Reidemeister-Franz torsion of compact orientable surfaces via pants decomposition

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Abstract. Let $\Sigma_{g,n}$ denote the compact orientable surface with genus $g \geq 2$ and boundary disjoint union of n circles. By using a particular pants decomposition of $\Sigma_{g,n}$, we obtain a formula that computes the Reidemeister-Franz torsion of $\Sigma_{g,n}$ in terms of the Reidemeister-Franz torsions of pairs of pants.

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1 Introduction

The Reidemeister-Franz torsion (or R-torsion) was introduced by Reidemeister to classify 3 dimensional lens spaces [5]. This invariant was later generalized by Franz to other dimensions [10] and shown to be a topological invariant by Kirby-Siebenmann [2]. The R-torsion is also an invariant of the basis of the homology of a manifold [3]. Moreover, for compact orientable Riemannian manifolds the R-torsion is equal to the analytic torsion [1].

Using the combinatorial definition of the Reidemeister torsion, Witten computed the volume of the moduli space $\mathcal{M}$ of gauge equivalence classes of flat connections on a compact Riemann surface [9]. The combinatorial torsion is equivalent to the Ray-Singer analytic torsion [1]. In the quantum field theory, one important ingredient was the ability to compute by decomposing a surface into elementary pieces. The pair of pants is a $(1+1)$-dimensional bordism, which corresponds to a product or co-product (depending on its orientation) in a 2-dimensional TQFT. Witten established a formula to compute the Ray-Singer analytic torsion of a pair of pants by using its cell decomposition. He also gave a cutting formula for orientable closed surface $\Sigma_{g,0}$ by decomposing an orientable surface $\Sigma_{g,0}$ of genus g into $2g-2$ pairs of pants.

The present paper provides a formula to compute the Reidemeister-Franz torsion of a pair of pants in terms of the determinant of the period matrix of the Poincaré dual basis of $H^1(\Sigma_{2,0})$. Then it expresses the Reidemeister-Franz torsion of orientable

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compact surface $\Sigma_{g,n}$ as the product of the Reidemeister-Franz torsions of pairs of pants.

For a manifold M and an integer η, we denote by $\mathbf{h}_\eta^M$ the basis of the homology $H_\eta(M) = H_\eta(M;\mathbb{R})$. Note that $\Sigma_{2,0}$ is the double of a pair of pants $\Sigma_{0,3}$ as in Figure 1. Let $\Delta_{0,2}(\Sigma_{2,0})$ be the matrix of the intersection pairing of $\Sigma_{2,0}$ in the bases $\mathbf{h}_0^{\Sigma_{2,0}}$, $\mathbf{h}_2^{\Sigma_{2,0}}$, and $\mathbf{h}^1_{\Sigma_{2,0}} = \{\omega_j\}_1^4$ denote the Poincaré dual basis of $H^1(\Sigma_{2,0})$ corresponding to $\mathbf{h}_1^{\Sigma_{2,0}}$. We first prove the following theorem for the R-torsion of the pair of pants $\Sigma_{0,3}$.

Theorem 1.1. *For a given basis $\mathbf{h}_i^{\Sigma_{0,3}}$, $i \in \{0,1\}$, there is a basis $\mathbf{h}_\eta^{\Sigma_{2,0}}$, $\eta \in \{0,1,2\}$ such that the following formula holds*

$$|\mathbb{T}(\Sigma_{0,3}, \{\mathbf{h}_i^{\Sigma_{0,3}}\}_0^1)| = \sqrt{\left|\frac{\det \Delta_{0,2}(\Sigma_{2,0})}{\det \wp(\mathbf{h}^1_{\Sigma_{2,0}}, \Gamma)}\right|},$$

where $\Gamma = \{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4\}$ is the canonical basis for $H_1(\Sigma_{2,0})$, i.e. $i \in \{1,2\}$, Γ_i intersects Γ_{i+2} once positively and does not intersect others, and $\wp(\mathbf{h}^1_{\Sigma_{2,0}}, \Gamma) = [\int_{\Gamma_i} \omega_j]$ is the period matrix of $\mathbf{h}^1_{\Sigma_{2,0}}$ with respect to the basis Γ.

By using the pants decomposition of $\Sigma_{g,n}$ as in Figure 2, we prove the following theorem.

Theorem 1.2. *Let $\mathbf{h}_\eta^{\Sigma_{g,n}}$ be a given basis for $\eta \in \{0,1\}$. Then there exists a basis $\mathbf{h}_\eta^{\Sigma_{0,3}^\nu}$ for each $\nu \in \{1, \ldots, 2g-2+n\}$ such that*

$$|\mathbb{T}(\Sigma_{g,n}, \{\mathbf{h}_\eta^{\Sigma_{g,n}}\}_0^1)| = \prod_{\nu=1}^{2g-2+n} |\mathbb{T}(\Sigma_{0,3}^\nu, \{\mathbf{h}_\eta^{\Sigma_{0,3}^\nu}\}_0^1)|,$$

where $\Sigma_{0,3}^\nu$ is the pair of pants in the decomposition labelled by ν.

2 R-torsion of a general chain complex

Let $C_* = (0 \to C_n \stackrel{\partial_n}{\to} C_{n-1} \to \cdots \to C_1 \stackrel{\partial_1}{\to} C_0 \to 0)$ be a chain complex of finite dimensional vector spaces over $\mathbb{R}$. Let $B_p(C_*) = \mathrm{Im}\partial_{p+1}$, $Z_p(C_*) = \mathrm{Ker}\partial_p$, and $H_p(C_*) = Z_p(C_*)/B_p(C_*)$ denote the p-th homology of the chain complex C_* for $p \in \{0, \ldots, n\}$. Then we have the following short exact sequences

$$0 \to Z_p(C_*) \stackrel{\mathrm{i}}{\to} C_p(C_*) \stackrel{\partial_p}{\to} B_{p-1}(C_*) \to 0, \tag{2.1}$$

$$0 \to B_p(C_*) \stackrel{\mathrm{i}}{\to} Z_p(C_*) \stackrel{\varphi_p}{\to} H_p(C_*) \to 0. \tag{2.2}$$

Here, i and φ_p are the inclusion and the natural projection, respectively. If we apply the Splitting Lemma to the above short exact sequences, then $C_p(C_*)$ can be expressed as the following direct sum

$$B_p(C_*) \oplus \ell_p(H_p(C_*)) \oplus s_p(B_{p-1}(C_*)).$$

Let $\mathbf{c_p}$, $\mathbf{b_p}$, and $\mathbf{h_p}$ be respectively bases of $C_p(C_*)$, $B_p(C_*)$, and $H_p(C_*)$. Then we obtain a new basis $\mathbf{b}_p \sqcup \ell_p(\mathbf{h}_p) \sqcup s_p(\mathbf{b}_{p-1})$ for $C_p(C_*)$.

Definition 2.1. The R-torsion of C_* with respect to bases $\{\mathbf{c}_p\}_0^n$, $\{\mathbf{h}_p\}_0^n$ is defined by

$$\mathbb{T}\left(C_*, \{\mathbf{c}_p\}_0^n, \{\mathbf{h}_p\}_0^n\right) = \prod_{p=0}^{n} \left[\mathbf{b}_p \sqcup \ell_p(\mathbf{h}_p) \sqcup s_p(\mathbf{b}_{p-1}), \mathbf{c}_p\right]^{(-1)^{(p+1)}}.$$

Here, $[\mathbf{b}_p \sqcup \ell_p(\mathbf{h}_p) \sqcup s_p(\mathbf{b}_{p-1}), \mathbf{c}_p]$ is the determinant of the change-base-matrix from basis $\mathbf{c}_p$ to $\mathbf{b}_p \sqcup \ell_p(\mathbf{h}_p) \sqcup s_p(\mathbf{b}_{p-1})$ of $C_p(C_*)$.

The R-torsion of a general chain complex C_* is an element of the dual of the vector space

$$\bigotimes_{p=0}^{n} (\det H_p(C_*))^{(-1)^p},$$

see [9, pp.185] and [6, Thm. 2.0.6].

For a smooth m-manifold M with a cell decomposition K, there is a chain complex

$$C_*(K) = (0 \to C_m(K) \overset{\partial_m}{\to} C_{m-1}(K) \to \cdots \to C_1(K) \overset{\partial_1}{\to} C_0(K) \to 0),$$

where ∂_i is the usual boundary operator. The R-torsion of M is defined as the R-torsion of its cellular chain complex $C_*(K)$ in the bases $\{\mathbf{c}_i\}_0^m$ and $\{\mathbf{h}_i\}_0^m$. Here, $\mathbf{c}_i$ is the geometric basis for the i-cells $C_i(K)$, $i \in \{0, \ldots, m\}$. By [6, Lem. 2.0.5], the R-torsion of M does not depend on the cell decomposition K. Thus, we write $\mathbb{T}(M, \{\mathbf{h}_i\}_0^m)$ instead of $\mathbb{T}(C_*(K), \{\mathbf{c}_i\}_0^m, \{\mathbf{h}_i\}_0^m)$. For details we refer to [6, 7, 8].

Corollary 2.1. *Let $Y = \mathbb{S}^1 \times [-\epsilon, +\epsilon]$ be a cylinder with boundary circles $\mathbb{S}^1 \times \{-\epsilon\}$ and $\mathbb{S}^1 \times \{+\epsilon\}$, where $\epsilon > 0$. Let $\mathbf{h}_i$ be a basis of $H_i(Y)$ for $i \in \{0, 1\}$. By Künneth formula, we have the isomorphisms:*

$$C_i(Y) \overset{\varphi_i}{\cong} C_i(\mathbb{S}^1)$$

$$H_i(Y) \overset{[\varphi_i]}{\cong} H_i(\mathbb{S}^1).$$

Then [7, Thm. 3.5] gives the following result

$$\left|\mathbb{T}\left(Y, \{\mathbf{h}_0, \mathbf{h}_1\}\right)\right| = \left|\mathbb{T}\left(\mathbb{S}^1, \{[\varphi_0](\mathbf{h}_0), [\varphi_1](\mathbf{h}_1)\}\right)\right| = 1.$$

3 Proofs of main results

For any manifold M, let $C_*(M)$ denote the associated cellular chain complex. Moreover, 0 denotes the trivial vector space.

Proof of Theorem 1.1. Note that $\Sigma_{2,0}$ is the double of $\Sigma_{0,3}$ (see, Figure 1). Let $\mathcal{B}$ be the intersection of the pairs of pants in $\Sigma_{2,0}$, so $\mathcal{B}$ is homeomorphic to the disjoint union of three circles, $\mathbb{S}_1 \amalg \mathbb{S}_2 \amalg \mathbb{S}_3$. Then there is the natural short exact sequence of the chain complexes

$$0 \to C_*(\mathcal{B}) \to C_*(\Sigma_{0,3}) \oplus C_*(\Sigma_{0,3}) \to C_*(\Sigma_{2,0}) \to 0. \tag{3.1}$$

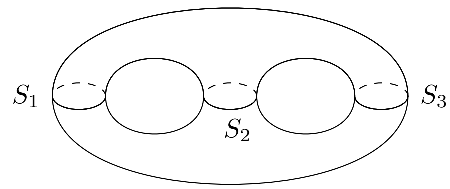

Figure 1: Double of the pair of pants $\Sigma_{0,3}$.

Associated with (3.1), we have the following Mayer-Vietoris sequence

$$\begin{aligned}\mathcal{H}_* : 0 \stackrel{\alpha}{\to} H_2(\Sigma_{2,0}) \stackrel{f}{\to} H_1(\mathcal{B}) \stackrel{g}{\to} H_1(\Sigma_{0,3}) \oplus H_1(\Sigma_{0,3}) \stackrel{h}{\to} H_1(\Sigma_{2,0}) \\ \stackrel{i}{\to} H_0(\mathcal{B}) \stackrel{j}{\to} H_0(\Sigma_{0,3}) \oplus H_0(\Sigma_{0,3}) \stackrel{k}{\to} H_0(\Sigma_{2,0}) \stackrel{\ell}{\to} 0.\end{aligned} \tag{3.2}$$

Let us denote by $C_p(\mathcal{H}_*)$ the vector spaces in (3.2) for $p \in \{0, \ldots, 6\}$ and consider the short exact sequences (2.1) and (2.2) for $\mathcal{H}_*$. Let us take the isomorphism $s_p : B_{p-1}(\mathcal{H}_*) \to s_p(B_{p-1}(\mathcal{H}_*))$ obtained by the First Isomorphism Theorem as a section of $C_p(\mathcal{H}_*) \to B_{p-1}(\mathcal{H}_*)$ for each p. By the exactness of $\mathcal{H}_*$, we get $Z_p(\mathcal{H}_*) = B_p(\mathcal{H}_*)$. Applying the Splitting Lemma to (2.2), we have

$$C_p(\mathcal{H}_*) = B_p(\mathcal{H}_*) \oplus s_p(B_{p-1}(\mathcal{H}_*)). \tag{3.3}$$

Then the R-torsion of $\mathcal{H}_*$ with respect to basis $\{\mathbf{h}_p\}_0^n$ is given as follows

$$\mathbb{T}\left(\mathcal{H}_*, \{\mathbf{h}_p\}_0^n, \{0\}_0^n\right) = \prod_{p=0}^{n} [\mathbf{h}'_p, \mathbf{h}_p]^{(-1)^{(p+1)}},$$

where $\mathbf{h}'_p = \mathbf{b}_p \sqcup s_p(\mathbf{b}_{p-1})$ for each p. In [3], Milnor proved that the R-torsion does not depend on bases $\mathbf{b}_p$ and sections s_p, ℓ_p. Therefore, we will choose a suitable bases $\mathbf{b}_p$ and sections s_p so that $\mathbb{T}(\mathcal{H}_*, \{\mathbf{h}_p\}_0^n, \{0\}_0^n) = 1$.

Let us consider the space $C_0(\mathcal{H}_*) = H_0(\Sigma_{2,0})$ in (3.3). Then $\mathrm{Im}(\ell) = 0$ yields

$$C_0(\mathcal{H}_*) = \mathrm{Im}(k) \oplus s_0(\mathrm{Im}(\ell)) = \mathrm{Im}(k). \tag{3.4}$$

Since $\{(\mathbf{h}_0^{\Sigma_{0,3}}, 0), (0, \mathbf{h}_0^{\Sigma_{0,3}})\}$ is the given basis of $H_0(\Sigma_{0,3}) \oplus H_0(\Sigma_{0,3})$,

$$\{a_{11} k(\mathbf{h}_0^{\Sigma_{0,3}}, 0) + a_{12} k(0, \mathbf{h}_0^{\Sigma_{0,3}})\}$$

can be taken as the basis $\mathbf{h}^{\mathrm{Im}(k)}$ of $\mathrm{Im}(k)$, where (a_{11}, a_{12}) is a non-zero vector. By (3.4), $\mathbf{h}^{\mathrm{Im}(k)}$ becomes the obtained basis $\mathbf{h}'_0$ of $C_0(\mathcal{H}_*)$. If we take the initial basis $\mathbf{h}_0$ (namely, $\mathbf{h}_0^{\Sigma_{2,0}}$) of $C_0(\mathcal{H}_*)$ as $\mathbf{h}'_0$, then

$$[\mathbf{h}'_0, \mathbf{h}_0] = 1. \tag{3.5}$$

If we use (3.3) for $C_1(\mathcal{H}_*) = H_0(\Sigma_{0,3}) \oplus H_0(\Sigma_{0,3})$, then we get

$$C_1(\mathcal{H}_*) = \mathrm{Im}(j) \oplus s_1(\mathrm{Im}(k)). \tag{3.6}$$

Note that $\{(\mathbf{h}_0^{\Sigma_{0,3}},0),(0,\mathbf{h}_0^{\Sigma_{0,3}})\}$ is the given basis $\mathbf{h}_1$ of $C_1(\mathcal{H}_*)$. Since $\mathrm{Im}(j)$ is a 1-dimensional subspace of 2-dimensional space $C_1(\mathcal{H}_*)$, there is a non-zero vector (a_{21},a_{22}) such that $\{a_{21}(\mathbf{h}_0^{\Sigma_{0,3}},0)+a_{22}(0,\mathbf{h}_0^{\Sigma_{0,3}})\}$ is a basis of $\mathrm{Im}(j)$. In the previous step, the basis of $\mathrm{Im}(k)$ was chosen as $\mathbf{h}^{\mathrm{Im}(k)}$ so

$$s_1(\mathbf{h}^{\mathrm{Im}(k)})=a_{11}(\mathbf{h}_0^{\Sigma_{0,3}},0)+a_{12}(0,\mathbf{h}_0^{\Sigma_{0,3}}).$$

Then we obtain a non-singular 2×2 matrix $A=[a_{ij}]$ with entries in $\mathbb{R}$. Let us choose the basis of $\mathrm{Im}(j)$ as

$$\mathbf{h}^{\mathrm{Im}(j)}=\{-(\det A)^{-1}[a_{21}(\mathbf{h}_0^{\Sigma_{0,3}},0)+a_{22}(0,\mathbf{h}_0^{\Sigma_{0,3}})]\}.$$

By (3.6), $\{\mathbf{h}^{\mathrm{Im}(j)},s_1(\mathbf{h}^{\mathrm{Im}(k)})\}$ becomes the obtained basis $\mathbf{h}_1'$ of $C_1(\mathcal{H}_*)$. Hence, we get

$$[\mathbf{h}_1',\mathbf{h}_1]=1. \tag{3.7}$$

Considering (3.3) for $C_2(\mathcal{H}_*)=H_0(\mathcal{B})$, we obtain

$$C_2(\mathcal{H}_*)=\mathrm{Im}(i)\oplus s_2(\mathrm{Im}(j)). \tag{3.8}$$

Recall that $\{\mathbf{h}_0^{\mathbb{S}_1},\mathbf{h}_0^{\mathbb{S}_2},\mathbf{h}_0^{\mathbb{S}_3}\}$ is the given basis $\mathbf{h}_2$ of $C_2(\mathcal{H}_*)$. Since $\mathrm{Im}(i)$ and $s_2(\mathrm{Im}(j))$ are respectively 2 and 1-dimensional subspaces of 3-dimensional space $C_2(\mathcal{H}_*)$, there are non-zero vectors (b_{i1},b_{i2},b_{i3}), $i\in\{1,2,3\}$ such that $\{\sum_{i=1}^3 b_{ji}\mathbf{h}_0^{\mathbb{S}_i}\}_{j=1}^2$ is a basis of $\mathrm{Im}(i)$ and

$$s_2(\mathbf{h}^{\mathrm{Im}(j)})=\sum_{i=1}^3 b_{3i}\mathbf{h}_0^{\mathbb{S}_i}$$

is a basis of $s_2(\mathrm{Im}(j))$. Then 3×3 real matrix $B=[b_{ij}]$ is invertible. Let us choose the basis of $\mathrm{Im}(i)$ as follows

$$\mathbf{h}^{\mathrm{Im}(i)}=\left\{(\det B)^{-1}\sum_{i=1}^3 b_{1i}\mathbf{h}_0^{\mathbb{S}_i},\ \sum_{i=1}^3 b_{2i}\mathbf{h}_0^{\mathbb{S}_i}\right\}.$$

By (3.8), $\{\mathbf{h}^{\mathrm{Im}(i)},s_2(\mathbf{h}^{\mathrm{Im}(j)})\}$ becomes the obtained basis $\mathbf{h}_2'$ of $C_2(\mathcal{H}_*)$ and we have

$$[\mathbf{h}_2',\mathbf{h}_2]=1. \tag{3.9}$$

Using (3.3), $C_3(\mathcal{H}_*)=H_1(\Sigma_{2,0})$ can be expressed as the following direct sum

$$C_3(\mathcal{H}_*)=\mathrm{Im}(h)\oplus s_3(\mathrm{Im}(i)). \tag{3.10}$$

Note that the basis of $H_1(\Sigma_{0,3})\oplus H_1(\Sigma_{0,3})$ is given as follows

$$\{(\mathbf{h}_{1,1}^{\Sigma_{0,3}},0),(0,\mathbf{h}_{1,1}^{\Sigma_{0,3}}),(\mathbf{h}_{1,2}^{\Sigma_{0,3}},0),(0,\mathbf{h}_{1,2}^{\Sigma_{0,3}})\}.$$

Since $\mathrm{Im}(h)$ is a 2-dimensional space, we can choose the basis of $\mathrm{Im}(h)$ as

$$\begin{aligned}\mathbf{h}^{\mathrm{Im}(h)}=\Big\{&c_{11}h(\mathbf{h}_{1,1}^{\Sigma_{0,3}},0)+c_{12}h(0,\mathbf{h}_{1,1}^{\Sigma_{0,3}})+c_{13}h(\mathbf{h}_{1,2}^{\Sigma_{0,3}},0)+c_{14}h(0,\mathbf{h}_{1,2}^{\Sigma_{0,3}}),\\ &c_{21}h(\mathbf{h}_{1,1}^{\Sigma_{0,3}},0)+c_{22}h(0,\mathbf{h}_{1,1}^{\Sigma_{0,3}})+c_{23}h(\mathbf{h}_{1,2}^{\Sigma_{0,3}},0)+c_{24}h(0,\mathbf{h}_{1,2}^{\Sigma_{0,3}})\Big\}.\end{aligned}$$

Here, $(c_{i1}, c_{i2}, c_{i3}, c_{i4})$ is a non-zero vector for $i \in \{1,2\}$. Using (3.10), we have that

$$\left\{\mathbf{h}^{\mathrm{Im}(h)}, s_3(\mathbf{h}^{\mathrm{Im}(i)})\right\}$$

is the obtained basis $\mathbf{h}'_3$ of $C_3(\mathcal{H}_*)$. If we take the initial basis $\mathbf{h}_3$ (namely, $\mathbf{h}_1^{\Sigma_{2,0}}$) of $C_3(\mathcal{H}_*)$ as $\mathbf{h}'_3$, then we get

$$[\mathbf{h}'_3, \mathbf{h}_3] = 1. \tag{3.11}$$

If we consider (3.3) for $C_4(\mathcal{H}_*) = H_1(\Sigma_{0,3}) \oplus H_1(\Sigma_{0,3})$, then we obtain

$$C_4(\mathcal{H}_*) = \mathrm{Im}(g) \oplus s_4(\mathrm{Im}(h)). \tag{3.12}$$

Recall that $\{(\mathbf{h}_{1,1}^{\Sigma_{0,3}}, 0), (0, \mathbf{h}_{1,1}^{\Sigma_{0,3}}), (\mathbf{h}_{1,2}^{\Sigma_{0,3}}, 0), (0, \mathbf{h}_{1,2}^{\Sigma_{0,3}})\}$ is the given basis $\mathbf{h}_4$ of $C_4(\mathcal{H}_*)$. In the previous step, $\mathbf{h}^{\mathrm{Im}(h)}$ was chosen as the basis of $\mathrm{Im}(h)$ so

$$\begin{aligned} s_4(\mathbf{h}^{\mathrm{Im}(h)}) = \Big\{ & c_{11}(\mathbf{h}_{1,1}^{\Sigma_{0,3}}, 0) + c_{12}(0, \mathbf{h}_{1,1}^{\Sigma_{0,3}}) + c_{13}(\mathbf{h}_{1,2}^{\Sigma_{0,3}}, 0) + c_{14}(0, \mathbf{h}_{1,2}^{\Sigma_{0,3}}), \\ & c_{21}(\mathbf{h}_{1,1}^{\Sigma_{0,3}}, 0) + c_{22}(0, \mathbf{h}_{1,1}^{\Sigma_{0,3}}) + c_{23}(\mathbf{h}_{1,2}^{\Sigma_{0,3}}, 0) + c_{24}(0, \mathbf{h}_{1,2}^{\Sigma_{0,3}})\Big\} \end{aligned}$$

is a basis of $s_4(\mathrm{Im}(h))$. As $\mathrm{Im}(g)$ is a 2-dimensional subspace of 4-dimensional space $C_4(\mathcal{H}_*)$, there are non-zero vectors $(c_{i1}, c_{i2}, c_{i3}, c_{i4})$ for $i \in \{3,4\}$ such that

$$\begin{aligned} \Big\{ & c_{31}(\mathbf{h}_{1,1}^{\Sigma_{0,3}}, 0) + c_{32}(0, \mathbf{h}_{1,1}^{\Sigma_{0,3}}) + c_{33}(\mathbf{h}_{1,2}^{\Sigma_{0,3}}, 0) + c_{34}(0, \mathbf{h}_{1,2}^{\Sigma_{0,3}}), \\ & c_{41}(\mathbf{h}_{1,1}^{\Sigma_{0,3}}, 0) + c_{42}(0, \mathbf{h}_{1,1}^{\Sigma_{0,3}}) + c_{43}(\mathbf{h}_{1,2}^{\Sigma_{0,3}}, 0) + c_{44}(0, \mathbf{h}_{1,2}^{\Sigma_{0,3}})\Big\} \end{aligned}$$

is a basis of $\mathrm{Im}(g)$ and $C = [c_{ij}]$ is the non-singular 4×4 real matrix. Thus, we can choose the basis of $\mathrm{Im}(g)$ as

$$\begin{aligned} \mathbf{h}^{\mathrm{Im}(g)} = \Big\{ & (\det C)^{-1}[c_{31}(\mathbf{h}_{1,1}^{\Sigma_{0,3}}, 0) + c_{32}(0, \mathbf{h}_{1,1}^{\Sigma_{0,3}}) + c_{33}(\mathbf{h}_{1,2}^{\Sigma_{0,3}}, 0) + c_{34}(0, \mathbf{h}_{1,2}^{\Sigma_{0,3}})], \\ & c_{41}(\mathbf{h}_{1,1}^{\Sigma_{0,3}}, 0) + c_{42}(0, \mathbf{h}_{1,1}^{\Sigma_{0,3}}) + c_{43}(\mathbf{h}_{1,2}^{\Sigma_{0,3}}, 0) + c_{44}(0, \mathbf{h}_{1,2}^{\Sigma_{0,3}})\Big\}. \end{aligned}$$

By (3.12), $\{\mathbf{h}^{\mathrm{Im}(g)}, s_4(\mathbf{h}^{\mathrm{Im}(h)})\}$ becomes the obtained basis $\mathbf{h}'_4$ of $C_4(\mathcal{H}_*)$ and the following equation holds

$$[\mathbf{h}'_4, \mathbf{h}_4] = 1. \tag{3.13}$$

Consider the space $C_5(\mathcal{H}_*) = H_1(\mathcal{B})$, then (3.3) becomes

$$C_5(\mathcal{H}_*) = \mathrm{Im}(f) \oplus s_5(\mathrm{Im}(g)). \tag{3.14}$$

Recall that the given basis $\mathbf{h}_5$ of $C_5(\mathcal{H}_*)$ is $\{\mathbf{h}_1^{\mathbb{S}_1}, \mathbf{h}_1^{\mathbb{S}_2}, \mathbf{h}_1^{\mathbb{S}_3}\}$. Since $\mathrm{Im}(f)$ and $s_5(\mathrm{Im}(g))$ are respectively 1 and 2-dimensional subspaces of 3-dimensional space $C_5(\mathcal{H}_*)$, there are non-zero vectors (d_{i1}, d_{i2}, d_{i3}), $i \in \{1,2,3\}$ such that $\{\sum_{i=1}^{3} d_{1i}\mathbf{h}_1^{\mathbb{S}_i}\}$ is a basis of $\mathrm{Im}(f)$ and

$$s_5(\mathbf{h}^{\mathrm{Im}(g)}) = \left\{\sum_{i=1}^{3} d_{2i}\mathbf{h}_1^{\mathbb{S}_i},\ \sum_{i=1}^{3} d_{3i}\mathbf{h}_1^{\mathbb{S}_i}\right\}$$

is a basis of $s_5(\mathrm{Im}(g))$. Then we get a non-singular 3×3 real matrix $D=[d_{ij}]$. Let us choose the basis of $\mathrm{Im}(f)$ as

$$\mathbf{h}^{\mathrm{Im}(f)} = \left\{ (\det D)^{-1} \sum_{i=1}^{3} d_{1i} \mathbf{h}_1^{\mathbb{S}_i} \right\}.$$

By (3.14), $\{\mathbf{h}^{\mathrm{Im}(f)}, s_5(\mathbf{h}^{\mathrm{Im}(g)})\}$ becomes the obtained basis $\mathbf{h}'_5$ of $C_5(\mathcal{H}_*)$. Hence, we get

$$[\mathbf{h}'_5, \mathbf{h}_5] = 1. \tag{3.15}$$

Finally, let us consider $C_6(\mathcal{H}_*) = H_2(\Sigma_{2,0})$. Since $\mathrm{Im}(\alpha)$ is trivial, (3.3) becomes

$$C_6(\mathcal{H}_*) = \mathrm{Im}(\alpha) \oplus s_6(\mathrm{Im}(f)) = s_6(\mathrm{Im}(f)). \tag{3.16}$$

From (3.16) it follows that $s_6(\mathbf{h}^{\mathrm{Im}(f)})$ is the obtained basis $\mathbf{h}'_6$ of $C_6(\mathcal{H}_*)$. If we take the initial basis $\mathbf{h}_6$ (namely, $\mathbf{h}_2^{\Sigma_{2,0}}$) of $C_6(\mathcal{H}_*)$ as $s_6(\mathbf{h}^{\mathrm{Im}(f)})$, then we have

$$[\mathbf{h}'_6, \mathbf{h}_6] = 1. \tag{3.17}$$

If we combine (3.5), (3.7), (3.9), (3.11), (3.13), (3.15), and (3.17), then we get

$$\mathbb{T}(\mathcal{H}_*, \{\mathbf{h}_p\}_0^6, \{0\}_0^6) = \prod_{p=0}^{6} [\mathbf{h}'_p, \mathbf{h}_p]^{(-1)^{(p+1)}} = 1. \tag{3.18}$$

As the natural bases in (3.1) are compatible, [3, Thm. 3.2] yields

$$\mathbb{T}(\Sigma_{0,3}, \{\mathbf{h}_i^{\Sigma_{0,3}}\}_0^1)^2 = \prod_{j=1}^{3} \mathbb{T}(\mathbb{S}_j, \{\mathbf{h}_i^{\mathbb{S}_j}\}_0^1)\, \mathbb{T}(\Sigma_{2,0}, \{\mathbf{h}_\eta^{\Sigma_{2,0}}\}_0^2)\, \mathbb{T}(\mathcal{H}_*, \{\mathbf{h}_p\}_0^6, \{0\}_0^6). \tag{3.19}$$

Considering [7, Thm. 3.5], (3.18), and (3.19), we obtain

$$|\mathbb{T}(\Sigma_{0,3}, \{\mathbf{h}_i^{\Sigma_{0,3}}\}_0^1)| = \sqrt{|\mathbb{T}(\Sigma_{2,0}, \{\mathbf{h}_\eta^{\Sigma_{2,0}}\}_0^2)|}. \tag{3.20}$$

By Poincaré Duality, Theorem 4.1 in [7] and (3.20), the main formula holds

$$|\mathbb{T}(\Sigma_{0,3}, \{\mathbf{h}_i^{\Sigma_{0,3}}\}_0^1)| = \sqrt{\left|\frac{\det \Delta_{0,2}(\Sigma_{2,0})}{\det \wp(\mathbf{h}^1_{\Sigma_{2,0}}, \Gamma)}\right|}.$$

□

A *pants decomposition* of $\Sigma_{g,n}$ is a finite collection of disjoint smoothly embedded circles cutting $\Sigma_{g,n}$ into pairs of pants $\Sigma_{0,3}$ and tori with one boundary circle $\Sigma_{1,1}$. The number of complementary components is

$$|\chi(\Sigma_{g,n})| = 2g - 2 + n.$$

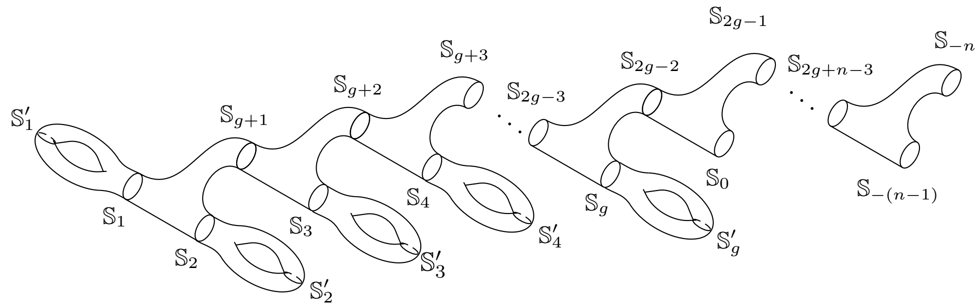

Figure 2: Compact orientable surface $\Sigma_{g,n}$ with genus $g \geq 2$ and bordered by $n \geq 1$ circles.

Proof of Theorem 1.2. Consider the decomposition of $\Sigma_{g,n}$, as in Figure 2, obtained by cutting the surface along the circles in the following order

$$\mathbb{S}_1, \dots, \mathbb{S}_g, \mathbb{S}_{g+1}, \dots, \mathbb{S}_{2g-3+n}.$$

This decomposition consists of

- the torus $\Sigma_{1,1}^{\nu}$ with boundary circle $\mathbb{S}_\nu$, $\nu \in \{1, \dots, g\}$,
- the pair of pants $\Sigma_{0,3}^{g+1}$ with boundaries $\mathbb{S}_1, \mathbb{S}_2, \mathbb{S}_{g+1}$,
- the pair of pants $\Sigma_{0,3}^{\nu+g}$ with boundaries $\mathbb{S}_{g+\nu}, \mathbb{S}_{\nu+1}, \mathbb{S}_{g+\nu-1}$, $\nu \in \{2, \dots, g-1\}$,
- the pair of pants $\Sigma_{0,3}^{\nu+g}$ with boundaries $\mathbb{S}_{g+\nu}, \mathbb{S}_{g+\nu-1}, \mathbb{S}_{g-\nu}$, $\nu \in \{g, \dots, g+n-3\}$,
- the pair of pants $\Sigma_{0,3}^{2g-2+n}$ with boundaries $\mathbb{S}_{2g+n-3}, \mathbb{S}_{-(n-1)}, \mathbb{S}_{-(n-2)}$.

Consider also the decomposition $\Sigma_{1,1}^{\nu} = Y_\nu \cup_{\partial Y_\nu} \Sigma_{0,3}^{\nu}$, $\nu \in \{1, \dots, g\}$, where Y_ν is the cylinder $\mathbb{S}'_\nu \times [-\varepsilon, +\varepsilon]$ and $\Sigma_{0,3}^{\nu}$ is the pair of pants with boundaries $\mathbb{S}'_\nu \times \{-\varepsilon\}$, $\mathbb{S}'_\nu \times \{\varepsilon\}$, $\mathbb{S}_\nu$ for sufficiently small $\varepsilon > 0$.

Case 1 : Consider the decomposition $\Sigma_{0,3} \cup_{\mathbb{S}_1} \Sigma_{0,n-1}$ of $\Sigma_{0,n}$ for $n \geq 4$, where $\Sigma_{0,3}$ and $\Sigma_{0,n-1}$ are glued along the common boundary circle $\mathbb{S}_1$. Then there is a short exact sequence of the chain complexes

$$0 \to C_*(\mathbb{S}_1) \to C_*(\Sigma_{0,3}) \oplus C_*(\Sigma_{0,n-1}) \to C_*(\Sigma_{0,n}) \to 0$$

and the corresponding Mayer-Vietoris sequence $\mathcal{H}_*$. By using the arguments stated in the proof of Theorem 1.1 for the given bases $\mathbf{h}_\eta^{\Sigma_{0,n}}$ and $\mathbf{h}_\eta^{\mathbb{S}_1}$, $\eta \in \{0, 1\}$, there exist bases $\mathbf{h}_\eta^{\Sigma_{0,3}}$ and $\mathbf{h}_\eta^{\Sigma_{0,n-1}}$ such that the R-torsion of $\mathcal{H}_*$ in the corresponding bases is 1 and the following formula holds

$$\begin{aligned} \mathbb{T}(\Sigma_{0,n}, \{\mathbf{h}_\eta^{\Sigma_{0,n}}\}_0^1) &= \mathbb{T}(\Sigma_{0,3}, \{\mathbf{h}_\eta^{\Sigma_{0,3}}\}_0^1)\, \mathbb{T}(\Sigma_{0,n-1}, \{\mathbf{h}_\eta^{\Sigma_{0,n-1}}\}_0^1) \\ &\quad \times \mathbb{T}(\mathbb{S}_1, \{\mathbf{h}_\eta^{\mathbb{S}_1}\}_0^1)^{-1}. \end{aligned} \tag{3.21}$$

By [7, Thm. 3.5] and (3.21), we obtain

$$|\mathbb{T}(\Sigma_{0,n},\{\mathbf{h}_\eta^{\Sigma_{0,n}}\}_0^1)| = |\mathbb{T}(\Sigma_{0,3},\{\mathbf{h}_\eta^{\Sigma_{0,3}}\}_0^1)||\mathbb{T}(\Sigma_{0,n-1},\{\mathbf{h}_\eta^{\Sigma_{0,n-1}}\}_0^1)|. \tag{3.22}$$

Applying (3.22) inductively, we get

$$|\mathbb{T}(\Sigma_{0,n},\{\mathbf{h}_\eta^{\Sigma_{0,n}}\}_0^1)| = \prod_{\nu=1}^{n-2}|\mathbb{T}(\Sigma_{0,3}^\nu,\{\mathbf{h}_\eta^{\Sigma_{0,3}^\nu}\}_0^1)|.$$

Case 2 : For the decomposition $\Sigma_{1,1} = Y\cup_{\partial Y}\Sigma_{0,3}$, where

$$Y = \mathbb{S}'\times[-\varepsilon,+\varepsilon],$$

$$\partial Y = \mathbb{S}'\times\{-\epsilon\}\sqcup\mathbb{S}'\times\{+\epsilon\},$$

and $\Sigma_{0,3}$ is the pair of pants with boundaries $\mathbb{S}'\times\{-\varepsilon\}$, $\mathbb{S}'\times\{\varepsilon\}$, $\mathbb{S}$ for sufficiently small $\varepsilon>0$, we have the following short exact sequence of the chain complexes

$$0\to C_*(\Sigma_{0,3}\cap Y)\to C_*(\Sigma_{0,3})\oplus C_*(Y)\to C_*(\Sigma_{1,1})\to 0 \tag{3.23}$$

and the corresponding Mayer-Vietoris sequence $\mathcal{H}_*$. If we follow the arguments in the proof of Theorem 1.1 for the given bases $\mathbf{h}_\eta^{\Sigma_{1,1}}$ and $\mathbf{h}_\eta^{\mathbb{S}'}$, $\eta\in\{0,1\}$, then we get the bases $\mathbf{h}_\eta^{\Sigma_{0,3}}$ and $\mathbf{h}_\eta^{Y}$ such that the R-torsion of $\mathcal{H}_*$ in the corresponding bases equals to 1 and the formula is valid

$$\mathbb{T}(\Sigma_{1,1},\{\mathbf{h}_\eta^{\Sigma_{1,1}}\}_0^1) = \mathbb{T}(\Sigma_{0,3},\{\mathbf{h}_\eta^{\Sigma_{0,3}}\}_0^1)\ \mathbb{T}(Y,\{\mathbf{h}_\eta^{Y}\}_0^1)\ \mathbb{T}(\mathbb{S}',\{\mathbf{h}_\eta^{\mathbb{S}'}\}_0^1)^{-2}.$$

From [7, Thm. 3.5] and Corollary 2.1 it follows

$$|\mathbb{T}(\Sigma_{1,1},\{\mathbf{h}_\eta^{\Sigma_{1,1}}\}_0^1)| = |\mathbb{T}(\Sigma_{0,3},\{\mathbf{h}_\eta^{\Sigma_{0,3}}\}_0^1)|.$$

Case 3 : Let $\Sigma_{g-1,1}\cup_{\mathbb{S}_1}\Sigma_{1,1}$ be the decomposition of $\Sigma_{g,0}$, $g\geq 2$, where $\Sigma_{1,1}$ and $\Sigma_{g-1,1}$ are glued along the common boundary circle $\mathbb{S}_1$. By the decomposition, there exists the natural short exact sequence

$$0\to C_*(\mathbb{S}_1)\to C_*(\Sigma_{g-1,1})\oplus C_*(\Sigma_{1,1})\to C_*(\Sigma_{g,0})\to 0$$

and its corresponding Mayer-Vietoris sequence

$$\begin{aligned}\mathcal{H}_*: 0\to H_2(\Sigma_{g,0})&\stackrel{\delta_2}{\to} H_1(\mathbb{S}_1)\stackrel{f}{\to} H_1(\Sigma_{g-1,1})\oplus H_1(\Sigma_{1,1})\stackrel{g}{\to} H_1(\Sigma_{g,0})\\ &\stackrel{\delta_1}{\to} H_0(\mathbb{S}_1)\stackrel{i}{\to} H_0(\Sigma_{g-1,1})\oplus H_0(\Sigma_{1,1})\stackrel{j}{\to} H_0(\Sigma_{g,0})\stackrel{k}{\to} 0.\end{aligned}$$

For the given bases $\mathbf{h}_\nu^{\Sigma_{g,0}}$ and $\mathbf{h}_\eta^{\mathbb{S}_1}$ with the condition $\delta_2(\mathbf{h}_2^{\Sigma_{g,0}}) = \mathbf{h}_1^{\mathbb{S}_1}$, $\nu\in\{0,1,2\}$, $\eta\in\{0,1\}$, if we use the arguments stated in the proof of Theorem 1.1, then we obtain the bases $\mathbf{h}_\eta^{\Sigma_{g-1,1}}$ and $\mathbf{h}_\eta^{\Sigma_{1,1}}$ such that the R-torsion of $\mathcal{H}_*$ in the corresponding bases becomes 1 and the following formula holds

$$\mathbb{T}(\Sigma_{g,0},\{\mathbf{h}_\nu^{\Sigma_{g,0}}\}_0^2) = \mathbb{T}(\Sigma_{g-1,1},\{\mathbf{h}_\eta^{\Sigma_{g-1,1}}\}_0^1)\ \mathbb{T}(\Sigma_{1,1},\{\mathbf{h}_\eta^{\Sigma_{1,1}}\}_0^1)\ \mathbb{T}(\mathbb{S}_1,\{\mathbf{h}_\eta^{\mathbb{S}_1}\}_0^1)^{-1}.$$

By [7, Thm. 3.5], we obtain

$$|\mathbb{T}(\Sigma_{g,0},\{\mathbf{h}_\nu^{\Sigma_{g,0}}\}_0^2)| = |\mathbb{T}(\Sigma_{g-1,1},\{\mathbf{h}_\eta^{\Sigma_{g-1,1}}\}_0^1)|\;|\mathbb{T}(\Sigma_{1,1},\{\mathbf{h}_\eta^{\Sigma_{1,1}}\}_0^1)|.$$

Case 4 : Consider the decomposition $\Sigma_{g,n} = \Sigma_{g-1,n+1} \cup_{\mathbb{S}_1} \Sigma_{1,1}$ for $g \geq 2$, $n \geq 1$, where $\Sigma_{1,1}$ and $\Sigma_{g-1,n+1}$ are glued along the common boundary circle $\mathbb{S}_1$. Then there is the natural short exact sequence of the chain complexes

$$0 \to C_*(\mathbb{S}_1) \to C_*(\Sigma_{g-1,n+1}) \oplus C_*(\Sigma_{1,1}) \to C_*(\Sigma_{g,n}) \to 0, \tag{3.24}$$

and the corresponding Mayer-Vietoris sequence $\mathcal{H}_*$. Using the arguments in the proof of Theorem 1.1 for the given bases $\mathbf{h}_\eta^{\Sigma_{g,n}}$ and $\mathbf{h}_\eta^{\mathbb{S}_1}$, $\eta \in \{0,1\}$, we get the bases $\mathbf{h}_\eta^{\Sigma_{g-1,n+1}}$ and $\mathbf{h}_\eta^{\Sigma_{1,1}}$ such that the R-torsion of $\mathcal{H}_*$ in the corresponding bases is 1 and

$$\mathbb{T}(\Sigma_{g,n},\{\mathbf{h}_\eta^{\Sigma_{g,n}}\}_0^1) = \mathbb{T}(\Sigma_{g-1,n+1},\{\mathbf{h}_\eta^{\Sigma_{g-1,n+1}}\}_0^1)\;\mathbb{T}(\Sigma_{1,1},\{\mathbf{h}_\eta^{\Sigma_{1,1}}\}_0^1)\;\mathbb{T}(\mathbb{S}_1,\{\mathbf{h}_\eta^{\mathbb{S}_1}\}_0^1)^{-1}.$$

By [7, Thm. 3.5], the R-torsion of $\Sigma_{g,n}$ satisfies the following formula

$$|\mathbb{T}(\Sigma_{g,n},\{\mathbf{h}_\eta^{\Sigma_{g,n}}\}_0^1)| = |\mathbb{T}(\Sigma_{g-1,n+1},\{\mathbf{h}_\eta^{\Sigma_{g-1,n+1}}\}_0^1)|\;|\mathbb{T}(\Sigma_{1,1},\{\mathbf{h}_\eta^{\Sigma_{1,1}}\}_0^1)|.$$

Applying the Cases 1-4 inductively, we have the following R-torsion formula for the compact orientable surfaces $\Sigma_{g,n}$, $g \geq 2$, $n \geq 0$

$$|\mathbb{T}(\Sigma_{g,n},\{\mathbf{h}_\eta^{\Sigma_{g,n}}\}_0^1)| = \prod_{\nu=1}^{2g-2+n} |\mathbb{T}(\Sigma_{0,3}^\nu,\{\mathbf{h}_\eta^{\Sigma_{0,3}^\nu}\}_0^1)|.$$

□

4 Applications

4.1 Compact 3-manifolds with boundary

Let N be a smooth compact orientable 3-manifold whose boundary consists of finitely many closed orientable surfaces $\partial N = \Sigma_{g_1,0} \sqcup \Sigma_{g_2,0} \sqcup \cdots \sqcup \Sigma_{g_m,0}$. Let $d(N)$ be the double of N. Consider the natural short exact sequence of the chain complexes

$$0 \to C_*(\partial N) \to C_*(N) \oplus C_*(N) \to C_*(d(N)) \to 0 \tag{4.1}$$

and the corresponding Mayer-Vietoris sequence $\mathcal{H}_*$. For the given bases $\mathbf{h}_\mu^N$, $\mathbf{h}_\nu^{\partial N}$, and $\mathbf{h}_\mu^{d(N)}$, $\nu \in \{0,1,2\}$, $\mu \in \{0,1,2,3\}$, we will denote the corresponding basis of $\mathcal{H}_*$ by $\mathbf{h}_n$, $n \in \{0,\ldots,11\}$. As the bases in the sequence (4.1) are compatible, [3, Thm. 3.2] yields

$$\mathbb{T}(N,\{\mathbf{h}_\mu^N\}_0^3)^2 = \mathbb{T}(\partial N,\{\mathbf{h}_\nu^{\partial N}\}_0^2)\;\mathbb{T}(d(N),\{\mathbf{h}_\mu^{d(N)}\}_0^3)\;\mathbb{T}(\mathcal{H}_*,\{\mathbf{h}_n\}_0^{11}). \tag{4.2}$$

By [7, Thm. 3.5] and (4.2), we have

$$|\mathbb{T}(N,\{\mathbf{h}_\mu^N\}_0^3)| = \sqrt{|\mathbb{T}(\partial N,\{\mathbf{h}_\nu^{\partial N}\}_0^2)||\mathbb{T}(\mathcal{H}_*,\{\mathbf{h}_n\}_0^{11})|}. \tag{4.3}$$

Note that ∂N is equal to $\Sigma_{g_1,0} \sqcup \Sigma_{g_2,0} \sqcup \cdots \sqcup \Sigma_{g_m,0}$. By [7, Lem. 1.4], we get

$$|\mathbb{T}(\partial N, \{\mathbf{h}_\nu^{\partial N}\}_0^2)| = \prod_{i=1}^{m} |\mathbb{T}(\Sigma_{g_i,0}, \{\mathbf{h}_\nu^{\Sigma_{g_i,0}}\}_0^2)|. \tag{4.4}$$

For each $i \in \{1, \ldots, m\}$, consider the given basis $\mathbf{h}_\nu^{\Sigma_{g_i,0}}$ for $\nu \in \{0,1,2\}$ and pants decompositions $\{\Sigma_{0,3}^{j,i}\}_{j=1}^{2g_i-2}$ of $\Sigma_{g_i,0}$. By using Theorem 1.2, we obtain the basis $\mathbf{h}_\eta^{\Sigma_{0,3}^{j,i}}$, $\eta \in \{0,1\}$, $j \in \{1, \ldots, 2g_i - 2\}$ such that

$$|\mathbb{T}(\partial N, \{\mathbf{h}_\nu^{\partial N}\}_0^2)| = \prod_{i=1}^{m} \prod_{j=1}^{2g_i-2} |\mathbb{T}(\Sigma_{0,3}^{j,i}, \{\mathbf{h}_\eta^{\Sigma_{0,3}^{j,i}}\}_0^1)|. \tag{4.5}$$

Equations (4.4) and (4.5) yield the following formula

$$|\mathbb{T}(N, \{\mathbf{h}_\mu^N\}_0^3)| = \sqrt{\prod_{i=1}^{m} \prod_{j=1}^{2g_i-2} |\mathbb{T}(\Sigma_{0,3}^{j,i}, \{\mathbf{h}_\eta^{\Sigma_{0,3}^{j,i}}\}_0^1)|\, |\mathbb{T}(\mathcal{H}_*, \{\mathbf{h}_i\}_0^{11})|}.$$

Corollary 4.1. *Let N be the handlebody of genus $g \geq 2$. Clearly, the boundary ∂N of N is an orientable closed surface $\Sigma_{g,0}$ and the double $d(N)$ of N is equal to $\#_g(\mathbb{S}\times\mathbb{S}^2)$. Then we have the short exact sequence*

$$0 \to C_*(\Sigma_{g,0}) \to C_*(N) \oplus C_*(N) \to C_*(d(N)) \to 0 \tag{4.6}$$

and the corresponding Mayer-Vietoris sequence $\mathcal{H}_*$. For the given bases $\mathbf{h}_\mu^{d(N)}$ and $\mathbf{h}_\mu^N$ $\mu \in \{0,1,2,3\}$, following the arguments above, there exists a basis $\mathbf{h}_i^{\Sigma_{g,0}}$ for $i \in \{0,1,2\}$ such that in the corresponding bases the R-torsion of $\mathcal{H}_*$ is 1 and from [7, Thm. 3.5] it follows

$$|\mathbb{T}(N, \{\mathbf{h}_\mu^N\}_0^3)| = \sqrt{|\mathbb{T}(\Sigma_{g,0}, \{\mathbf{h}_i^{\Sigma_{g,0}}\}_0^2)|}.$$

Let us consider the pants decomposition $\{\Sigma_{0,3}^j\}_{j=1}^{2g-2}$ of $\Sigma_{g,0}$. By Theorem 1.2, there exists the basis $\mathbf{h}_\eta^{\Sigma_{0,3}^j}$ for each $j \in \{1, \ldots, 2g-2\}$ and $\eta \in \{0,1\}$ such that the following formula holds

$$|\mathbb{T}(N, \{\mathbf{h}_\mu^N\}_0^3)| = \sqrt{\prod_{j=1}^{2g-2} |\mathbb{T}(\Sigma_{0,3}^j, \{\mathbf{h}_\eta^{\Sigma_{0,3}^j}\}_0^1)|}.$$

4.2 Product of $2d$-manifolds and compact 3-manifolds with boundary $\Sigma_{g,0}$

Let M be a smooth closed orientable $2d$-manifold ($d \geq 1$) and N an smooth compact orientable 3-manifold whose boundary consists of closed orientable surface $\Sigma_{g,0}$ ($g \geq 2$). Let X be the product manifold $M \times N$ and $d(X)$ denote the double of X. Clearly, the boundary of X is $M \times \Sigma_{g,0}$. Consider the natural short exact sequence of the chain complexes

$$0 \to C_*(M \times \Sigma_{g,0}) \to C_*(X) \oplus C_*(X) \to C_*(d(X)) \to 0 \tag{4.7}$$

and the Mayer-Vietoris sequence $\mathcal{H}_*$ corresponding to (4.7). Let $\mathbf{h}_i^X$, $\mathbf{h}_i^{d(X)}$, $\mathbf{h}_k^M$, and $\mathbf{h}_\ell^{\Sigma_{g,0}}$ be given bases for $i \in \{0, \cdots, 2d+3\}$, $k \in \{0, \ldots, 2d\}$, $\ell \in \{0,1,2\}$. Let $\mathbf{h}_\nu^{M\times\Sigma_{g,0}}$ denote the basis $\bigoplus_i \mathbf{h}_i^M \otimes \mathbf{h}_{\nu-i}^{\Sigma_{g,0}}$ of $H_\nu(M \times \Sigma_{g,0})$, $\nu \in \{0, \ldots, 2d+2\}$. For $n \in \{0, \ldots, 6d+11\}$, let $\mathbf{h}_n$ be the corresponding basis of $\mathcal{H}_*$. Let $\{\Sigma_{0,3}^j\}_{j=1}^{2g-2}$ be the pants decomposition of $\Sigma_{g,0}$. Note that the bases in the sequence (4.7) are compatible. Thus, by [7, Lem. 1.4], we obtain

$$\begin{aligned} \mathbb{T}(X, \{\mathbf{h}_i^X\}_0^{2d+3})^2 = {} & \mathbb{T}(M \times \Sigma_{g,0}, \{\mathbf{h}_\nu^{M\times\Sigma_{g,0}}\}_0^{2d+2})\, \mathbb{T}(d(X), \{\mathbf{h}_i^{d(X)}\}_0^{2d+3}) \\ & \times \mathbb{T}(\mathcal{H}_*, \{\mathbf{h}_n\}_0^{6d+11}). \end{aligned} \tag{4.8}$$

From [7, Thm. 3.5] and (4.8) it follows that

$$|\mathbb{T}(X, \{\mathbf{h}_i^X\}_0^{2d+3})| = |\mathbb{T}(M \times \Sigma_{g,0}, \{\mathbf{h}_\nu^{M\times\Sigma_{g,0}}\}_0^{2d+2})|^{1/2}\, |\mathbb{T}(\mathcal{H}_*, \{\mathbf{h}_n\}_0^{6d+11})|^{1/2}. \tag{4.9}$$

By [4, Thm. 3.1], the R-torsion of $M \times \Sigma_{g,0}$ satisfies the equality

$$\begin{aligned} |\mathbb{T}(M \times \Sigma_{g,0}, \{\mathbf{h}_\nu^{M\times\Sigma_{g,0}}\}_0^{2d+2})| = {} & |\mathbb{T}(M, \{\mathbf{h}_k^M\}_0^{2d})|^{\chi(\Sigma_{g,0})} \\ & \times |\mathbb{T}(\Sigma_{g,0}, \{\mathbf{h}_\ell^{\Sigma_{g,0}}\}_0^2)|^{\chi(M)}. \end{aligned} \tag{4.10}$$

Here, χ is the Euler characteristic. Then equations (4.9) and (4.10) yield

$$\begin{aligned} |\mathbb{T}(X, \{\mathbf{h}_i^X\}_0^{2d+3})| = {} & |\mathbb{T}(M, \{\mathbf{h}_k^M\}_0^{2d})|^{\chi(\Sigma_{g,0})/2}\, |\mathbb{T}(\Sigma_{g,0}, \{\mathbf{h}_\ell^{\Sigma_{g,0}}\}_0^2)|^{\chi(M)/2} \\ & \times |\mathbb{T}(\mathcal{H}_*, \{\mathbf{h}_n\}_0^{6d+11})|^{1/2}. \end{aligned} \tag{4.11}$$

Since $\{\Sigma_{0,3}^j\}_{j=1}^{2g-2}$ is the pants decomposition of $\Sigma_{g,0}$ as in Theorem 1.2, there exists a basis $\mathbf{h}_\eta^{\Sigma_{0,3}^j}$ of $H_\eta(\Sigma_{0,3}^j)$ for $j \in \{1, \ldots, 2g-2\}$, $\eta \in \{0,1\}$ so that

$$|\mathbb{T}(\Sigma_{g,0}, \{\mathbf{h}_\ell^{\Sigma_{g,0}}\}_0^2)| = \prod_{j=1}^{2g-2} |\mathbb{T}(\Sigma_{0,3}^j, \{\mathbf{h}_\eta^{\Sigma_{0,3}^j}\}_0^1)|. \tag{4.12}$$

Equations (4.11) and (4.12) yield

$$\begin{aligned} |\mathbb{T}(X, \{\mathbf{h}_i^X\}_0^{2d+3})| = {} & \prod_{j=1}^{2g-2} |\mathbb{T}(\Sigma_{0,3}^j, \{\mathbf{h}_\eta^{\Sigma_{0,3}^j}\}_0^1)|^{\frac{\chi(M)}{2}}\, |\mathbb{T}(M, \{\mathbf{h}_k^M\}_0^{2d})|^{\frac{\chi(\Sigma_{g,0})}{2}} \\ & \times |\mathbb{T}(\mathcal{H}_*, \{\mathbf{h}_n\}_0^{6d+1})|^{1/2}. \end{aligned}$$

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