Bounds for generalized normalized δ-Casorati curvature for submanifold in real space forms endowed with a quarter-symmetric connection

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Abstract. In this article, we obtain some optimal inequalities involving the scalar curvature and the Casorati curvature of submanifolds in a real space forms with a quarter-symmetric connection. Moreover, we prove that the equality holds if and only if the submanifolds are invariantly quasi-umbilical.

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1 Introduction

In 1993, Chen [3] initiated the theory of δ-invariants and established a sharp inequality for a submanifold into the real space form using the scalar curvature and the sectional curvature, both being intrinsic invariants, and squared mean curvature, the main extrinsic invariant. Chen [2] established simple relationships between the main intrinsic invariants and the main extrinsic invariants of a submanifold in real space forms with any codimension. Now it has become one of the most interesting research topics in differential geometry of submanifolds. Instead of concentrating on the sectional curvature with the extrinsic squared mean curvature, the Casorati curvature of a submanifold in a Riemannian manifold was considered as an extrinsic invariant defined as the normalized square of the length of the second fundamental form.

The notion of Casorati curvature extends the concept of the principal direction of a hypersurface of a Riemannian manifold. It was preferred by Casorati over the traditional Gauss curvature. Several geometers [4, 5, 8, 16, 17, 18, 19, 20, 23, 24] found geometrical meaning and the importance of the Casorati curvature. Therefore, it attracts the geometers to obtain optimal inequalities for the Casorati curvatures of submanifolds in different ambient spaces. Decu, Haesen and Verstraelen introduced

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the normalized δ-Casorati curvatures $\delta_C(n-1)$ and $\hat{\delta}_C(n-1)$ and established inequalities involving $\delta_C(n-1)$ and $\hat{\delta}_C(n-1)$ for submanifolds in real space forms [4]. Moreover, the same authors proved in [5] an inequality in which the scalar curvature is estimated from above by the normalized Casorati curvatures. Recently, Lee et al. in [11] obtained optimal inequalities for submanifolds in real space forms, endowed with a semi-symmetric metric connection. Many authors obtained the optimal inequalities for the Casorati curvatures of submanifolds in different ambient spaces [9, 10, 12, 21, 22, 30]. The idea of a semi-symmetric linear connection in a differentiable manifold was introduced by Friedmann and Schouten in [6]. Later, Hayden [7] introduced the idea of a metric connection with torsion in a Riemannian Manifold.

Yano [29] studied semi-symmetric metric connection in a Riemannian manifold. Many other geometers have used this idea of connection in different ambient spaces such as real space forms, complex space forms, Sasakian space forms and so on (see [13, 15]). The paper is structured as follows: Section 2 is devoted to preliminaries. Section 3 deals with the study of Casorati curvatures for any submanifold of n-dimension. In Section 4, we establish two sharp inequalities that relate the normalized scalar curvature with generalized normalized δ-Casorati curvature for any submanifold in a real space form with quarter-symmetric connection with some immediate consequences.

2 Preliminaries

Let $\mathcal{N}$ be a Riemannian manifold with Riemannian metric g. A linear connection $\bar{\nabla}$ on $\mathcal{N}$ is called a quarter-symmetric connection if its torsion tensor T given by

$$T(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y]$$

satisfies

$$T(X,Y) = \pi(Y)\varphi X - \pi(X)\varphi Y,$$

where π is a 1-form and $\mathcal{V}$ is a vector field such that $\pi(X) = g(X,\mathcal{V})$ and φ is a (1,1) tensor field. If $\bar{\nabla} g = 0$, then $\bar{\nabla}$ is known as quarter-symmetric metric connection and $\bar{\nabla} g \neq 0$, then $\bar{\nabla}$ is known as quarter symmetric non-metric connection. In this setting, it is shown in [25], one can easily obtain a special quarter-symmetric connection defined as

$$\bar{\nabla}_X Y = \bar{\tilde{\nabla}}_X Y + \psi_1 \pi(Y) X - \psi_2 g(X,Y)\mathcal{V}. \tag{2.1}$$

This is a general class of connection in the sense of (2.1) can be obtained as

1. when $\psi_1 = \psi_2 = 1$, then the above connection reduces to semi-symmetric metric connection.

2. when $\psi_1 = 1$ and $\psi_2 = 0$. then the above connection reduces to semi-symmetric non metric connection.

The curvature tensor with respect to $\bar{\nabla}$ is given by

$$\bar{\mathcal{R}}(X,Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z. \tag{2.2}$$

The curvature tensor $\tilde{\bar{\mathcal{R}}}$ can be defined in the same way. Let

$$\alpha(X,Y) = (\tilde{\bar{\nabla}}_X\pi)(Y) - \psi_1\pi(X)\pi(Y) + \frac{\psi_2}{2}g(X,Y)\pi(\mathcal{V}),$$

and

$$\beta(X,Y) = \frac{\pi(\mathcal{V})}{2}g(X,Y) + \pi(X)\pi(Y)$$

are $(0,2)$ tensors. Then the curvature tensor of $\mathcal{N}$ is given by [26]

$$\begin{aligned}\bar{\mathcal{R}}(X,Y,Z,W) &= \tilde{\bar{\mathcal{R}}}(X,Y,Z,W) + \psi_1\alpha(X,Z)g(Y,W) - \psi_1\alpha(Y,Z)g(X,W)\\ &\quad +\psi_2\alpha(Y,W)g(X,Z) - \psi_2\alpha(X,W)g(Y,Z)\\ &\quad +\psi_2(\psi_1-\psi_2)g(X,Z)\beta(Y,W) - \psi_2(\psi_1-\psi_2)g(Y,Z)\beta(X,W).\end{aligned} \tag{2.3}$$

For simplicity, we denote by $tr(\alpha) = \beta_1$ and $tr(\beta) = \beta_2$.

Let $\mathcal{M}$ be an m-dimensional submanifold of a Riemannian manifold $\mathcal{N}$ and ∇, $\tilde{\nabla}$ be the induced quarter symmetric-metric connection and Levi-Civita connection $\mathcal{M}$, respectively. Then the Gauss formula are given by:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad X,Y \in \Gamma(T\mathcal{M}), \tag{2.4}$$

$$\tilde{\bar{\nabla}}_X Y = \tilde{\nabla}_X Y + \tilde{h}(X,Y), \quad X,Y \in \Gamma(T\mathcal{M}). \tag{2.5}$$

Here $\tilde{h}$ is the second fundamental form given by

$$h(X,Y) = \tilde{h}(X,Y) - \psi_2 g(X,Y)\mathcal{V}^{\perp},$$

where $\mathcal{V}^{\perp}$ is the normal component of the vector field $\mathcal{V}$ on $\mathcal{M}$.
Furthermore, the equation Gauss is given by [26]

$$\begin{aligned}\bar{\mathcal{R}}(X,Y,Z,W) &= \mathcal{R}(X,Y,Z,W) - g(h(X,W),h(Y,Z)) + g(h(Y,W),h(X,Z))\\ &\quad +(\psi_1-\psi_2)g(h(Y,Z),\mathcal{V}^{\perp})g(X,W)\\ &\quad +(\psi_2-\psi_1)g(h(X,Z),\mathcal{V}^{\perp})g(Y,W).\end{aligned} \tag{2.6}$$

Let $\mathcal{N}^{n+p}$ be a real space form of constant sectional curvature c endowed with quarter-symmetric connection. The curvature tensor $\tilde{\bar{\mathcal{R}}}$ with respect to the Levi–Civita connection $\tilde{\bar{\nabla}}$ on $\mathcal{N}^{n+p}(c)$ is expressed by

$$\tilde{\bar{\mathcal{R}}}(X,Y,Z,W) = c\big\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\big\}, \tag{2.7}$$

for any $X,Y,Z,W \in \Gamma(T\mathcal{N})$.

From (2.3) and (2.7), we get

$$\begin{aligned}\bar{\mathcal{R}}(X,Y,Z,W) &= c\big\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\big\} + \psi_1\alpha(X,Z)g(Y,W)\\ &\quad -\psi_1\alpha(Y,Z)g(X,W) + \psi_2\alpha(Y,W)g(X,Z) - \psi_2\alpha(X,W)g(Y,Z)\\ &\quad +\psi_2(\psi_1-\psi_2)g(X,Z)\beta(Y,W) - \psi_2(\psi_1-\psi_2)g(Y,Z)\beta(X,W).\end{aligned} \tag{2.8}$$

3 Casorati Curvatures

In this section, we study the Casorati curvature of any submanifold $\mathcal{M}$ of dimension n in $(n+p)$-dimensional Riemannian manifold $\mathcal{N}$ of real space forms with a semi-symmetric non-non metric connection. Consider the local orthonormal tangent frame $\{E_1, ..., E_n\}$ of the tangent bundle $T\mathcal{M}$ of $\mathcal{M}$ and a local orthonormal normal frame $\{E_{n+l}, ..., E_{n+p}\}$ of the normal bundle $T^{\perp}\mathcal{M}$ of $\mathcal{M}$ in $\mathcal{N}^{n+p}$. At any $p \in \mathcal{M}$ the scalar curvature tau at that point is given by

$$\tau = \sum_{1 \leq i < j \leq k+l} R(E_i, E_j, E_j, E_i) \tag{3.1}$$

and the normalized scalar curvature ρ of $\mathcal{M}$ is defined as

$$\rho = \frac{2\tau}{n(n-1)}. \tag{3.2}$$

The mean curvature $\mathcal{H}$ of submanifold is given by

$$\mathcal{H} = \frac{1}{n} \sum_{i=1}^{n} h(E_i, E_i).$$

Conveniently, let us put

$$h_{ij}^r = g(h(E_i, E_j), E_r)$$

for any $i, j = \{1, ..., n\}$ and $r = \{n+1, ..., n+p\}$, the squared mean curvature is given by

$$||\mathcal{H}||^2 = \frac{1}{n^2} \sum_{r=n+1}^{n+p} \big\{ \sum_{i,j=1}^{n} h_{ii}^r \big\}^2 \tag{3.3}$$

and the Casorati curvature $\mathcal{C}$ is defined as the squared norm of second fundamental form h, given by

$$\mathcal{C} = \frac{1}{(n)} ||h||^2, \tag{3.4}$$

where

$$||h||^2 = \frac{1}{n} \sum_{n+1}^{n+p} \sum_{i,j=1}^{n} (h_{ij}^r)^2.$$

If we suppose that $\mathcal{L}$ is an s-dimensional subspace of $T\mathcal{M}$, $s \geq 2$ and $\{E_1, ..., E_s\}$ is an orthonormal basis of $\mathcal{L}$, then the scalar curvature of the s-plane section $\mathcal{L}$ is given as

$$\tau = \sum_{1 \leq i < j \leq s} \kappa(e_i \wedge e_j)$$

and the Casorati curvature $\mathcal{C}$ of subspace $\mathcal{L}$ is as follows

$$\mathcal{C}(\mathcal{L}) = \frac{1}{s}\sum_{r=n+1}^{n+p}\sum_{i,j=1}^{s}(h_{ij}^r)^2.$$

A point $p \in \mathcal{M}$ is said to be an invariantly quasi-umbilical point if there exist $n+p-n-1$ mutually orthogonal unit vectors $\xi_{n+1},...,\xi_{n+p}$, such that the shape operator with respect to all the directions ξ_α have an eigenvalue of multiplicity $n-1$ and that for each ξ_α the distinguished eigen direction is the same. The submanifold is said to be an invariantly quasi-umbilical submanifold if each of its points is an invariantly quasi-umbilical point.

The normalized δ-Casorati curvature $\delta_C(n-1)$ and $\tilde{\delta_C}(n-1)$ are defined as

$$[\delta_C(n-1]_x = \frac{1}{2}\mathcal{C}_x + \frac{n+1}{2n} inf\{\mathcal{C}(\mathcal{L})|\mathcal{L} : a\ hyperplane\ of\ T_x\mathcal{M}\}, \tag{3.5}$$

and

$$[\hat{\delta_C}(n-1]_x = \frac{1}{2}\mathcal{C}_x + \frac{2n-1}{2n} sup\{\mathcal{C}(\mathcal{L})|\mathcal{L} : a\ hyperplane\ of\ T_x\mathcal{M}\}. \tag{3.6}$$

For a positive real number $t \neq n(n-1)$, put

$$b(t) = \frac{1}{nt}(n-1)(n+t)(n^2-n-t), \tag{3.7}$$

then the generalized normalized δ-casortai curvatures $\delta_C(t;n-1)$ and $\tilde{\delta}_C(t;n-1)$ are given as

$$[\delta_C(t;n-1)]_x = t\mathcal{C}_x + b(t) inf\{\mathcal{C}(\mathcal{L})|\mathcal{L} : a\ hyperplane\ of\ T_x\mathcal{M}\},$$

if $0 < t < n^2 - n$, and

$$[\hat{\delta}_C(t;n-1)]_x = t\mathcal{C}_x + b(t) sup\{\mathcal{C}(\mathcal{L})|\mathcal{L} : a\ hyperplane\ of\ T_x\mathcal{M}\},$$

if $t > n^2 - n$.

For any orthonormal basis $\{E_1,...,E_n\}$ of the tangent space $T_x\mathcal{M}^n$, the scalar curvature τ at x is given by

$$\tau(x) = \sum_{1\leq i<j\leq n} K(E_i \wedge E_j). \tag{3.8}$$

Finally, in this section we recall the following well-known algebraic lemma for later use

Lemma 3.1. *[1] let $n \geq 2$ and a_1 and $a_1,...,a_n,b$ real numbers such that*

$$\Big(\sum_{i=1}^n a_i\Big)^2 = (n-1)\Big(\sum_{i=1}^n a_i^2 + b\Big).$$

Then $2a_1a_2 \geq b$, with equality holding if and only if

$$a_1 + a_2 = a_3 = ... = a_n.$$

4 Main Inequalities

Theorem 4.1. *let $\mathcal{M}^n$, $n \geq 3$, be an n-dimensional submanifold of $n+p$-dimensional real space form $\mathcal{N}^{n+p}$ of constant sectional curvature c endowed with a quarter-symmetric. Then for the generalized normalized δ-Casorati curvatures, we have the following optimal relationships:*

(i) For any real number t such that $0 < t < n(n-1)$:

$$\rho \leq \frac{\delta_C(t;n-1)}{n(n-1)} + c - \big\{(\psi_1+\psi_2)\frac{\beta_1}{n} + \psi_2(\psi_1-\psi_2)\frac{\beta_2}{n} - (\psi_2-\psi_1)\pi(\mathcal{H})\big\}. \tag{4.1}$$

(ii) For any real number $t > n(n-1)$:

$$\rho \leq \frac{\hat{\delta}_C(t;n-1)}{n(n-1)} + c - \big\{(\psi_1+\psi_2)\frac{\beta_1}{n} + \psi_2(\psi_1-\psi_2)\frac{\beta_2}{n} - (\psi_2-\psi_1)\pi(\mathcal{H})\big\}. \tag{4.2}$$

Moreover, the equality holds in (4.1) and (4.20) if and only if $\mathcal{M}$ is invariantly quasi-umbilical with trivial normal connection in $\mathcal{N}$, such that with respect to suitable tangent orthonormal frame $\{E_1, ..., E_n\}$ and normal orthonormal frame $\{E_{n+1}, ..., E_{n+p}\}$, the shape operator $\mathcal{S}_r \equiv \mathcal{S}_{e_r}$, $r \in \{n+1, ..., n+p\}$, take the following form

$$\mathcal{S}_{n+1} = \begin{pmatrix} b & 0 & 0 & . & . & . & 0 & 0 \\ 0 & b & 0 & . & . & . & 0 & 0 \\ 0 & 0 & b & . & . & . & 0 & 0 \\ . & . & . & . & & & . & . \\ . & . & . & & . & & . & . \\ . & . & . & & & . & . & . \\ 0 & 0 & 0 & . & . & . & b & . \\ 0 & 0 & 0 & . & . & . & 0 & \frac{n(n-1)}{t}b \end{pmatrix} \quad \mathcal{S}_{n+2} = ... = \mathcal{S}_{n+p} = 0. \tag{4.3}$$

Proof. Let $x \in \mathcal{M}$ and $\{E_1, ..., E_n\}$ and $\{E_{n+1}, ..., E_{n+p}\}$ be the orthonormal basis of $T_x\mathcal{M}$ and $T_x^\perp\mathcal{M}$, respectively at any point $x \in \mathcal{M}$. Putting $X = W = E_i$ and $Y = Z = E_j$ into (2.8) with (2.6) and considering $i \neq j$,Then we obtain

$$\begin{aligned} \sum_{i,j=1}^{n} R(E_i,E_j,E_j,E_i) = & \sum_{i,j=1}^{n} \Big\{ c\big\{g(E_i,E_i)g(E_j,E_j) - g(E_i,E_j)g(E_j,E_i)\big\} \\ & +\psi_1\alpha(E_i,E_j)g(E_j,E_i) - \psi_1\alpha(E_j,E_j)g(E_i,E_i) \\ & +\psi_2 g(E_i,E_j)\alpha(E_j,E_i) - \psi_2 g(E_j,E_j)\alpha(E_i,E_i) \\ & +\psi_2(\psi_1-\psi_2)g(E_i,E_j)\beta(E_j,E_i) - \psi_2(\psi_1-\psi_2)g(E_j,E_j)\beta(E_i,E_i), \\ & -(\psi_1-\psi_2)g(h(E_j,E_j),\mathcal{V}^\perp)g(E_i,E_i) \\ & -(\psi_2-\psi_1)g(h(E_i,E_j),\mathcal{V}^\perp)g(E_j,E_i) \\ & +g(h(E_j,E_j),h(E_i,E_i)) - g(h(E_i,E_j),h(E_j,E_i) \Big\}. \end{aligned} \tag{4.4}$$

By summation after $i \leq i, j \leq n$, it follows that

$$\begin{aligned} 2\tau(x) &= n^2\mathcal{H} - n\mathcal{C} + n(n-1)c + (\psi_1+\psi_2)(1-n)\beta_1 + \psi_2(\psi_1-\psi_2)(1-n)\beta_1 \\ &\quad +(\psi_2-\psi_1)n(n-1)\pi(\mathcal{H}). \end{aligned} \tag{4.5}$$

We define the following function , denoted by $\mathcal{Q}$, which is a quadratic polynomial in the components of second fundamental form

$$\begin{aligned} \mathcal{Q} &= t\mathcal{C} + b(t)\mathcal{C}(\mathcal{L}) - 2\tau(x) + n(n-1)c + (\psi_1+\psi_2)(1-n)\beta_1 + \psi_2(\psi_1 \\ &\quad -\psi_2)(1-n)\beta_1 + (\psi_2-\psi_1)n(n-1)\pi(\mathcal{H}), \end{aligned} \tag{4.6}$$

where $\mathcal{L}$ is the hyperplane of $T_x\mathcal{M}$. Without loss of generality, we suppose that $\mathcal{L}$ is spanned by $\{E_1, ..., E_{n-1}\}$, it follow 4.6 that

$$\mathcal{Q} = \frac{(n+t)}{n}\sum_{r=n+1}^{n+p}\sum_{i,j=1}^{n}(h_{ij}^r)^2 + \frac{b(t)}{(n-1)}\sum_{r=n+1}^{n+p}\sum_{i,j=1}^{n-1}(h_{ij}^r)^2 - \sum_{r=n+1}^{n+p}\big(\sum_{i=1}^{n}(h_{ii}^r)^2,$$

which can be easily written as

$$\begin{aligned} \mathcal{Q} &= \sum_{r=n+1}^{n+p}\sum_{i,j=1}^{n-1}\left[\left(\frac{n+t}{n} + \frac{b(t)}{n-1}\right)(h_{ii}^r)^2 + 2\frac{n+t}{n}(h_{in}^r)^2\right] \\ &\quad + \sum_{r=n+1}^{n+p}\left[2\big(\frac{n+t}{n} + \frac{b(t)}{n-1} - 1\big)\sum_{i<j}^{n-1}(h_{ij}^r)^2 - 2\sum_{i<j}^{n} h_{ii}^r h_{jj}^r \right. \\ &\quad \left. + \frac{t}{n}(h_{nn}^r)^2\right]. \end{aligned} \tag{4.7}$$

From (4.7), we can see the critical points

$$h^c = \{h_{11}^{n+1}, h_{12}^{n+1}, ..., h_{nn}^{n+1}, ..., h_{11}^{n+p}, ..., h_{nn}^{n+p}\}$$

of $\mathcal{Q}$ are the solutions of the followings system of homogeneous equations:

$$\begin{cases} \frac{\partial \mathcal{Q}}{\partial h_{ii}} = 2\big(\frac{n+t}{n} + \frac{b(t)}{n-1} - 1\big)h_{ii}^r - 2\sum_{q=1}^{n} h_{qq}^r = 0 \\ \frac{\partial \mathcal{Q}}{\partial h_{nn}} = \frac{2t}{n}h_{nn}^r - 2\sum_{q=1}^{n-1} h_{qq}^r = 0 \\ \frac{\partial \mathcal{Q}}{\partial h_{ij}} = 4\big(\frac{n+t}{n} + \frac{b(t)}{n-1}\big)h_{ij}^r \\ \frac{\partial \mathcal{Q}}{\partial h_{in}} = 4\big(\frac{n+t}{n}\big)h_{in}^r, \end{cases} \tag{4.8}$$

where $i, j = \{1, 2, ..., n-1\}$, $i \neq j$ and $r \in \{n+1, ..., n+p\}$. Hence, every solution h^c and $h_{ij}^r = 0$ for $i \neq j$ and the corresponding determinant to the first two equations of the above system is zero. Moreover, the Hessian matrix of $\mathcal{Q}$ is of the following form

$$H(\mathcal{Q}) = \begin{pmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & H_3 \end{pmatrix}.$$

$$H_1 = \begin{pmatrix} 2\left(\frac{n+t}{n}+\frac{b(t)}{n-1}\right)-2 & -2 & \cdots & -2 & -2 \\ -2 & 2\left(\frac{n+t}{n}+\frac{b(t)}{n-1}\right)-2 & \cdots & -2 & -2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -2 & -2 & \cdots & 2\left(\frac{n+t}{n}+\frac{b(t)}{n-1}\right)-2 & -2 \\ -2 & -2 & \cdots & -2 & \frac{2t}{n} \end{pmatrix} \tag{4.9}$$

H_2 and H_3 are the diagonal matrices and 0 is the null matrix of the respective dimensions, given by

$$\begin{aligned} H_2 &= diag\Big[4(\frac{n+t}{n}+\frac{b(t)}{n-1}-1), 4(\frac{n+t}{n}+\frac{b(t)}{n-1}-1), \dots, \\ &\quad 4(\frac{n+t}{n}+\frac{b(t)}{n-1}-1)\Big], \end{aligned}$$

and

$$H_3 = diag\Big[4\frac{(n+t)}{n}, 4\frac{(n+t)}{n}, \dots, 4\frac{(n+t)}{n}\Big].$$

Hence, we find that $\mathcal{H}(\mathcal{Q})$ has the following eigenvalues:
$\lambda_{11}=0$, $\lambda_{22}=2(\frac{2t}{n}+\frac{b(t)}{n-1})$, $\lambda_{33}=.....=\lambda_{nn}=2(\frac{n+t}{n}+\frac{b(t)}{n-1})$, $\lambda_{ij}=4(\frac{n+t}{n}+\frac{b(t)}{n-1})$, $\lambda_{in}=4(\frac{n+t}{n})$, $\forall i,j\in\{1,2,...,n-1\}$, $i\neq j$.

Thus, $\mathcal{Q}$ is parabolic and reaches at minimum $\mathcal{Q}(h^c)=0$ for some solution h^c of the system (4.8). Therefore, $\mathcal{Q}\geq 0$ and hence we have

$$\begin{aligned} 2\tau(x) &\leq t\mathcal{C}+b(t)\mathcal{C}(\mathcal{L})+n(n-1)c+(\psi_1+\psi_2)(1-n)\beta_1+\psi_2(\psi_1-\psi_2)(1-n)\beta_1 \\ &\quad +(\psi_2-\psi_1)n(n-1)\pi(\mathcal{H}), \end{aligned} \tag{4.10}$$

whereby, we obtain

$$\begin{aligned} \rho &\leq \frac{t}{n(n-1)}\mathcal{C}+\frac{b(t)}{n(n-1)}\mathcal{C}(\mathcal{L})+c-\frac{1}{n}(\psi_1+\psi_2)\beta_1-\frac{1}{n}\psi_2(\psi_1-\psi_2)\beta_1 \\ &\quad +(\psi_2-\psi_1)\pi(\mathcal{H}), \end{aligned} \tag{4.11}$$

For each tangent hyperplane $\mathcal{L}$ of $\mathcal{M}$. If we take infimum over all tangent hyperplanes $\mathcal{L}$, the result trivially follows. Moreover, the equality sign holds if and only if

$$h^r_{ij}=0, \quad \forall i,j\in\{1,...,n\}, \quad i\neq j, \quad r\in\{n+1,...,n+p\} \tag{4.12}$$

and

$$h^r_{nn}=2h^r_{11}=...=2h^r_{n-1n-1}, \quad \forall r\in\{n+1,...,n+p\}. \tag{4.13}$$

From (4.12) and (4.13), we conclude that the equality sign holds if and only if the submanifold M is invariantly quasi-umbilical submanifold with normal connection N^{n+p}

such that shape operator takes the (4.21) with respect to the orthonormal tangent and orthonormal normal frames.

In the same manner, we can establish the inequality (4.20) as a second part of the theorem. □

Corollary 4.2. *let $\mathcal{M}^n$, $n \geq 3$, be an n-dimensional submanifold of $n+p$-dimensional real space form $\mathcal{N}^{n+p}$ of constant sectional curvature c endowed with a quarter-symmetric. Then, we have*

(i) The normalized δ-Casorati curvature $\delta_C(n-1)$

$$\rho \leq \delta_C(n-1) + c - \big\{(\psi_1+\psi_2)\frac{\beta_1}{n} + \psi_2(\psi_1-\psi_2)\frac{\beta_2}{n} - (\psi_2-\psi_1)\pi(\mathcal{H})\big\}. \tag{4.14}$$

Moreover, the equality holds in if and only if $\mathcal{M}$ is invariantly quasi-umbilical with trivial normal connection in $\mathcal{N}$, such that with respect to suitable tangent orthonormal frame $\{E_1, ..., E_n\}$ and normal orthonormal frame $\{E_{n+1}, ..., E_{n+p}\}$, the shape operator $\mathcal{S}_r \equiv \mathcal{S}_{e_r}$, $r \in \{n+1, ..., n+p\}$, take the following form

$$\mathcal{S}_{n+1} = \begin{pmatrix} b & 0 & 0 & . & . & . & 0 & 0 \\ 0 & b & 0 & . & . & . & 0 & 0 \\ 0 & 0 & b & . & . & . & 0 & 0 \\ . & . & . & . & & & . & . \\ . & . & . & & . & & . & . \\ . & . & . & & & . & . & . \\ 0 & 0 & 0 & . & . & . & b & . \\ 0 & 0 & 0 & . & . & . & 0 & 2b \end{pmatrix}, \quad \mathcal{S}_{n+2} = ... = \mathcal{S}_{n+p} = 0. \tag{4.15}$$

(ii) The normalized δ-Casorati curvature $\hat{\delta}_C(n-1)$

$$\rho \leq \hat{\delta}_C(n-1) + c - \big\{(\psi_1+\psi_2)\frac{\beta_1}{n} + \psi_2(\psi_1-\psi_2)\frac{\beta_2}{n} - (\psi_2-\psi_1)\pi(\mathcal{H})\big\}. \tag{4.16}$$

Moreover, the equality holds if and only if $\mathcal{M}$ is invariantly quasi-umbilical with trivial normal connection in $\mathcal{N}$, such that with respect to suitable tangent orthonormal frame $\{E_1, ..., E_n\}$ and normal orthonormal frame $\{E_{n+1}, ..., E_{n+p}\}$, the shape operator $\mathcal{S}_r \equiv \mathcal{S}_{e_r}$, $r \in \{n+1, ..., n+p\}$, take the following form

$$\mathcal{S}_{n+1} = \begin{pmatrix} 2b & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 2b & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 2b & . & . & . & 0 & 0 \\ . & . & . & . & & & . & . \\ . & . & . & & . & & . & . \\ . & . & . & & & . & . & . \\ 0 & 0 & 0 & . & . & . & 2b & . \\ 0 & 0 & 0 & . & . & . & 0 & b \end{pmatrix}, \quad \mathcal{S}_{n+2} = ... = \mathcal{S}_{n+p} = 0. \tag{4.17}$$

Proof. (i) One can easily see that

$$[\frac{\delta_C(n(n-1);n-1)}{2}]_x = n(n-1)[\delta_C(n-1)], \tag{4.18}$$

ar any point $x \in \mathcal{M}$. Therefore, putting $t = \frac{n(n-1)}{2}$ in (4.1) and taking into account (4.18), we have our assertion. Similarly, we can prove (ii). □

For semi-symmetric metric connection $\psi_1 = \psi_2 = 1$, we have the result in [11]. And for semi-symmetric non-metric connection $\psi_1 = 1$ and $\psi_2 = 0$, we have the following:

Theorem 4.3. *let $\mathcal{M}^n$, $n \geq 3$, be an n-dimensional submanifold of an $n+p$-dimensional real space form $\mathcal{N}^{n+p}$ of constant sectional curvature c endowed with a quarter-symmetric. Then for the generalized normalized δ-Casorati curvature, we have the following optimal relationships:*

(i) For any real number t such that $0 < t < n(n-1)$:

$$\rho \leq \frac{\delta_C(t;n-1)}{n(n-1)} + c - \frac{\beta_1}{n} - \pi(\mathcal{H}). \tag{4.19}$$

(ii) For any real number $t > n(n-1)$:

$$\rho \leq \frac{\hat{\delta}_C(t;n-1)}{n(n-1)} + c - \frac{\beta_1}{m} - \pi(\mathcal{H}). \tag{4.20}$$

Moreover, the equality holds in (4.1) and (4.20) if and only if $\mathcal{M}$ is invariantly quasi-umbilical with trivial normal connection in $\mathcal{N}$, such that with respect to suitable tangent orthonormal frame $\{E_1, ..., E_n\}$ and normal orthonormal frame $\{E_{n+1}, ..., E_{n+p}\}$, the shape operator $\mathcal{S}_r \equiv \mathcal{S}_{e_r}$, $r \in \{n+1, ..., n+p\}$, take the following form

$$\mathcal{S}_{n+1} = \begin{pmatrix} b & 0 & 0 & . & . & . & 0 & 0 \\ 0 & b & 0 & . & . & . & 0 & 0 \\ 0 & 0 & b & . & . & . & 0 & 0 \\ . & . & . & . & & & . & . \\ . & . & . & & . & & . & . \\ . & . & . & & & . & . & . \\ 0 & 0 & 0 & . & . & . & b & . \\ 0 & 0 & 0 & . & . & . & 0 & \frac{n(n-1)}{t}b \end{pmatrix}, \quad \mathcal{S}_{n+2} = ... = \mathcal{S}_{n+p} = 0. \tag{4.21}$$

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