On the geometrization of vector fields (I)

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Abstract. We look for new geometric invariants associated, in canonical ways, to vector fields on differentiable manifolds, as special sets of affine connections and (semi)-Riemannian metrics. This approach is motivated by the need of geometrization of autonomous ODEs systems (and, to some extent, of the non-autonomous ones).

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1 Introduction

The main problem. Let M be a m-dimensional differentiable manifold and $I \subseteq \mathbb{R}$ an open interval. Consider a non-autonomous (i.e. time dependent) system of first order ODEs on M

$$\frac{dx}{dt}(t) = X(t, x(t)) \tag{1.1}$$

with unknown function $x = x(t) : I \to M$, smooth given right-side function $X : I \times M \to TM$ and real parameter $t \in I$.

First goal (minimal): We are looking for new (differential, affine differential, metric) invariants for (1.1), which may be deduced from X and (eventually from) the initial data, and which may spread light onto the properties of the solutions of (1.1).

Second goal (more ambitious): We are looking for a canonical and hopefully unique (differential, affine differential, metric) geometry for (1.1), which may be deduced from X and which may spread light onto the properties of all the "free falling" trajectories in M. That is, from *a force* on M we try to derive the (or a !) *geometry* of M.

Our geometrization program, begun in 2006 ([6],[7],[8],[9], [10], [11], [12], [13]), is twofold. **The first approach** is to start with a given vector field ξ on a differential manifold M and associate to it *a set* of geometric objects (metrics, connections, tensor

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fields of a special type) on M, in a canonical manner, with properties derived from the properties of ξ; this set is a differential invariant and catches both the behaviour (properties) of ξ and M.

Our second approach is to start with a vector field and impose enough relevant hypothesis on its trajectories in order to obtain a canonical and, hopefully, unique Riemannian metric or affine connections w.r.t. which the vector field is auto-parallel. The number of the conditions in the hypothesis grows with the dimension of the manifold M.

When (1.1) is autonomous (i.e. time independent), the function X does not depend on t and defines a tangent vector field on M. We associated ([12], [13]) the set $\mathcal{C}(M, X)$ of all linear connections on M, such that the trajectories of X are auto-parallel curves with respect to them. If X do not vanish, this set is non-void ([12]). We know also ([12]) that $\mathcal{C}(M, X)$ is closed w.r.t. transposition and symmetrization, due to the fact that the auto-parallel curves (i.e. integral curves of ξ) are the same for a connection and its transposed and symmetrized ones.

Other useful invariants are ([12], [13]) the sets of connections in $\mathcal{C}(M, X)$ which are symmetric, metric and symmetric or make X divergence-free, denoted by $\mathcal{C}_s(M, X)$; $\mathcal{C}_{LC}(M, X)$; $\mathcal{C}_d(M, X)$, respectively.

In §2, we review the main notions and focus on the obstructions brought by the singularities of the involved vector fields. We also suggest a method on how to avoid singularities, by "blowing up" them on a "cylinder".

In §3, we define a new invariant associated to an autonomous non-singular vector field X: the set $\mathcal{C}_{par}(M, X)$ of all the linear connections in M such that X be parallel w.r.t. them. We consider the particular cases of the Newtonian gravitational vector field (§4). In §5 we discuss possible extensions of the theory developed sofar, from autonomous to non-autonomous vector fields, and depict two different strategies. The Appendix (§6) contains a discussion about the various indices of nullity which may be associated to the curvature tensor field and to the torsion tensor field of a linear connection (and to their covariant derivatives and contracted tensor fields). These numbers provide invariants needed for a classification of connections and for a classification of differentiable manifolds, both with finite number of families. We use these classifications to show how they induce classifications of vector fields, via the invariants described in the previous paragraphs.

2 Singular vector fields vs regular vector fields

As pointed out in the previous section, the study of a first order autonomous ODEs system on M is equivalent with the study of a vector field ξ on M. The set $\mathcal{C}(M, \xi)$ is a differential invariant and each of its elements is an affine invariant. Moreover, the absence of singular points of ξ guarantees the non-voidness of $\mathcal{C}(M, \xi)$. *Could this property hold also for singular vector fields ?* Unfortunately, the answer is negative, as the following results will show.(The hasty reader may jump to Proposition 2.5 directly.) Lets begin with some enlighting examples.

Example 2.1. Consider the vector field $\xi \in \mathcal{X}(\mathbb{R})$, $\xi = x\partial_x$. Obviously, ξ has a unique singularity in 0. It follows that $\mathcal{C}(\mathbb{R}\setminus\{0\},\xi)=\{\nabla\}$, where $\nabla_{\partial_x}\partial_x=-\frac{1}{x}\partial_x$. The connection ∇ cannot be extended on the whole $\mathbb{R}$.

Remark 2.2. More generally, consider the vector field $\xi\in\mathcal{X}(\mathbb{R})$, $\xi=f(x)\partial_x$, where $f:\mathbb{R}\to\mathbb{R}$ is smooth and vanishes in (and only in) 0. As in the previous example, we determine $\mathcal{C}(\mathbb{R}\setminus\{0\},\xi)=\{\nabla\}$, where $\nabla_{\partial_x}\partial_x=-\frac{f'(x)}{f(x)}\partial_x$ and f' is the derivative of f.

Suppose now, moreover, that ∇ can be extended on the whole $\mathbb{R}$. Using the theorem of l'Hospital and an induction argument, it follows that all the derivatives of f in 0 vanish.

(i) The standard flat function $f(x)=e^{-\frac{1}{x^2}}$ for $x\neq 0$ and $f(0)=0$ does not verify the additional condition for ∇.

(ii) If, moreover, f is quasi-analytic (in particular real analytic), then (by definition) f is identically null, contradiction !

(iii) The only chance to get an affine connection in $\mathcal{C}(\mathbb{R},\xi)$ would be to find a smooth non quasi-analytic function f, with a unique singularity in 0, with all the derivatives vanishing in 0, such that

$$\lim_{x\to 0, x\neq 0}\frac{f'(x)}{f(x)}=0$$

W.r.t.g. we may suppose there exists an $\epsilon>0$ such that $f>0$ on $(0,\epsilon)$. Then

$$\lim_{x\to 0, x>0} lnf(x)=-\infty$$

This shows that the derivative of the function $ln\circ f$ cannot behave near 0_+ as previously depicted. This contradiction finishes the proof of the following

Proposition 2.3. *For $\xi\in\mathcal{X}(\mathbb{R})$, with exactly one singularity, the set $\mathcal{C}(\mathbb{R},\xi)$ is void.*

The local character of the proof allows an easy generalization, by the

Corollary 2.4. *For $\xi\in\mathcal{X}(\mathbb{R})$, with a discrete set of singularities, the set $\mathcal{C}(\mathbb{R},\xi)$ is void.*

This method is hard to extend to higher dimensional manifolds and to more general types of singular vector fields, as it rests on an ad hoc trick. Fortunately, we can use a direct geometric method, for proving the

Proposition 2.5. *Let ξ be a singular vector field on a (connected) differentiable manifold M different from the null vector field. Then $\mathcal{C}(M,\xi)=\varnothing$.*

Proof. Suppose, ad absurdum, there exists $\nabla\in\mathcal{C}(M,\xi)$. Let $p\in M$ be a singular point for ξ and α_p an integral curve for ξ (defined on some open interval of $\mathbb{R}$ containing 0), such that $\alpha_p(0)=p$. The vector field ξ is parallel along α_p and $\xi_p=0$, so ξ vanishes along α_p. It follows that ξ vanishes on M (which is supposed to be connected), contradiction ! □

Remark 2.6. If we try to relax the hypothesis and look for concurrent, recurrent or torse-forming properties (in the non-null singular case), we get no such connections as well. It seems that the behaviour of vector fields arround singular points escapes the geometrization using some parallel transport conditions.

On another hand, when away from singular points, vector fields may be locally transformed (by local diffeomorphisms) such that their flow looks like parallel lines in $\mathbb{R}^n$. Then, what is the relevance of our constructions ? It consists in the possibility to depict the behaviour of all auto-parallel curves (of the "Universe" subject to some plausible assumptions), not only the trajectories of ξ.

Remark 2.7. Let M be a n-dimensional differentiable manifold and ξ a singular vector field on M. Let ϵ be a fixed positive real number. We define $\xi^{(\epsilon)} \in \mathcal{X}(M \times \mathbb{R})$ by $\xi^{(\epsilon)} := \xi + \epsilon \partial_t$, where t is the global coordinate function on $\mathbb{R}$. Then, the vector field $\xi^{(\epsilon)}$ is no more singular and it "approximates" ξ (when ϵ tends to zero). We apply to $\xi^{(\epsilon)}$ the machinery for non-singular vector fields, and associate the respective invariants as $\mathcal{C}(M \times \mathbb{R}, \xi^{(\epsilon)})$, etc.

For example, to the singular vector field $\xi = x\partial_x$ in $\mathcal{X}(\mathbb{R})$, we associate the non-singular vector fields $\xi^{(\epsilon)} := x\partial_x + \epsilon\partial_t$ in $\mathcal{X}(\mathbb{R}^2)$, for any $\epsilon > 0$.

3 Invariants associated to autonomous first order ODEs

Let ξ be a non-singular vector field on a (connected) n-dimensional differentiable manifold M. We denote by $\mathcal{C}_{par}(M, \xi)$ the set of all the affine connections ∇ on M, such that ξ is ∇-parallel.

Theorem 3.1. $\mathcal{C}_{par}(M, \xi) \neq \varnothing$.

Proof. Consider a parallelizable open set U of M and take there a basis of vector fields of the form $\{X_1, ..., X_n\}$, with X_1 the restriction of ξ on U. The connection $\nabla^{(-)} \in \mathcal{C}_{par}(U, \xi)$, because

$$\nabla^{(-)}_{X_i} X_j = 0$$

for all $i, j = \overline{1, n}$. An affine sum of connections w.r.t. which ξ is parallel has the same property. So, a partition of unity argument shows there exist a global connection in $\mathcal{C}_{par}(M, \xi)$. □

Remark 3.2. (i) One knows ([4]) that the ∇-parallelism of ξ implies the "partial" vanishing of the curvature tensor field R of ∇, i.e.,

$$R(X, Y)\xi = 0, \tag{3.1}$$

for every vector fields $X, Y \in \mathcal{X}(M)$. From the first Bianchi's identity it follows that the torsion tensor field T of ∇ satisfies the following identity

$$R(\xi, X)Y - R(\xi, Y)X = \sum_{circXY\xi} \{T(T(X, Y), \xi) + (\nabla_X T)(Y, \xi)\}.$$

In particular, for curvature-flat connections in $\mathcal{C}_{par}(M, \xi)$, the Bianchi's identities reduce to

$$\sum_{circXY\xi} \{T(T(X, Y), \xi) + (\nabla_X T)(Y, \xi)\} = 0.$$

(ii) The set $\mathcal{C}_{par}(M,\xi)$ is an affine sub-module of $\mathcal{C}(M,\xi)$, which is an affine sub-module of $\mathcal{C}(M)$. As all of them are infinite dimensional, the following example may provide a hint about their "dimension" and "codimension".

(iii) Relations (3.1) tells us that each non-singular vector field ξ admits a connection in $\mathcal{C}_{par}(M,\xi)$ whose index of (curvature) nullity is at least one.

We define *the index of nullity of* ξ to be the maximal index of nullity w.r.t. the set $\mathcal{C}_{par}(M,\xi)$. This number is a differential invariant and is between 1 and n $(= dimM)$ (By contrast, the index of nullity of a connection is only an affine differential invariant). The index of nullity classifies the (non-singular) vector fields into n distinct families.

(iv) By analogy with (iii), we may define other types of indices of nullity for ξ, following the hints in the Appendix. They may apply not only for $\mathcal{C}_{par}(M,\xi)$, but also for other sets of remarkable connections associated to ξ. All these indices may be afterthat used for different classifications of the (non-singular) vector fields.

Remark 3.3. Consider the particular case when $M := G$ is a n-dimensional Lie group with the Lie algebra $L(G)$.

(i) Let ξ be a non-null left invariant vector field on G. We choose a basis of left invariant vector fields $E_1 = \xi, E_2, ..., E_n$ and the canonical Cartan-Schouten (bi-invariant and curvature flat) connection ∇^- associated to it. It is obvious that $\nabla^{(-)} \in \mathcal{C}_{par}(G,\xi)$. In this case, $(\nabla^{(-)}_X T)(Y,Z) = 0$, for all vector fields on G and the restriction of T to $L(G)$ is $-[,]$. It follows that, in this case, the first Bianchi's identity reduces to the identity of Jacobi.

The curvature flatness is not necessary, as the following example will show. We define the left invariant connection $\nabla^{(1)} \in \mathcal{C}_{par}(G,\xi)$, with arbitrary (constant) coefficients in the given basis of $L(G)$, with the only restriction

$$\nabla^{(1)}_{E_i} E_1 = 0,$$

for all $i = \overline{1,n}$. That means the components $\Gamma^i_{j1} = 0$ for all $i,j = \overline{1,n}$. In general, $\nabla^{(1)}$ is not curvature flat, nor symmetric.

We impose, in addition, that $\Gamma^j_{1i} = c^j_{1i}$ and $\Gamma^k_{\alpha\beta} = \Gamma^k_{\beta\alpha} + c^k_{\alpha\beta}$ for all $i,j,k = \overline{1,n}$ and $\alpha,\beta = \overline{1,n-1}$, $\alpha > \beta$, where c^j_{ik} are the structure constants w.r.t. the fixed basis in $L(G)$ and $\Gamma^k_{\alpha\beta}$ are arbitrary constants (for $\alpha \leq \beta$). Under these assumptions, it follows that the connection is symmetric (even if, in general, it is not curvature flat).

(ii) Consider the (left invariant) vector field $\xi := \frac{\partial}{\partial x^1}$ on $\mathbb{R}^2$ (as a Lie group). The set of all left invariant affine connections on $\mathbb{R}^2$ is a vector space of dimension 8. The set of all left invariant affine connections in $\mathcal{C}(\mathbb{R}^2,\xi)$ is a vector space of dimension 6. The set of all left invariant affine connections in $\mathcal{C}_{par}(\mathbb{R}^2,\xi)$ is a vector subspace of dimension 4. If we require the connections to be symmetric, the dimensions decrease to 6, 4 and 2, respectively.

Generalization: replace $\mathbb{R}^2$ by a Lie group G of arbitrary dimension n, and fix an arbitrary left invariant vector field ξ. Then the respective dimensions are: n^3, $n^3 - n$

and $n^3 - n^2$, respectively (for arbitrary left invariant connections); $\frac{n^2(n+1)}{2}$, $\frac{n(n^2+n-2)}{2}$ and at most $\frac{n^2(n-1)}{2}$, respectively (for symmetric left invariant connections).

(iii) In (i) we gave examples of non-symmetric curvature flat connection and of non-flat symmetric connections in $\mathcal{C}_{par}(G, \xi)$. We may ask if there exists a symmetric and curvature flat connection (i.e. an affine structure). The existence of affine structures is a difficult unsolved problem which deserved a lot of work in the recent decades ([1, 2, 3, 5]). In this context, our new defined invariants related to the various indices of nullity may be used to refine this problem.

(iv) There are several methods to introduce the (bi-invariant classical) Cartan-Schouten connections $\nabla^{(-)}$, $\nabla^{(+)}$ and $\nabla^{(0)}$ on a Lie group G: by ad hoc definition as operators from $\mathcal{X}(G)\times\mathcal{X}(G)$ to $\mathcal{X}(G)$; starting with a basis in the Lie algebra of G and imposing their properties in this basis (and afterthat showing that the properties are characteristic); by the intrinsic and basis-free characterizations due to their behaviour as operators from $L(G)\times L(G)$ to $L(G)$ (i.e. the null one, [,] and $\frac{1}{2}$ [,] respectively).

In what follows, we give other *intrinsic basis-free* characterizations, whose proofs are obvious. *On a Lie group there exists a unique bi-invariant connection:* $\nabla^{(-)}$ *which is flat and has the torsion tensor field* -[,]; $\nabla^{(+)}$ *which is flat and has the torsion tensor field* [,]; $\nabla^{(0)}$ *which is symmetric and has the curvature tensor field* $\frac{1}{4}$[[,],].

4 Application: the Newtonian gravitational vector field

Let $M = \mathbf{R}^2\backslash\{0\}$ and m be a positive constant (with signification of mass); denote by (r, φ) the polar coordinates on M and by $\xi = -mr^{-2}\partial r$ the "Newtonian gravitational vector field" on M.

In [9] and [10] we studied $C_{LC}(M, \xi)$; in [9] we extended the study to $C_s(M, \xi)$. In what follows, we shall restrict the study to $C_{par}(M, \xi)$ and to $C_{s,par}(M, \xi)$.

Remark 4.1. (i) Denote $|^i_{jk}|$ the coefficients, in polar coordinates, for an arbitrary linear connection $\nabla \in C_{par}(M, \xi)$. We deduce, as only constraints, that

$$|^1_{11}| = 2r^{-1}\ ,\ |^2_{11}| = 0\ ,\ |^1_{21}| = |^2_{21}| = 0 \tag{3.1}$$

The coefficients $|^1_{12}|$, $|^2_{12}|$, $|^2_{22}|$, $|^1_{22}|$ are arbitrary functions (depending on r and φ). (See also [12].) We have $div^{\nabla}\xi = 0$ and $R^{\nabla}(\partial_r, \partial_\varphi)\partial_r = 0$.

(ii) As a consequence, the coefficients of a (symmetric) connection in $\mathcal{C}_{s,par}(M, \xi)$ satisfy

$$|^1_{11}| = 2r^{-1}\ ,\ |^2_{11}| = 0\ ,\ |^1_{21}| = |^2_{21}| = |^1_{12}| = |^2_{12}| = 0 \tag{3.2}$$

and the coefficients $|^2_{22}|$, $|^1_{22}|$ are arbitrary functions (depending on r and φ). Such a connection ∇ is completely controlled by $\nabla_{\partial_\varphi}\partial_\varphi$; we deduce that the geometric model is completely determined by the ∇- acceleration of the angular coordinate movement.

In the following we list some general properties.

- The geodesics equations are

$$r'' + 2r^{-1}(r')^2 + \left|{}^{1}_{22}\right| \cdot (\varphi')^2 = 0 \ , \ \varphi'' + \left|{}^{2}_{22}\right| \cdot (\varphi')^2 = 0 \tag{3.3}$$

- The curvature tensor field is given by $R^{\nabla}(\partial_r, \partial_\varphi)\partial_r = 0$ and

$$R^{\nabla}(\partial_r, \partial_\varphi)\partial_\varphi = (\partial_r \left|{}^{1}_{22}\right| + 2r^{-1} \cdot \left|{}^{1}_{22}\right|)\partial_r + \partial_r \left|{}^{2}_{22}\right| \cdot \partial_\varphi \tag{3.4}$$

- The Ricci tensor field has the components

$$Ric^{\nabla}(\partial_r, \partial_r) = Ric^{\nabla}(\partial_\varphi, \partial_r) = 0$$

$$Ric^{\nabla}(\partial_r, \partial_\varphi) = \partial_r \left|{}^{2}_{22}\right| \ , \ Ric^{\nabla}(\partial_\varphi, \partial_\varphi) = \partial_r \left|{}^{1}_{22}\right| + \frac{2}{r} \cdot \left|{}^{1}_{22}\right|$$

- The divergence operator $div^{\nabla}(\partial_\varphi) = \left|{}^{2}_{22}\right|$, $div^{\nabla}(\partial_r) = 2r^{-1}$, $div^{\nabla}(\partial_\xi) = 0$.

(iii) Imposing the additional condition $div^{\nabla}(\partial_\varphi) = 0$ leads to the fact that all the coefficients of ∇ in $\mathcal{C}_{s,par}(M, \xi)$ are determined but $\left|{}^{1}_{22}\right|$. Moreover, in this case, the geodesics of ∇ are of the form $(r(t), \varphi(t))$, with $\varphi(t) = at + b$ (with constant real numbers a and b) and $r = r(t)$ is the solution of the equation

$$r'' + 2r^{-1}(r')^2 + a^2 \left|{}^{1}_{22}\right| = 0 \tag{3.5}$$

where $\left|{}^{1}_{22}\right|$ is an arbitrary function of r and φ (and which controlls -in this case- the behaviour of the geodesics).

The following result details some other particular cases.

Proposition 4.2. *Let $M = \mathbf{R}^2 \backslash \{0\}$, $\xi = -mr^{-2}\partial r$ the Newtonian gravitational vector field on M and $\nabla \in \mathcal{C}_{s,par}(M, \xi)$.*

(i) *The following assertions are equivalent:* (i)$_1$ $R^{\nabla} = 0$; (i)$_2$ $Ric^{\nabla} = 0$; (i)$_3$ ∂_φ *is* ∇*-parallel*; (i)$_4$ $\left|{}^{2}_{22}\right| = \left|{}^{1}_{22}\right| = 0$; (i)$_5$ $div^{\nabla}\partial_\varphi = 0$ *and* $\Gamma^1_{22} = 0$.

(*In this case, the geodesics of* ∇ *are radial curves, circles and spirals.*)

(ii) *Suppose* $\Gamma^2_{22} = 0$. *Then:* (ii)$_1$ $div^{\nabla}\partial_\varphi = 0$; (ii)$_2$ $R^{\nabla}(\partial_r, \partial_\varphi)\partial_\varphi = \partial_r(\Gamma^1_{22})\partial_r$; (ii)$_3$ $R^{\nabla} = 0$ *if and only* Γ^1_{22} *does not depend on the variable* r;

(iii) *Suppose* $\Gamma^2_{22} = 0$ *and*

$$\Gamma^1_{22}(r, \varphi) = -(2r^3 sin\varphi - \varphi^2 + 1)(12\varphi sin\varphi + 12\varphi^2 cos\varphi - 2\varphi^3 sin\varphi - 2) -$$

$$-2(6\varphi^2 sin\varphi + 2\varphi^3 cos\varphi - 2\varphi)^2.$$

Then the geodesics are of the form $\varphi(t) = t$, $r(t) = 2t^3 sint - t^2 + 1$, *see Figure 1.*

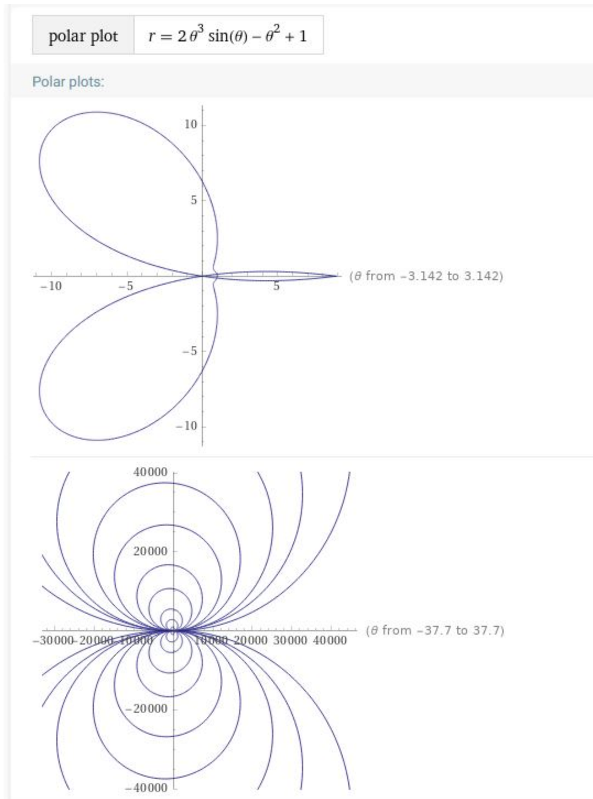

Figure 1. Geodesics in (iii)

(iv) *Suppose* $\Gamma^2_{22} = 0$ *and* $\Gamma^1_{22}(r, \varphi) = -9\varphi^2$. *Then we have geodesics of the form* $\varphi(t) = t$, $r(t) = t^2$ (double spirals).

(v) *Suppose* $\nabla_{\partial_\varphi \partial_\varphi = \partial_\varphi}$. *Then* $R^\nabla = 0$ and *the geodesics are exponential-like spirals.*

(vi) *Suppose* $\nabla_{\partial_\varphi}\partial_\varphi = \partial_r$, *or* $\nabla_{\partial_\varphi}\partial_\varphi = \partial_r + \partial_\varphi$, *or* $\nabla_{\partial_\varphi}\partial_\varphi = \varphi\partial_\varphi$, *or* $\nabla_{\partial_\varphi}\partial_\varphi = \varphi\partial_r$. *Then* $R^\nabla = 0$.

(vii) *If* $\nabla_{\partial_\varphi}\partial_\varphi = r\partial_r$, *or if* $\nabla_{\partial_\varphi}\partial_\varphi = r\partial_r + \varphi\partial_\varphi$, *then* $R^\nabla(\partial_r, \partial_\varphi)\partial_\varphi = \partial_r$. *Moreover,* $Ric^\nabla(\partial_r, \partial_\varphi) = 0$ *and* $Ric^\nabla(\partial_\varphi, \partial_\varphi) = 3$.

(viii) *If* $\nabla_{\partial_\varphi}\partial_\varphi = r\partial_\varphi$, *or if* $\nabla_{\partial_\varphi}\partial_\varphi = \varphi\partial_r + r\partial_\varphi$, *then* $R^\nabla(\partial_r, \partial_\varphi)\partial_\varphi = \partial_\varphi$. *Moreover,* $Ric^\nabla(\partial_r, \partial_\varphi) = 1$ *and* $Ric^\nabla(\partial_\varphi, \partial_\varphi) = 0$.

Proof. The equivalences in (i) are obvious, as well as the claims concerning the curvature tensor field in (ii)-(viii).

We integrate the system of equations (3.3) whose solutions are the geodesics (i.e. the auto-parallel curves of ∇).

In the case (i), the geodesics of ∇ were determined in [10], as follows:

$$[r(t)]^3 = at + b \quad , \quad \varphi(t) = ct + d$$

with arbitrary $a, b, c, d \in \mathbb{R}$.

Consider only non-degenerated auto-parallel curves, i.e. with $a^2 + c^2 > 0$.

The first family of curves contains the (segments of) radial curves: $(c = 0)$, i.e.

$$[r(t)]^3 = at + b \quad , \quad \varphi(t) = d$$

The second family contains the (arcs of) circles: $(a = 0)$, i.e.,

$$[r(t)]^3 = b \quad , \quad \varphi(t) = ct + d$$

The third family of generic curves ($a \neq 0$, $c \neq 0$) contains bounded "spirals", given by implicit equations of the form $r^3 = A\varphi + B$, with arbitrary constants $A \neq 0$, B.

In (iii), (iv) and (v) the geodesics are derived in a similar way. □

Remark 4.3. We know ([10]) that if $\nabla \in \mathcal{C}_{LC}(M, \xi)$, then there exist $\gamma = \gamma(r, \varphi)$ a nowhere vanishing differentiable function and a differentiable function $\beta = \beta(\varphi)$, such that

$$|_{11}^{1}| = 2r^{-1} \quad , \quad |_{11}^{2}| = 0 \quad , \quad |_{12}^{1}| = -2m\beta r^{-3} + m\beta\gamma^{-1}r^{-2}\partial_r\gamma$$

$$|_{12}^{2}| = \gamma^{-1}\partial_r\gamma - 2r^{-1} \quad , \quad |_{22}^{2}| = \gamma^{-1}\partial_\varphi\gamma - m\beta\gamma^{-1}r^{-2}\partial_r\gamma + 2m\beta r^{-3}$$

$$|_{22}^{1}| = m\beta\gamma^{-1}r^{-2}\partial_\varphi\gamma - m\beta' r^{-2} + 2m^2\beta^2 r^{-5} +$$
$$+ 2m^4\gamma^2 r^{-9} - m^2\beta^2\gamma^{-1}r^{-4}\partial_r\gamma - m^4\gamma r^{-8}\partial_r\gamma$$

We remark that $r^2\ |_{12}^{1}| = -m\beta\ |_{12}^{2}|$.

Suppose now we have $\nabla \in \mathcal{C}_{LC}(M, \xi) \cap \mathcal{C}_{s,par}(M, \xi)$. From $|_{12}^{1}| = |_{12}^{2}| = 0$ we derive $r\partial_r\gamma = 2\gamma$, hence there exists a nowhere vanishing function $a = a(\varphi)$ such that $\gamma(r, \varphi) = a(\varphi)r^2$. We replace in the remaining two coefficients and get

$$|_{22}^{2}| = a^{-1}\partial_\varphi a \ , \ |_{22}^{1}| = ma^{-1}r^{-2}(\beta a' - a\beta')$$

The function φ is a solution of $\varphi'' + a'a^{-1}(\varphi')^2 = 0$.

Particular cases: (i) $a = constant$ and $\beta(\varphi) = const$. It follows that $|_{22}^{2}| = |_{22}^{1}| = 0$ and we recover Prop. 4.2.,i.

(ii) $a = 1$ and $\beta(\varphi) = \varphi$. It follows that $|_{22}^{2}| = 0$ and $|_{22}^{1}| = -mr^{-2}$. The geodesics are spirals of the form $\varphi(t) = ct + d$ (for some constants c and d) and $r(t) = (\frac{3m}{2})^{\frac{1}{3}} \cdot (t^2 + 2et + f)^{\frac{1}{3}}$ (for some constants e and f).

5 Invariants associated to non-autonomous first order ODEs

Consider now an m-dimensional differentiable manifold M and an open subset $I \subseteq \mathbb{R}$. Given a non-autonomous system of first order ODEs (1.1), we can associate a "parameterized" vector field $\eta : I \times M \to TM$, $\pi \circ \eta = pr_2$, where $\pi : TM \to M$ and $pr_2 : I \times M \to M$ are the canonical projections. Conversely, any such "parameterized" vector field induces a non-autonomous ODEs (1.1).

The geometrization of η: the first approach. We may view η as an "autonomous" vector field in the $(m+1)$-dimensional differentiable manifold $I \times M$, i.e. $\eta \in \mathcal{X}(I \times M)$. In $I \times M$ we apply the machinery described in the previous paragraphs, by constructing special connections in $\mathcal{C}(I \times M)$. adapted to η. All the invariants are obtained as objects on $I \times M$.

The geometrization of η : the second approach. For each $t \in I$, we denote $\eta_t : M \to TM$, $\eta_t(x) := \eta(t,x)$. We identify η with the family of vector fields $\{\eta_t \mid t \in I\} \subset \mathcal{X}(M)$. For each η_t we construct a special adapted connection $\nabla^{(t)} \in \mathcal{C}(M)$. All the invariants are obtained as families of objects on M.

The two approaches may provide quite different outcomes, as the following simple example shows.

Example 5.1. Let $M := \mathbb{R}$, $I := (0, \infty)$ and the non-autonomous vector field $\eta : I \times M \to TM$ on M, given by $\eta = t\partial_x$.

Any connection $\nabla^{(t)} \in \mathcal{C}(M)$ is trivial, for $t \in I$. Instead, there exist many non-trivial connections on $I \times M$, associated to the autonomous vector field $\eta \in \mathcal{X}(I \times M)$.

6 Appendix: Indices of curvature nullity and of torsion nullity

Let V and W be two real vector spaces of finite dimension and $f : V \to W$ a linear map. The index of nullity of f is the dimension of the nullity space ("kernel") Ker f.

Let M be a n-dimensional differentiable manifold and $\nabla \in \mathcal{C}(M)$. Denote R and T the curvature and the torsion tensor fields of ∇, respectively. The existence of a connection with null curvature and torsion (an affine structure) is an open, difficult and longstanding problem (see [1, 2, 3, 5]). This is why we may consider intermediary steps and look for different indices of nullity associated to R and T, together with the respective nullity spaces.

Let $p \in M$ be a fixed point. The (curvature) nullity subspace at p is ([14]) $\{v \in T_pM \mid R(\cdot,\cdot)v = 0\}$. Its dimension is called ([14]) the (curvature) nullity index at p.

We introduce now some new subspaces and indices, which are more useful in the context of affine differential geometry than of Riemannian one.

Definition 6.1.
The *middle (curvature) nullity subspace* at p is $\{v \in T_pM \mid R(\cdot, v)\cdot = 0\}$.
We call its dimension *the middle (curvature) nullity index* at p.

The *(curvature) nullity 2-set* at p is $\{(v,w) \in T_pM \times T_pM \mid R(\cdot,v)w = 0\}$ and the *(curvature) nullity 2-subspace* at p is *span*$\{(v,w) \in T_pM \times T_pM \mid R(\cdot,v)w = 0\}$. We call its dimension *the (curvature) 2-nullity index* at p.

We define the *(torsion) nullity subspace* at p as $\{v \in T_pM \mid T(\cdot,v) = 0\}$ and we call its dimension the *(torsion) nullity index* at p.

Remark 6.2. (i) Similar nullity sets, subspaces and indices are defined for the covariant derivatives of R and T, of any (fixed) order.

Next we consider these invariants in all the points of M.

If all of them are "maximal", then (M, ∇) is an affine manifold. When their "upper limits" are smaller than their "maximums" (all of them or only some), then these new (affine differential) invariants measure the obstruction to the existence of affine structures.

(ii) The next step is to consider the previous invariants for *all* the connections on M, and to take extremals of the respective indices over $\mathcal{C}(M)$. We shall obtain differential invariants of the manifold M.

(iii) Consider, in a similar way, the index of nullity at the right and at the left, respectively, for the Ricci tensor field at the point p. As the Ricci tensor field is not symmetric, in general, these two indices may differ. Afterthat, consider nullity indices for the covariant derivatives of the Ricci tensor field (in a point and/or on M).

Remark 6.3. (Classifications of linear connections on M) Let M be fixed. Suppose each of the previous defined indices to be maximal and constant on M (in general, they are maximal and constant on an open subset of M). In this case, each and all of them allow clasifications the linear connections ∇ in a finite number of families. Probably, this is not a complete set of invariants (for a classification of linear connections on M, up to an affine diffeomorphism).

Remark 6.4. (Classifications of differentiable manifolds M). For each differentiable manifold M, we consider all the possible values of the previous nullity indices, which may be associated to linear connections on M: $\{i_1, ..., i_k\}$. This provides a "fingerprint" of each manifold and a classification of the differentiable manifolds by differential invariants, classification with a finite number of families. Probably, this is not a complete set of invariants (for a classification of differentiable manifolds, up to a diffeomorphism).

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