X}_i$ are the horizontal lift of the vector fields X_i on $\mathbf{M}$, $\hat{g}^s$ is the induced metric on the fibers and $S_V X = \nabla^c_V X - \overline{\nabla}^c_V X$. To see (4.3) consider

$$\begin{aligned}(\nabla^c_V g^s)(X,Y) &= Vg^s(X,Y) - g^s(\nabla^c_V X, Y) - g^s(X, \nabla^c_V Y)\\ &= g^s(\overline{\nabla}^c_V X, Y) - g^s(X, \nabla^c_V Y)\\ &= -g^s(S_V X, Y).\end{aligned}$$

Similarly, we can prove the other equations. Now suppose $(T\mathbf{M}, \nabla^c, g^s)$ is a statistical manifold, then $\nabla^c g^s$ is symmetric. From (4.3) and (4.4), we get

$$\mathcal{H}(S_V X) = A_X V - \overline{A}_X V.$$

From (4.5) and (4.6), we get $\mathcal{V}(S_X V) = T_V X - \overline{T}_V X$, from (4.7) $\hat{\nabla}^c \hat{g}^s$ is symmetric, so $(T_p\mathbf{M}, \hat{\nabla}^c, \hat{g}^s)$ is a statistical manifold for each $p \in \mathbf{M}$. Also, from (4.8) $(\mathbf{M}, \nabla, g)$ is a statistical manifold.
Conversely, if all the four conditions hold then from the above equations $\nabla^c g^s$ is symmetric, so $(T\mathbf{M}, \nabla^c, g^s)$ is a statistical manifold. □

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