Quasi bi-slant submanifolds of para-Kenmotsu manifolds

Rajendra Prasad, Abdul Haseeb and Shweta Singh

Abstract. In this paper, we study quasi bi-slant submanifolds of a para-Kenmotsu manifold. We obtain the necessary and sufficient conditions for integrability of the distributions which are involved in the definition of such manifold. We also prove that the slant distributions which defines a totally umbilical foliation on submanifold of a para-Kenmotsu manifold is either invariant or anti-invariant.

M.S.C. 2010: 53C12, 53D15.
Key words: Quasi bi-slant submanifold, para-Kenmotsu manifold, integrability conditions, totally umbilical foliations.

1 Introduction

The study of submanifolds of an almost contact manifold is one of the utmost interesting topic in differential geometry. According to the behaviour of the tangent bundle of a submanifold with respect to the action of an almost contact structure ϕ of the ambient manifold, there are two well known classes of submanifolds, namely; invariant and anti-invariant submanifolds. Chen [2], introduced the notion of slant submanifolds of an almost Hermitian manifold. The contact [10] and K-contact [11] versions of slant submanifolds were given by Lotta. Since then many research articles have been appeared on the existence of different contact and Lorentzian manifolds (we refer to see [1, 5, 6, 12, 14, 15]). Analogous to the definition of Kenmotsu manifold [8], Welyczko [17] introduced para-Kenmotsu structure for three dimensional normal almost para contact metric structures. The similar notion called $p-$Kenmotsu structure was studied by Sinha and Prasad [16].

The purpose of this paper is to study quasi bi-slant submanifolds of para-Kenmotsu manifolds which includes the classes of slant submanifolds, semi-slant submanifolds, hemi-slant submanifolds and bi-slant submanifolds as its particular cases. Primarily, the hemi-slant submanifolds were known as anti-slant submanifolds. Later, F. Sahin [13] named these submanifolds as hemi-slant submanifolds. Hemi-slant submanifolds are one of the classes of bi-slant submanifolds.

Balkan Journal of Geometry and Its Applications, Vol.26, No.2, 2021, pp. 100-111.

The paper is organized as follows: In section 2, we mention the basic definitions and properties of para-Kenmotsu manifolds. In section 3, we define quasi bi-slant submanifolds and obtain some useful lemmas. Section 4 deals with the necessary and sufficient conditions for integrability of the distributions. In section 5, we prove that the slant distributions which defines a totally umbilical foliation on submanifolds of a para-Kenmotsu manifold is either invariant or anti-invariant.

2 Preliminaries

Let $\overline{\mathcal{N}}$ be a $(2m+1)$ dimensional smooth manifold, Φ a tensor field of type $(1,1)$, ξ a vector field and η a 1-form. We say that (Φ, ξ, η) is an almost para contact structure on $\overline{\mathcal{N}}$ if [18]

$$\Phi^2 = I - \eta \otimes \xi,\ \eta(\xi) = 1,\ \Phi(\xi) = 0,\ \eta \circ \Phi = 0,\ rank(\Phi) = 2m, \tag{2.1}$$

$$g(U, \xi) = \eta(U),\ g(\xi, \xi) = 1, \tag{2.2}$$

and Φ is a skew-symmetric operator, i. e., $g(\Phi U, V) = -g(U, \Phi V)$.
If an almost paracontact manifold admits a pseudo Riemannian metric g of signature $(m+1, m)$ satisfying

$$g(\Phi U, \Phi V) = -g(U, V) + \eta(U)\eta(V) \tag{2.3}$$

called almost para contact metric manifold. Examples of almost para contact metric structure are given in [4] and [7].

Definition 2.1. An almost para contact metric manifold $\overline{\mathcal{M}}(\Phi, \xi, \eta, g)$ is para-Kenmotsu manifold if the Levi-Civita connection $\overline{\nabla}$ of g satisfies

$$(\overline{\nabla}_U \Phi)V = g(\Phi U, V)\xi - \eta(V)\Phi U \tag{2.4}$$

for all $U, V \in \Gamma(T\overline{\mathcal{M}})$.

From (2.4), we have

$$\overline{\nabla}_U \xi = U - \eta(U)\xi. \tag{2.5}$$

Example. Let $\overline{\mathcal{M}} = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. We set

$$\Phi := \frac{\partial}{\partial y} \otimes dx + \frac{\partial}{\partial x} \otimes dy,\ \xi := -\frac{\partial}{\partial z},\ \eta := -dz,$$

$$g := dx \otimes dx - dy \otimes dy + dz \otimes dz.$$

Then (Φ, ξ, η, g) defines a para-Kenmotsu structure on $\mathbb{R}^3$.

Assume $\overline{\mathcal{M}}$ a submanifold of a para-Kenmotsu manifold $\overline{\mathcal{N}}$. Let g and ∇ be the induced Riemannian metric and connection on $\overline{\mathcal{M}}$, respectively. Then the Gauss and Weingarten formulae are given respectively, by

$$\overline{\nabla}_U V = \nabla_U V + \nu(U, V), \tag{2.6}$$

$$\overline{\nabla}_U N = -\mathcal{A}_N U + \nabla^{\perp}_U N \tag{2.7}$$

for all U, V on $T\overline{\mathcal{M}}$ and $N \in T^{\perp}\overline{\mathcal{M}}$, where $\nabla^{\perp}$ is the normal connection and $\mathcal{A}$ is the shape operator of $\overline{\mathcal{M}}$ with respect to the unit normal vector N. The second fundamental form ν and the shape operator $\mathcal{A}$ are related by

$$g(\nu(U,V),N) = g(\mathcal{A}_N U, V). \tag{2.8}$$

Now, for any $U \in \Gamma(T\overline{\mathcal{M}})$ and $X \in \Gamma(T^{\perp}\overline{\mathcal{M}})$, we write

$$\Phi U = BU + CU, \tag{2.9}$$

$$\Phi X = bX + cX. \tag{2.10}$$

For any $U, V \in \Gamma(T\overline{\mathcal{M}})$, it is easy to observe that

$$g(BU,V) = -g(U,BV). \tag{2.11}$$

The covariant derivatives of the endomorphisms Φ, B and C are defined respectively by

$$(\overline{\nabla}_U \Phi)V = \overline{\nabla}_U \Phi V - \Phi \overline{\nabla}_U V,\ \forall\ U,V \in \Gamma(T\overline{\mathcal{M}}), \tag{2.12}$$

$$(\overline{\nabla}_U B)V = \nabla_U BV - B\nabla_U V,\ \forall\ U,V \in \Gamma(T\overline{\mathcal{M}}), \tag{2.13}$$

$$(\overline{\nabla}_U C)V = \nabla_U CV - C\nabla_U V,\ \forall\ U,V \in \Gamma(T\overline{\mathcal{M}}). \tag{2.14}$$

Since $\xi \in T\overline{\mathcal{M}}$, therefore for any $U \in \Gamma(T\overline{\mathcal{M}})$ by virtue of (2.6) and (2.7) we have

$$\nabla_U \xi = U - \eta(U)\xi, \tag{2.15}$$

and

$$\nu(U,\xi) = 0.$$

Making use of (2.5), (2.7), (2.8), (2.10), (2.11) and (2.13)-(2.15), we obtain

$$(\overline{\nabla}_U B)V = b\nu(U,V) + \mathcal{A}_{CV} U + g(BU,V)\xi - \eta(Y)BU, \tag{2.16}$$

$$(\overline{\nabla}_U C)V = c\nu(U,V) - \nu(U,BV) - \eta(V)CU. \tag{2.17}$$

A submanifold $\overline{\mathcal{M}}$ of an almost para contact metric manifold $\overline{\mathcal{N}}$ is said to be totally umbilical if

$$\nu(U,V) = g(U,V)\mathcal{H}, \tag{2.18}$$

where $\mathcal{H}$ is mean curvature vector of $\overline{\mathcal{M}}$. Further, $\overline{\mathcal{M}}$ is totally geodesic if $\nu(U,V) = 0$ and minimal if $\mathcal{H} = 0$ [9].

Definition 2.2. Let $\overline{\mathcal{M}}$ be a submanifold of $\overline{\mathcal{N}}$. Then $\overline{\mathcal{M}}$ is called invariant submanifold of $\overline{\mathcal{N}}$ if $\Phi(T_u\overline{\mathcal{M}}) \subset T_u\overline{\mathcal{M}}$, for any $u \in \overline{\mathcal{M}}$. It follows $\Phi(T_u\overline{\mathcal{M}}^{\perp}) \subset T_u\overline{\mathcal{M}}^{\perp}$, for any $u \in \overline{\mathcal{M}}$. Indeed for any $X \in \Gamma(T\overline{\mathcal{M}}^{\perp})$, $g(U, \Phi X) = -g(\Phi U, X) = 0$, for any $U \in \Gamma(T\overline{\mathcal{M}})$.

Definition 2.3. Let $\overline{\mathcal{M}}$ be a submanifold of $\overline{\mathcal{N}}$. Then $\overline{\mathcal{M}}$ is called anti invariant submanifold of $\overline{\mathcal{N}}$ if $\Phi(T_u\overline{\mathcal{M}}) \subset T_u\overline{\mathcal{M}}^{\perp}$, for any $u \in \overline{\mathcal{M}}$.

Definition 2.4. For any $u \in \overline{\mathcal{M}}$ and $U \in T_u\overline{\mathcal{M}}$ such that U, ξ are linearly independent, the angle $\theta(u) \in [0, \frac{\pi}{2}]$ between ΦU and $T_u\overline{\mathcal{M}}$ is constant θ, that is θ does not depend on the choice of U and $u \in \overline{\mathcal{M}}$. θ is called the slant angle of $\overline{\mathcal{M}}$ in $\overline{\mathcal{N}}$. Invariant and anti-invariant submanifolds are slant submanifolds with slant angle θ equal to 0 and $\frac{\pi}{2}$, respectively [3]. A slant submanifold which is neither invariant nor anti-invariant is called a proper slant submanifold.

We mention the following result for later use:
Let $\overline{\mathcal{M}}$ be a submanifold of an almost contact metric manifold $\overline{\mathcal{N}}$ such that $\xi \in T\overline{\mathcal{M}}$. Then, $\overline{\mathcal{M}}$ is slant iff there exist a constant $\lambda \in [0, 1]$ such that

$$B^2 = -\lambda(I - \eta \otimes \xi). \tag{2.19}$$

3 Quasi bi-slant submanifolds of a para-Kenmotsu manifold

In this section, we define the concept of quasi bi-slant submanifolds of a para-Kenmotsu manifold and obtain some related results for later use.

Assume $\overline{\mathcal{M}}$ is a submanifold of a para-Kenmotsu manifold $\overline{\mathcal{N}}$. Let g and ∇ represents the induced Riemannian metric and connection on $\overline{\mathcal{M}}$, respectively.

Definition 3.1. A submanifold $\overline{\mathcal{M}}$ of a para-Kenmotsu manifold $\overline{\mathcal{N}}(\Phi, \xi, \eta, g)$ is called quasi bi-slant if there exist four orthogonal distributions D, D_1, D_2 and ξ of $\overline{\mathcal{M}}$, at the point $p \in \overline{\mathcal{M}}$ such that
1. $T\overline{\mathcal{M}} = D \oplus D_1 \oplus D_2 \oplus < \xi >$,
2. $\Phi(D) = D$ i.e., D is invariant,
3. $\Phi(D_1) \perp D_2$, and $\Phi(D_2) \perp D_1$
4. The distributions D_1 and D_2 are slant with angles θ_1 and θ_2 respectively.

We easily observe that
(a) $If dim D \neq 0$, $dim D_1 = 0$ and $dim D_2 = 0$, then $\overline{\mathcal{M}}$ is an invariant submanifold,
(b) If dim $D = 0$, $dim D_1 = 0$, and $\theta_2 = \frac{\pi}{2}$, then $\overline{\mathcal{M}}$ is an anti-invariant submanifold,
(c) If dim $D = 0$, $dim D_1 \neq 0$, $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{2}$ and $dim D_2 \neq 0$, then $\overline{\mathcal{M}}$ is semi-invariant submanifold,
(d) If $dim D = 0$, $dim D_1 = 0$ and $0 < \theta_2 < \frac{\pi}{2}$, then $\overline{\mathcal{M}}$ is slant submanifold,
(e) $dim D = 0$, $dim D_1 \neq 0$, $dim D_2 \neq 0$, $\theta_1 = 0$ and $0 < \theta_2 < \frac{\pi}{2}$, then $\overline{\mathcal{M}}$ is semi-slant submanifold.
(f) If $dim D = 0$, $dim D_1 \neq 0$, $\theta_1 = \frac{\pi}{2}$ and $0 < \theta_2 < \frac{\pi}{2}$ and $dim D_2 \neq 0$, then $\overline{\mathcal{M}}$ is hemi-slant submanifold, (g) $dim D = 0$, $dim D_1 \neq 0$,and $dim D_2 \neq 0$, θ_1 and θ_2 are

different from 0 and $\frac{\pi}{2}$, then $\overline{\mathcal{M}}$ is bi-slant submanifold.

If $dimD_1 \neq 0$, $dimD_2 \neq 0$, $dimD_3 \neq 0$ and $\theta_1, \theta_2 \neq 0, \frac{\pi}{2}$, then $\overline{\mathcal{M}}$ is called a proper quasi bi-slant submanifold.

Let $\overline{\mathcal{M}}$ be a quasi bi-slant submanifold of a para-Kenmotsu manifold $\overline{\mathcal{N}}$. Then for any $U \in \Gamma(T\overline{\mathcal{M}})$, we have

$$U = JU + KU + LU + \eta(U)\xi, \tag{3.1}$$

where J, K, L denote the projections on the distribution D, D_1, and D_2, respectively. Now making use of (2.9) and (3.1), we get

$$\Phi U = BJU + BKU + BLU + CKU + CLU. \tag{3.2}$$

Here since $\Phi D = D$, we have $CJU = 0$, this means, for any $U \in \Gamma(T\overline{\mathcal{M}})$, we have

$$BU = BJU + BKU + BLU,$$

$$CU = CKU + CLU.$$

Thus we get

$$\Phi(T\overline{\mathcal{M}}) = D \oplus TD_1 \oplus TD_2, \tag{3.3}$$

and

$$T^{\perp}\overline{\mathcal{M}} = CD_1 \oplus CD_2 \oplus \mu, \tag{3.4}$$

where μ is the orthogonal complement of $CD_1 \oplus CD_2$ in $T^{\perp}\overline{\mathcal{M}}$ and it is invariant with respect to Φ.

Lemma 3.1. *Let $\overline{\mathcal{M}}$ be a quasi bi-slant submanifold of a para-Kenmotsu manifold $\overline{\mathcal{N}}$. Then*

(i) $B^2U = (\cos^2\theta_1)U$,

(ii) $g(BU, BV) = -(\cos^2\theta_1)g(U, V)$,

(iii) $g(CU, CV) = -(\sin^2\theta_1)g(U, V)$,

(iv) $g(U, \xi) = \eta(U) = 0$

for any $U, V \in \Gamma(D_1)$, where θ_1 denotes the slant angle of D_1.

Lemma 3.2. *Let $\overline{\mathcal{M}}$ be a quasi bi-slant submanifold of a para-Kenmotsu manifold $\overline{\mathcal{N}}$. Then*

(i) $B^2U = (\cos^2\theta_2)U$,

(ii) $g(BU, BV) = -(\cos^2\theta_2)g(U, V)$,

(iii) $g(CU, CV) = -(\sin^2\theta_2)g(U, V)$,

(iv) $g(U, \xi) = \eta(U) = 0$

for any $U, V \in \Gamma(D_2)$, where θ_2 denotes the slant angle of D_2.

4 Integrability of the distributions

In this section we give some necessary and sufficient conditions for the integrability of the distributions.

Theorem 4.1. *Let $\overline{\mathcal{M}}$ be a proper quasi bi-slant submanifold of a para-Kenmotsu manifold $\overline{\mathcal{N}}$. Then the invariant distribution D is integrable if and only if*

$$g(\nabla_U BV - \nabla_V BU, BKW + BLW) = g(\nu(V, BU) - \nu(U, BV), CKW + CLW) \tag{4.1}$$

for any $U, V \in \Gamma(D)$ and $W \in \Gamma(D_1 \oplus D_2)$.

Proof. The distribution D is integrable if and only if $g([U, V], \xi) = 0$ and $g([U, V], W) = 0$ for any $U, V \in \Gamma(D)$, $W \in \Gamma(D_1 \oplus D_2)$ and $\xi \in \Gamma(T\overline{\mathcal{M}})$.
Since $\overline{\mathcal{M}}$ is submanifold of a para-Kenmotsu manifold $\overline{\mathcal{N}}$, we immediately have $g([U, V], \xi) = 0$. Thus D is integrable if and only if $g([U, V], W) = 0$. Now for any $U, V \in D$ and $W = KW + LW \in \Gamma(D_1 \oplus D_2)$, from (2.3) we obtain

$$\begin{aligned}
g([U, V], W) &= -g(\Phi\overline{\nabla}_U V, \Phi W) + \eta(\overline{\nabla}_U V)\eta(W) + g(\Phi\overline{\nabla}_V U, \Phi W) - \eta(\overline{\nabla}_V U)\eta(W)\\
&= -g(\overline{\nabla}_U \Phi V - (\overline{\nabla}_U \Phi)V, \Phi W) + g(\overline{\nabla}_V \Phi U - (\overline{\nabla}_V \Phi)U, \Phi W)\\
&= -g(\overline{\nabla}_U \Phi V, \Phi W) + g((\overline{\nabla}_U \Phi)V, \Phi W) + g(\overline{\nabla}_V \Phi U, \Phi W)\\
&- g((\overline{\nabla}_V \Phi)U, \Phi W)\\
&= -g(\overline{\nabla}_U \Phi V, \Phi W) + g(\overline{\nabla}_V \Phi U, \Phi W) + g(g(\Phi U, V)\xi - \eta(V)\Phi U, \Phi W)\\
&- g(g(\Phi V, U)\xi - \eta(U)\Phi V, \Phi W).
\end{aligned}$$

Using (2.2) and fact that $CU = 0$ and $CV = 0$, we have

$$\begin{aligned}
g([U, V], W) = &-g(\overline{\nabla}_U BV, \Phi W) + g(\overline{\nabla}_V BU, \Phi W)\\
&- g(\Phi U, V)g(\Phi\xi, W) + g(\Phi V, U)g(\Phi\xi, W).
\end{aligned}$$

By using (2.6) and fact that $\Phi\xi = 0$, we obtain

$$\begin{aligned}
g([U, V], W) &= -g(\overline{\nabla}_U BV, \Phi W) - g(\nu(U, BV), \Phi W)\\
&+ g(\overline{\nabla}_V BU, \Phi W) + g(\nu(V, BU), \Phi W)\\
&= -g(\overline{\nabla}_U BV - \overline{\nabla}_V BU, \Phi W) - g(\nu(U, BV) - \nu(V, BU), \Phi W)\\
&= -g(\nabla_U BV - \nabla_V BU, BW) - g(\nu(U, BV) - \nu(V, BU), CW)\\
&= -g(\nabla_U BV - \nabla_V BU, BKW + BLW)\\
&- g(\nu(U, BV) - \nu(V, BU), CKW + CLW).
\end{aligned}$$

This completes the proof. □

For the slant distribution D_1, we have

Theorem 4.2. *Let $\overline{\mathcal{M}}$ be a proper quasi bi-slant submanifold of a para-Kenmotsu manifold $\overline{\mathcal{N}}$. Then the slant distribution D_1 is integrable if and only if*

$$g(\mathcal{A}_{CV}U - \mathcal{A}_{CU}V, BJW + BLW) = g(\mathcal{A}_{BCV}U - \mathcal{A}_{BCU}V, W) + g(\nabla^{\perp}_U CV - \nabla^{\perp}_V CU, CLW)$$

for any $U, V \in \Gamma(D_1)$ and $W \in \Gamma(D \oplus D_2)$.

Proof. The distribution D_1 is integrable on $\overline{\mathcal{M}}$ if and only if $g([U,V],\xi) = 0$ and $g([U,V],W) = 0$ for any $U,V \in \Gamma(D_1)$, $W \in \Gamma(D \oplus D_2)$ and $\xi \in \Gamma(T\overline{\mathcal{M}})$. The first case is trivial. Thus D_1 is integrable if and only if $g([U,V],W) = 0$. Now for any $U,V \in D_1$ and $W = JW + LW \in \Gamma(D \oplus D_2)$, from (2.3) and (2.6), we obtain

$$\begin{aligned} g([U,V],W) &= -g(\Phi\overline{\nabla}_U V, \Phi W) + \eta(\overline{\nabla}_U V)\eta(W) + g(\Phi\overline{\nabla}_V U, \Phi W) - \eta(\overline{\nabla}_V U)\eta(W) \\ &= -g(\overline{\nabla}_U \Phi V - (\overline{\nabla}_U \Phi)V, \Phi W) + g(\overline{\nabla}_V \Phi U - (\overline{\nabla}_V \Phi)U, \Phi W) \\ &= -g(\overline{\nabla}_U \Phi V, \Phi W) + g(\overline{\nabla}_V \Phi U, \Phi W) \\ &= -g(\overline{\nabla}_U BV, \Phi W) + g(\overline{\nabla}_V BU, \Phi W) - g(\overline{\nabla}_U CV, \Phi W) + g(\overline{\nabla}_V CU, \Phi W) \\ &= g(\overline{\nabla}_U \Phi BV, W) - g(\overline{\nabla}_V \Phi BU, W) - g(\overline{\nabla}_U CV, \Phi W) + g(\overline{\nabla}_V CU, \Phi W) \\ &= g(\overline{\nabla}_U B^2 V, W) + g(\overline{\nabla}_U BCV, W) - g(\overline{\nabla}_U CV - \overline{\nabla}_V CU, \Phi W) \\ &\quad - g(\overline{\nabla}_V B^2 U, W) - g(\overline{\nabla}_V BCU, W). \end{aligned}$$

By using Lemma 3.2 and (2.7), we get

$$\begin{aligned} g([U,V],W) &= \cos^2\theta_1 g([U,V],W) + g(\overline{\nabla}_U BCV - \overline{\nabla}_V BCU, W) \\ &\quad - g(\overline{\nabla}_U CV - \overline{\nabla}_V CU, \Phi W) \\ &= -g(-\mathcal{A}_{CV}U + \mathcal{A}_{CU}V, BJW + BLW) - g(\nabla^{\perp}_U CV - \nabla^{\perp}_V CU, CLW) \\ &\quad + \cos^2\theta_1 g([U,V],W) + g(-\mathcal{A}_{BCV}U + \mathcal{A}_{BCU}V, W), \end{aligned}$$

which leads to

$$\begin{aligned} \sin^2\theta_1 g([U,V],W) &= -g(\mathcal{A}_{BCV}U - \mathcal{A}_{BCU}V, W) + g(\mathcal{A}_{CV}U - \mathcal{A}_{CU}V, BJW + BLW) \\ &\quad - g(\nabla^{\perp}_U CV - \nabla^{\perp}_V CU, CLW). \end{aligned}$$

Thus the proof is completed. □

From Theorem 4.2, we have the following sufficient conditions for the slant distribution D_1 to be integrable.

Theorem 4.3. *Let $\overline{\mathcal{M}}$ be a proper quasi bi-slant submanifold of a para-Kenmotsu manifold $\overline{\mathcal{N}}$. If*

$$\nabla^{\perp}_U CV - \nabla^{\perp}_V CU \in CD_1 \oplus \mu,$$

$$\mathcal{A}_{BCV}U - \mathcal{A}_{BCU}V \in D_1$$

and

$$\mathcal{A}_{CV}U - \mathcal{A}_{CU}V \in D_1$$

for any $U,V \in \Gamma(D_1)$, then the slant distribution D_1 is integrable.

Theorem 4.4. *Let $\overline{\mathcal{M}}$ be a proper quasi bi-slant submanifold of a para-Kenmotsu manifold $\overline{\mathcal{N}}$. Then the slant distribution D_2 is integrable if and only if*

$$g(\mathcal{A}_{CV}U - \mathcal{A}_{CU}V, \Phi W) = g(\mathcal{A}_{BCV}U - \mathcal{A}_{BCU}V, W) + g(\nabla^{\perp}_U CV - \nabla^{\perp}_V CU, CKW)$$

for any $U,V \in \Gamma(D_2)$ and $W \in \Gamma(D \oplus D_1)$.

From Theorem 4.4, we have the following sufficient conditions for the slant distribution D_2 to be integrable.

Theorem 4.5. *Let $\overline{\mathcal{M}}$ be a proper quasi bi-slant submanifold of a para-Kenmotsu manifold $\overline{\mathcal{N}}$. If*

$$\nabla^{\perp}_{U}CV - \nabla^{\perp}_{V}CU \in CD_2 \oplus \mu,$$

$$\mathcal{A}_{BCV}U - \mathcal{A}_{BCU}V \in D_2$$

and

$$\mathcal{A}_{CV}U - \mathcal{A}_{CU}V \in D_2$$

for any $U, V \in \Gamma(D_2)$, then the slant distribution D_2 is integrable.

5 Totally umbilical foliations

In this section, we prove that slant distributions, which defines a totally umbilical foliation on submanifold of a para-Kenmotsu manifold is either invariant or anti-invariant.

Theorem 5.1. *Let $\overline{\mathcal{M}}$ be a proper quasi bi-slant submanifold of a para-Kenmotsu manifold $\overline{\mathcal{N}}$. And let the slant distribution D_1 is totally umbilical foliation on $\overline{\mathcal{M}}$. Then either one of the following statement is true:*

(i) D_1 is invariant,

(ii) D_1 is anti-variant,

(iii) D_1 is totally geodesic,

(iv)D_1 is proper slant, then $\mathcal{H} \in \Gamma(CD_2 \oplus \mu)$, where $\mathcal{H}$ is a mean curvature vector of D_1.

Proof. Let D_1 is totally umbilical foliation on $\overline{\mathcal{M}}$ and $U \in D_1$, then we have

$$\nu(PU, PU) = g(PU, PU)\mathcal{H} = \cos^2\theta_1\{-\|U\|^2\}\mathcal{H}.$$

By using (2.6), we obtain

$$\cos^2\theta_1\{-\|U\|^2\}\mathcal{H} = \overline{\nabla}_{BU}BU - \nabla_{BU}BU.$$

From (2.9) we have

$$\begin{aligned}\cos^2\theta_1\{-\|U\|^2\}\mathcal{H} &= \overline{\nabla}_{BU}\Phi U - \overline{\nabla}_{BU}CU - \nabla_{BU}BU\\ &= (\overline{\nabla}_{BU}\Phi)U + \Phi\overline{\nabla}_{BU}U + \mathcal{A}_{CU}BU - \nabla^{\perp}_{BU}CU - \nabla_{BU}BU\\ &= g(\Phi BU, U)\xi + \Phi(\nabla_{BU}U + \nu(U, BU)) + \mathcal{A}_{CU}BU\\ &- \nabla^{\perp}_{BU}CU - \nabla_{BU}BU.\end{aligned}$$

From (2.9), (2.11) and (2.18) and the fact that U and BU are orthogonal vector fields on $\overline{\mathcal{M}}$, we arrive at

$$\cos^2\theta_1\{-\|U\|^2\}\mathcal{H} = -g(BU, BU)\xi + B\nabla_{BU}U + C\nabla_{BU}U + \mathcal{A}_{CU}BU - \nabla^{\perp}_{BU}CU - \nabla_{BU}BU.$$

Then applying (2.19) and Lemma 3.2, we obtain

$$\begin{aligned} \cos^2\theta_1\{-\|U\|^2\}\mathcal{H} = \; & \cos^2\theta_1\{\|U\|^2\}\xi + B\nabla_{BU}U + C\nabla_{BU}U + \mathcal{A}_{CU}BU \\ & -\nabla^{\perp}_{BU}CU - \nabla_{BU}BU. \end{aligned} \tag{5.1}$$

Taking the inner product of (1) with BU, we get

$$0 = g(B\nabla_{BU}U, BU) + g(\mathcal{A}_{CU}BU, BU) - g(\nabla_{BU}BU, BU). \tag{5.2}$$

Also we can write

$$g(B\nabla_{BU}U, BU) = -\cos^2\theta_1 g(\nabla_{BU}U, U). \tag{5.3}$$

Simplifying the third term of (5.2), we infer

$$\begin{aligned} g(\nabla_{BU}BU, BU) & = g(\overline{\nabla}_{BU}BU, BU) = \tfrac{1}{2}BUg(BU, BU) \\ & = \tfrac{1}{2}BU\{-\cos^2\theta_1 g(U, U)\} = -\tfrac{1}{2}\cos^2\theta_1 BUg(U, U) \\ & = -\tfrac{1}{2}\cos^2\theta_1\{2g(\nabla_{BU}U, U)\} = -\cos^2\theta_1 g(\nabla_{BU}U, U). \end{aligned} \tag{5.4}$$

From (5.3) and (5.4), we obtain

$$g(B\nabla_{BU}U, BU) = g(\nabla_{BU}BU, BU).$$

Using this fact in (5.2), we infer

$$0 = g(\mathcal{A}_{CU}BU, BU) = g(\nu(BU, BU), CU).$$

Since the slant distribution D_1 is a totally umbilical foliation, then from (2.19) and Lemma 3.2 we obtain

$$0 = -\cos^2\theta_1\{\|X\|^2\}g(\mathcal{H}, CU). \tag{5.5}$$

Thus, from (5.5), we conclude that either $\theta = \frac{\pi}{2}$, that is, D_1 is anti-invariant which is part (ii) or $\mathcal{H} \perp CU$, for all $U \in \Gamma(D_1)$, i.e; $\mathcal{H} \in \Gamma(CD_2 \oplus \mu)$ which is the last part of the theorem or $\mathcal{H} = 0$, i.e., D_1 is totally geodesic, which is part (iii) or $CU = 0$, i.e., D_1 is invariant, which is part (i).
This completes the proof of the theorem. □

In a similar way to above theorem we conclude the following:

Theorem 5.2. *Let $\overline{\mathcal{M}}$ be a proper quasi bi-slant submanifold of a para-Kenmotsu manifold $\overline{\mathcal{N}}$. And let the slant distribution D_2 is totally umbilical foliation on $\overline{\mathcal{M}}$. Then either one of the following statement is true:*

(i) D_2 is invariant,

(ii) D_2 is anti-variant,

(iii) D_2 is totally geodesic,

(iv) D_2 is proper slant, then $\mathcal{H} \in \Gamma(CD_1 \oplus \mu)$, where $\mathcal{H}$ is a mean curvature vector of D_2.

Theorem 5.3. *Let $\overline{\mathcal{M}}$ be a proper quasi bi-slant submanifold of $\overline{\mathcal{N}}$. And let D_1 is proper slant distribution which defines a totally umbilical foliation on $\overline{\mathcal{M}}$. Then D_1 is totally geodesic.*

Proof. For any $U, V \in \Gamma(D_1)$, we have

$$\overline{\nabla}_U \Phi V - \Phi \overline{\nabla}_U V = g(\Phi U, V)\xi - \eta(V)\Phi U.$$

By using (2.6) and (2.9), we get

$$\overline{\nabla}_U BV + \overline{\nabla}_U CV - \Phi(\nabla_U V + \nu(U, V)) = g(BU, V)\xi - \eta(V)BU - \eta(V)CU,$$

$$g(BU, V)\xi = \nabla_U BV + \nu(U, BV) - \mathcal{A}_{CV}U + \nabla^{\perp}_U CV - B(\nabla_U V) - C(\nabla_U V) - \Phi\nu(U, V).$$

As D_1 is totally umbilical foliation on $\overline{\mathcal{M}}$, then

$$g(BU, V)\xi = \nabla_U BV + g(U, BV)\mathcal{H} - \mathcal{A}_{CV}U + \nabla^{\perp}_U CV - B(\nabla_U V) - C(\nabla_U V) - g(U, V)\Phi\mathcal{H}.$$

Taking the inner product with $\Phi\mathcal{H}$ and from the fact that $\mathcal{H} \in \Gamma(CD_2 \oplus \mu)$, we obtain

$$g(\nabla^{\perp}_U CV, \Phi\mathcal{H}) = -g(U, V)\|\mathcal{H}\|^2,$$

$$g(CV, \nabla^{\perp}_U \Phi\mathcal{H}) = g(U, V)\|\mathcal{H}\|^2. \tag{5.6}$$

Now, for any $X \in \Gamma(D_1)$, we have

$$\overline{\nabla}_U \Phi\mathcal{H} = (\overline{\nabla}_U \Phi)\mathcal{H} + \Phi\overline{\nabla}_U \mathcal{H}.$$

Using the fact $\mathcal{H} \in \Gamma(CD_2 \oplus \mu)$, we get

$$-\mathcal{A}_{\Phi\mathcal{H}}U + \nabla^{\perp}_U \Phi\mathcal{H} = -B\mathcal{A}_{\mathcal{H}}U - C\mathcal{A}_{\mathcal{H}}U + \Phi\nabla^{\perp}_U \mathcal{H}. \tag{5.7}$$

Also for any $X \in \Gamma(D_1)$, we have

$$\begin{aligned} g(\nabla^{\perp}_U \mathcal{H}, CU) &= g(\overline{\nabla}_U \mathcal{H}, CU) \\ &= -g(\mathcal{H}, \overline{\nabla}_U BU) \\ &= -g(\mathcal{H}, (\overline{\nabla}_U \Phi)U) - g(\mathcal{H}, \Phi\overline{\nabla}_U U) + g(\mathcal{H}, \nu(U, BU)) \\ &= g(\Phi\mathcal{H}, \overline{\nabla}_U U) \\ g(\nabla^{\perp}_U \mathcal{H}, CU) &= g(\Phi\mathcal{H}, \mathcal{H})\|U\|^2 = 0. \end{aligned}$$

This means that

$$\nabla^{\perp}_U \mathcal{H} \in \Gamma(CD_2 \oplus \mu). \tag{5.8}$$

Now taking the inner product in (5.7) with CV for any $V \in \Gamma(D_1)$, we get

$$g(\nabla^{\perp}_U \Phi\mathcal{H}, CV) = -g(C\mathcal{A}_{\mathcal{H}}U, CV) + g(\Phi\nabla^{\perp}_U \mathcal{H}, CV).$$

Using (5.8), and from Lemma 3.2 and (5.6), we obtain

$$g(U, V)\|\mathcal{H}\|^2 = \sin^2\theta_1 g(\mathcal{A}_{\mathcal{H}}U, V), \qquad g(U, V)\|\mathcal{H}\|^2 = \sin^2\theta_1 g(U, V)\|\mathcal{H}\|^2$$
$$\implies \cos^2\theta_1 g(U, V)\|\mathcal{H}\|^2 = 0.$$

Since D_1 is a proper slant distribution which defines a totally umbilical foliation on $\overline{\mathcal{M}}$, we conclude that $\mathcal{H} = 0$, i.e., D_1 is totally geodesic. This completes the proof. □

In a similar way to the above theorem, we can conclude the following:

Theorem 5.4. *Let $\overline{\mathcal{M}}$ be a proper quasi bi-slant submanifold of $\overline{\mathcal{N}}$ and let D_2 be a proper slant distribution which defines a totally umbilical foliation on $\overline{\mathcal{M}}$. Then D_2 is totally geodesic.*

Acknowledgements. The authors are thankful to the editor and anonymous referees for their valuable suggestions towards the improvement of the paper.

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Author's address:

Rajendra Prasad
Department of Mathematics and Astronomy,
University of Lucknow,
Lucknow-226007, India.
Email: rp.manpur@rediffmail.com

Abdul Haseeb
Department of Mathematics,
Faculty of Science, Jazan University,
Jazan-2097, Kingdom of Saudi Arabia.
Email: haseeb@jazanu.edu.sa , malikhaseeb80@gmail.com

Shweta Singh
Department of Mathematics and Astronomy,
University of Lucknow,
Lucknow-226007, India.
Email: singhshweta037@gmail.com