η-Ricci-Yamabe solitons on anti-invariant submanifolds of trans-Sasakian manifold admitting Zamkovoy connection

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Abstract. The present paper deals with the study of η-Ricci-Yamabe solitons on anti-invariant submanifolds of trans-Sasakian manifold with respect to a new non-metric affine connection called Zamkovoy connection. An η-Ricci-Yamabe soliton and also two more solitons arose as its particular cases are studied on Ricci flat, concircularly flat, M-projectively flat and pseudo projectively flat anti-invariant submanifolds of trans-Sasakian manifold with respect to the aforesaid connection. At last, some conclusions are made after observing all the results and an example of an anti-invariant submanifold of a trans-Sasakian manifold is given in which all the results can be verified very easily.

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1 Introduction

The concepts of Ricci flow and Yamabe flow were introduced simultaneously by R. S. Hamilton in 1988[17]. Ricci soliton and Yamabe soliton emerged as limits of solutions of Ricci flow and Yamabe flow respectively. These solitons are equivalent in dimension 2 but in greater dimensions, these two do not agree since Yamabe soliton preserves the conformal class of the metric but Ricci soliton does not in general. In 2019, S. Guler and M. Crasmareanu[16] introduced a new geometric flow called Ricci-Yamabe flow as a scalar combination of Ricci flow and Yamabe flow. *Ricci-Yamabe flow of type* (p, q) is an evolution for the metrics on Riemannian or semi-Riemannian manifolds defined as[16]

$$\frac{\partial}{\partial t}g(t) = -2pRic(t) + qr(t)g(t),\ g(0) = g_0, \tag{1.1}$$

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where $p,\ q$ are scalars. Due to the signs of $p,\ q$, this flow can also be a Riemannian flow or semi-Riemannian flow or singular Riemannian flow. Naturally, Ricci-Yamabe soliton emerged as the limit of solutions of Ricci-Yamabe flow. An interpolation soliton, called Ricci-Bourguignon soliton corresponding to Ricci-Bourguignon flow, was considered and further studied by G. Catino and L. Mazzieri ([7], [8]), but this soliton depends on a single scalar.

Ricci-Yamabe soliton of type (p, q) on Riemannian complex (M, g) is represented by the quintuplet (g, V, λ, p, q) satisfying the following equation−

$$L_V g + 2pS + (2\lambda - qr)g = 0, \tag{1.2}$$

where $L_V g$ is the Lie derivative of the Riemannian metric g along the vector field V, r is the scalar curvature, S is the Ricci curvature tensor and $\lambda,\ p,\ q$ are scalars. This soliton is called shrinking, steady or expanding according as $\lambda < 0,\ \lambda = 0$ or $\lambda > 0$ respectively. Ricci-Yamabe soliton of type $(0, q)$ and $(p, 0)$ are called *q-Yamabe soliton* and *p-Ricci soliton* respectively. These solitons are studied by many geometers([4], [5], [6], [11], [13], [15], [37], [38]).

J. T. Cho and M. Kimura introduced the notion of η-Ricci soliton as an advance extension of Ricci soliton in 2009[12]. Analogously in 2020[39], M. D. Siddiqi and M. A. Akyol introduced the concept of η-Ricci-Yamabe soliton as a generalization of Ricci-Yamabe soliton. *η-Ricci-Yamabe soliton of type* (p, q) is represented by the sextuplet $(g, V, \lambda, \mu, p, q)$ on a Riemannian manifold M satisfying the following equation−

$$L_V g + 2pS + (2\lambda - qr)g + 2\mu\eta \otimes \eta = 0, \tag{1.3}$$

where $L_V g$ is the Lie derivative of the Riemannian metric g along the vector field V, r is the scalar curvature, S is the Ricci curvature tensor, $\eta \otimes \eta$ is a $(0, 2)$-tensor field and $\lambda,\ \mu,\ p,\ q$ are scalars. The soliton is called shrinking, steady or expanding according as $\lambda < 0,\ \lambda = 0$ or $\lambda > 0$ respectively. η-Ricci-Yamabe soliton of type $(0, q)$ and $(p, 0)$ are called *q-η-Yamabe soliton* and *p-η-Ricci soliton* respectively. Recently, in 2021, G. Somashekhara et al. studied η-Ricci-Yamabe solitons on submanifolds of some indefinite almost contact manifolds concerning Riemannian and quarter symmetric metric connection[40].

The notion of Zamkovoy connection was introduced by S. Zamkovoy in 2009[45]. Later A. Biswas and K. K. Baishya applied this connection on generalized pseudo Ricci symmetric Sasakian manifolds[1] and on almost pseudo symmetric Sasakian manifolds[2]. This connection was further studied by A. M. Blaga in 2015[3]. In 2020, A. Mandal and A. Das worked in detail on various curvature tensors of Sasakian and Lorentzian para-Sasakian manifolds admitting this new connection([20], [21], [22], [23]), and recently in 2021, they discussed LP-Sasakian manifolds equipped with this connection and conharmonic curvature tensor[24]. Also, most recently in 2021, the author studied curvature tensors and Ricci solitons with respect to this connection in anti-invariant submanifolds of trans-Sasakian manifold[18].

For an n-dimensional almost contact metric manifold $M(\phi, \xi, \eta, g)$ consisting of a $(1,1)$-tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g with the Riemannian connection ∇, *Zamkovoy connection* ∇^* is defined as[45]

$$\nabla^*_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi + \eta(X)\phi Y. \tag{1.4}$$

Curvature is the central subject in Riemannian geometry. It measures distance between a manifold and a Euclidean space.

K. Yano introduced the notion of *concircular curvature tensor* C of type (1,3) on Riemannian manifold for an n-dimensional manifold M as[43]

$$C(X,Y)Z = R(X,Y)Z - \tfrac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]$$

for all smooth vectors fields $X, Y, Z \in \chi(M)$, where R is the Riemannian curvature tensor of type (1,3) and r is the scalar curvature.

Hence if we consider C^* as the concircular curvature tensor with respect to Zamkovoy connection, then for a $(2n+1)$-dimensional manifold M we have

$$C^*(X,Y)Z = R^*(X,Y)Z - \tfrac{r^*}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y] \tag{1.5}$$

for all smooth vectors fields $X, Y, Z \in \chi(M)$, where R^* is the curvature tensor and r^* is the scalar curvature with respect to Zamkovoy connection.

Definition 1.1. A $(2n+1)$-dimensional manifold M is called *Ricci flat* with respect to Zamkovoy connection if $S^*(X,Y) = 0 \;\; \forall X, Y \in \chi(M)$, where S^* is the Ricci curvature tensor with respect to Zamkovoy connection.

Definition 1.2. [23] A $(2n+1)$-dimensional manifold M is called *concircularly flat* with respect to Zamkovoy connection if $C^*(X,Y)Z = 0 \;\; \forall X, Y, Z \in \chi(M)$.

G. P. Pokhariyal and R. S. Mishra introduced the notion of M-projective curvature tensor on a Riemannian manifold in 1971[31]. Later R. H. Ojha studied its properties([27], [28], [29]). This curvature tensor was further discussed by many geometers([9], [10], [20], [32], [35]). The *M-projective curvature tensor* $\bar{M}$ of rank 3 on an n-dimensional manifold M is given by[31]

$$\bar{M}(X,Y)Z = R(X,Y)Z - \tfrac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y] - \tfrac{1}{2(n-1)}[g(Y,Z)QX - g(X,Z)QY]$$

for all smooth vectors fields $X, Y, Z \in \chi(M)$, where Q is the Ricci operator.

Thus for a $(2n+1)$-dimensional manifold, considering $\bar{M}^*$ as the M-projective curvature tensor with respect to Zamkovoy connection we get

(1.6) $\bar{M}^*(X,Y)Z = R^*(X,Y)Z - \frac{1}{4n}[S^*(Y,Z)X - S^*(X,Z)Y] - \frac{1}{4n}[g(Y,Z)Q^*X -$

$g(X,Z)Q^*Y]$,

where Q^* is the Ricci operator with respect to Zamkovoy connection.

Definition 1.3. [20] A $(2n+1)$-dimensional manifold M is called *M-projectively flat* with respect to Zamkovoy connection if $\bar{M}^*(X,Y)Z = 0 \ \ \forall X,Y,Z \in \chi(M)$.

B. Prasad introduced the notion of pseudo projective curvature tensor in a Riemannian manifold of dimension $n > 2$ in 2002[33]. Its properties were further studied by many geometers on various manifolds([22], [25], [26], [34], [42]). The *pseudo projective curvature tensor* $\bar{P}$ of rank 3 on an n-dimensional manifold M is given by[33]

$$\bar{P}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] + cr[g(Y,Z)X - g(X,Z)Y]$$

for all smooth vectors fields $X,Y,Z \in \chi(M)$, where $a,\ b,\ c$ are non-zero constants related as $c = -\frac{1}{n}\left(\frac{a}{n-1} + b\right)$.

Thus for a $(2n+1)$-dimensional manifold, considering $\bar{P}^*$ as the pseudo projective curvature tensor with respect to Zamkovoy connection we get

$$(1.7)\bar{P}^*(X,Y)Z = aR^*(X,Y)Z + b[S^*(Y,Z)X - S^*(X,Z)Y] + cr^*[g(Y,Z)X - g(X,Z)Y],$$

where $a,\ b,\ c$ are non-zero constants related as

$$c = -\tfrac{1}{2n+1}\left(\tfrac{a}{2n} + b\right). \tag{1.8}$$

Definition 1.4. [22] A $(2n+1)$-dimensional manifold M is called *pseudo projectively flat* with respect to Zamkovoy connection if $\bar{P}^*(X,Y)Z = 0 \ \ \forall X,Y,Z \in \chi(M)$.

In 1977, anti-invariant submanifolds of Sasakian space forms[44] were discussed by K. Yano and M. Kon. In 1985, H. B. Pandey and A. Kumar investigated about anti-invariant submanifolds of almost para-contact manifolds[30]. Recently in 2020, the author and A. Bhattacharyya studied anti-invariant submanifolds of some indefinite almost contact and paracontact manifolds[19].

Let φ be a differentiable map from a manifold M into a manifold $\tilde{M}$ and let the dimensions of $M, \tilde{M}$ be n, m respectively. If at each point p of M, $(\varphi_*)_p$ is a 1-1 map, i.e., if rankφ=n, then φ is called an *immersion* of M into $\tilde{M}$.

If an immersion φ is one-one, i.e., if $\varphi(p) \neq \varphi(q)$ for $p \neq q$, then φ is called an *imbedding* of M into $\tilde{M}$.

If the manifolds $M, \tilde{M}$ satisfy the following two conditions, then M is called a *submanifold* of $\tilde{M}-$
(i) $M \subset \tilde{M}$,
(ii) the inclusion map i from M into $\tilde{M}$ is an imbedding of M into $\tilde{M}$.

A submanifold M is called *anti-invariant* if $X \in T_x(M) \Rightarrow \phi X \in T_x^{\perp}(M)\ \ \forall x \in M$, where $T_x(M),\ T_x^{\perp}(M)$ are respectively the tangent space and the normal space at $x \in M$. Thus in an anti-invariant submanifold M, we have $\forall X, Y \in \chi(M)$,

$$g(X, \phi Y) = 0. \tag{1.9}$$

Motivated by the works mentioned above, in this paper, the study has been done on η-Ricci-Yamabe solitons on anti-invariant submanifolds of trans-Sasakian manifold with respect to Zamkovoy connection. This paper is divided into three sections. After the introduction, in the preliminaries section, definition and some properties of a trans-Sasakian manifold of type (α, β) are given. After preliminaries, there remains the third section which concerns the main topic and it is further subdivided into four subsections dealing with the study of η-Ricci-Yamabe soliton, q-η-Yamabe soliton and p-η-Ricci soliton on an anti-invariant submanifold of a trans-Sasakian manifold, where the submanifold is (i) Ricci flat, (ii) concircularly flat, (iii) M-projectively flat and (iv) pseudo projectively flat respectively with respect to Zamkovoy connection. At last, three conclusions are made after observing all the results of the four subsections and an example of an anti-invariant submanifold of a trans-Sasakian manifold is given in which all the results can be verified very easily.

2 Preliminaries

Let $\tilde{M}$ be an odd dimensional differentiable manifold equipped with a metric structure (ϕ, ξ, η, g) consisting of a (1,1) tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g satisfying the following relations−

$$\phi^2 X = -X + \eta(X)\xi,\ \ \eta(\xi) = 1,\ \ \eta \circ \phi = 0,\ \ \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(\phi X, Y) = -g(X, \phi Y),\ \ \eta(X) = g(X, \xi)\ \ \forall X, Y \in \chi(\tilde{M}), \tag{2.3}$$

then $\tilde{M}$ is called *almost contact metric manifold*[14].

An odd dimensional almost contact metric manifold $\tilde{M}(\phi, \xi, \eta, g)$ is called *trans-Sasakian manifold of type* (α, β) (α, β are smooth functions on $\tilde{M}$) if $\forall X, Y \in \chi(\tilde{M})$ [14]

$$(\nabla_X \phi)Y = \alpha[g(X, Y)\xi - \eta(Y)X] + \beta[g(\phi X, Y)\xi - \eta(Y)\phi X], \tag{2.4}$$

$$\nabla_X \xi = -\alpha\phi X + \beta[X - \eta(X)\xi]. \tag{2.5}$$

In a trans-Sasakian manifold $\tilde{M}^{2n+1}$ of type (α, β), we have the following relations[14]−

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta[g(X, Y) - \eta(X)\eta(Y)], \tag{2.6}$$

$$R(X,Y)\xi = (\alpha^2-\beta^2)[\eta(Y)X-\eta(X)Y]+2\alpha\beta[\eta(Y)\phi X-\eta(X)\phi Y] \\ +[(Y\alpha)\phi X-(X\alpha)\phi Y+(Y\beta)\phi^2 X-(X\beta)\phi^2 Y], \tag{2.7}$$

$$R(\xi,Y)X = (\alpha^2-\beta^2)[g(X,Y)\xi-\eta(X)Y]+2\alpha\beta[g(\phi X,Y)\xi+\eta(X)\phi Y]+(X\alpha)\phi Y \\ +g(\phi X,Y)(grad\ \alpha)-g(\phi X,\phi Y)(grad\ \beta)+(X\beta)[Y-\eta(Y)\xi], \tag{2.8}$$

$$S(X,\xi) = [2n(\alpha^2-\beta^2)-\xi\beta]\eta(X)-(\phi X)\alpha-(2n-1)(X\beta), \tag{2.9}$$

$$Q\xi = [2n(\alpha^2-\beta^2)-\xi\beta]\xi+\phi(grad\ \alpha)-(2n-1)(grad\ \beta). \tag{2.10}$$

Now we state the following lemma−

Lemma 2.1. [36] *In a $(2n+1)$-dimensional trans-Sasakian manifold of type (α,β), if $\phi(grad\ \alpha)=(2n-1)(grad\ \beta)$, then $\xi\beta=0$.*

Using (2.5), (2.6) on (1.4) we get the expression of Zamkovoy connection on $\tilde{M}$ as−

$$\nabla^*_X Y = \nabla_X Y+\eta(X)\phi Y+\alpha\eta(Y)\phi X-\beta\eta(Y)X+\beta g(X,Y)\xi-\alpha g(\phi X,Y)\xi \tag{2.11}$$

with torsion tensor

$$T^*(X,Y) = (1-\alpha)[\eta(X)\phi Y-\eta(Y)\phi X]+\beta[\eta(X)Y-\eta(Y)X]+2\alpha g(X,\phi Y)\xi. \tag{2.12}$$

Again, we have

$$(\nabla^*_X g)(Y,Z) = \nabla^*_X g(Y,Z)-g(\nabla^*_X Y,Z)-g(Y,\nabla^*_X Z).$$

Then, using (2.11) in the above equation we obtain $\nabla^* g = 0$, i.e. Zamkovoy connection is a metric compatible connection on $\tilde{M}$.

Now applying (1.9) in (2.11) and (2.12) respectively we get the expression of Zamkovoy connection on an anti-invariant submanifold M of $\tilde{M}$ as−

$$\nabla^*_X Y = \nabla_X Y+\eta(X)\phi Y+\alpha\eta(Y)\phi X-\beta\eta(Y)X+\beta g(X,Y)\xi \tag{2.13}$$

with torsion tensor

$$T^*(X,Y) = (1-\alpha)[\eta(X)\phi Y-\eta(Y)\phi X]+\beta[\eta(X)Y-\eta(Y)X].$$

Setting $Y=\xi$ in (2.13) and then using (2.5) we obtain

$$\nabla^*_X \xi = 0. \tag{2.14}$$

Applying (2.4), (2.5) and (2.13) on the following equation

$$R^*(X,Y)Z = \nabla^*_X \nabla^*_Y Z - \nabla^*_Y \nabla^*_X Z - \nabla^*_{[X,Y]} Z$$

we get $\forall X, Y, Z \in \chi(M)$,

$$(2.15)\quad R^*(X,Y)Z = R(X,Y)Z + \alpha^2[\eta(X)Y - \eta(Y)X]\eta(Z) + \beta[\eta(X)\phi Y - \eta(Y)\phi X]\eta(Z) + \beta^2[g(Y,Z)X - g(X,Z)Y] + \alpha\beta[g(X,Z)\phi Y - g(Y,Z)\phi X] + \beta[\nabla_Y g(X,Z) - \nabla_X g(Y,Z)]\xi.$$

Consequently, if $\xi\beta = 0$ and $\dim(M) = 2n+1$, then we have

$$S^*(Y,Z) = S(Y,Z) - 2n\alpha^2\eta(Y)\eta(Z) + 2n\beta^2 g(Y,Z) \tag{2.16}$$

which implies that

$$Q^*Y = QY - 2n\alpha^2\eta(Y)\xi + 2n\beta^2 Y, \tag{2.17}$$

$$r^* = r - 2n\alpha^2 + 2n(2n+1)\beta^2. \tag{2.18}$$

3 η-Ricci-Yamabe solitons on anti-invariant submanifolds of trans-Sasakian manifold with respect to Zamkovoy connection

This section deals with the main topic, i.e. η-Ricci-Yamabe solitons on anti-invariant submanifolds of trans-Sasakian manifold with respect to Zamkovoy connection. This section is further subdivided into four subsections but before proceeding to these subsections, here two theorems are proved concerning the nature of a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) when an η-Ricci-Yamabe soliton of type (p, q) is considered on it with respect to Zamkovoy connection ∇^* given by the equation (2.13).

Let $(g, \xi, \lambda, \mu, p, q)$ be an η-Ricci-Yamabe soliton on M with respect to Zamkovoy connection ∇^*, then from (1.3) we have $\forall Y, Z \in \chi(M)$

$$(L^*_\xi g)(Y,Z) + 2pS^*(Y,Z) + (2\lambda - qr^*)g(Y,Z) + 2\mu\eta(Y)\eta(Z) = 0$$

$$\Rightarrow g(\nabla^*_Y \xi, Z) + g(\nabla^*_Z \xi, Y) + 2pS^*(Y,Z) + (2\lambda - qr^*)g(Y,Z) + 2\mu\eta(Y)\eta(Z) = 0.$$

Using (2.14) in the above equation we get

$$S^*(Y,Z) = \left(\tfrac{qr^* - 2\lambda}{2p}\right)g(Y,Z) - \left(\tfrac{\mu}{p}\right)\eta(Y)\eta(Z). \tag{3.1}$$

Hence we can state the following theorem−

Theorem 3.1. *Let $(g, \xi, \lambda, \mu, p, q)$ be an η-Ricci-Yamabe soliton on an anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to Zamkovoy connection, then M is η-Einstein with respect to Zamkovoy connection.*

Again if $\xi\beta = 0$, then using (2.16) and (2.18) in (3.1) we obtain

$$S(Y,Z) = \big[\tfrac{q\{r-2n\alpha^2+2n(2n+1)\beta^2-2\lambda\}}{2p} - 2n\beta^2\big]g(Y,Z) + \big[2n\alpha^2 - \big(\tfrac{\mu}{p}\big)\big]\eta(Y)\eta(Z).$$

Thus applying lemma 2.1 we have the following theorem−

Theorem 3.2. *Let $(g, \xi, \lambda, \mu, p, q)$ be an η-Ricci-Yamabe soliton on a $(2n + 1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to Zamkovoy connection, then M is η-Einstein with respect to Riemannian connection, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

3.1 η-Ricci-Yamabe solitons on Ricci flat anti-invariant submanifolds

Here η-Ricci-Yamabe soliton of type (p, q), q-η-Yamabe soliton and p-η-Ricci soliton on a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is discussed, where M is Ricci flat with respect to Zamkovoy connection ∇^* given by the equation (2.13).

Let $(g, \xi, \lambda, \mu, p, q)$ be an η-Ricci-Yamabe soliton on M, then from (1.3) we have $\forall Y, Z \in \chi(M)$

$$(L_\xi g)(Y,Z) + 2pS(Y,Z) + (2\lambda - qr)g(Y,Z) + 2\mu\eta(Y)\eta(Z) = 0$$

$$\Rightarrow g(\nabla_Y \xi, Z) + g(\nabla_Z \xi, Y) + 2pS(Y,Z) + (2\lambda - qr)g(Y,Z) + 2\mu\eta(Y)\eta(Z) = 0.$$

Using (2.5) and then applying (1.9) in the above equation we get

$$pS(Y,Z) + \big(\lambda + \beta - \tfrac{qr}{2}\big)g(Y,Z) + (\mu - \beta)\eta(Y)\eta(Z) = 0.$$

Setting $Z = \xi$ in the above equation we obtain

$$pS(Y,\xi) = \big(\tfrac{qr}{2} - \lambda - \mu\big)\eta(Y). \tag{3.2}$$

Now if $\xi\beta = 0$ and M is Ricci flat with respect to ∇^*, then from (2.16) we have

$$S(Y,Z) = 2n\alpha^2\eta(Y)\eta(Z) - 2n\beta^2 g(Y,Z).$$

Setting $Z = \xi$ in the above equation and multiplying both sides by p we obtain

$$pS(Y,\xi) = 2np(\alpha^2 - \beta^2)\eta(Y). \tag{3.3}$$

Equating (3.2) and (3.3) we get

$$\lambda = \tfrac{qr}{2} - \mu - 2np(\alpha^2 - \beta^2). \tag{3.4}$$

Hence from (3.4) and applying lemma 2.1 we conclude the following theorem−

Theorem 3.1.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is Ricci flat with respect to Zamkovoy connection, then an η-Ricci-Yamabe soliton $(g, \xi, \lambda, \mu, p, q)$ on M is shrinking, steady or expanding according as $\frac{qr}{2} - \mu < 2np(\alpha^2 - \beta^2)$, $\frac{qr}{2} - \mu = 2np(\alpha^2 - \beta^2)$ or $\frac{qr}{2} - \mu > 2np(\alpha^2 - \beta^2)$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

Now from (3.4) we get, when $p = 0$ then $\lambda = \frac{qr}{2} - \mu$, and when $q = 0$ then $\lambda = -\mu + 2np(\beta^2 - \alpha^2)$. Thus from theorem 3.1.1 we respectively conclude the following results−

Corollary 3.1.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is Ricci flat with respect to Zamkovoy connection, then a q-η-Yamabe soliton $(g, \xi, \lambda, \mu, q)$ on M is shrinking, steady or expanding according as $qr < 2\mu$, $qr = 2\mu$ or $qr > 2\mu$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

Corollary 3.1.2. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is Ricci flat with respect to Zamkovoy connection, then a p-η-Ricci soliton $(g, \xi, \lambda, \mu, p)$ on M is shrinking, steady or expanding according as $2np(\beta^2 - \alpha^2) < \mu$, $2np(\beta^2 - \alpha^2) = \mu$ or $2np(\beta^2 - \alpha^2) > \mu$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

3.2 η-Ricci-Yamabe solitons on concircularly flat anti-invariant submanifolds

This subsection deals with the study of η-Ricci-Yamabe soliton of type (p, q), q-η-Yamabe soliton and p-η-Ricci soliton on a $(2n+1)$-dimensional concircularly flat anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to Zamkovoy connection ∇^* given by the equation (2.13).

Since M is concircularly flat with respect to ∇^*, from (1.5) we have

$$R^*(X, Y)Z = \tfrac{r^*}{2n(2n+1)}[g(Y, Z)X - g(X, Z)Y]$$

which implies that

$$S^*(Y, Z) = \left(\tfrac{r^*}{2n+1}\right)g(Y, Z). \tag{3.5}$$

Let $\xi\beta = 0$, hence using (2.16) and (2.18) in (3.5) we obtain

$$S(Y,Z) = \left(\tfrac{r-2n\alpha^2}{2n+1}\right)g(Y,Z) + 2n\alpha^2\eta(Y)\eta(Z). \tag{3.6}$$

Putting $Z=\xi$ in (3.6) and then multiplying both sides by p we get

$$pS(Y,\xi) = p\left(\tfrac{r+4n^2\alpha^2}{2n+1}\right)\eta(Y). \tag{3.7}$$

Next, let (g,ξ,λ,μ,p,q) be an η-Ricci-Yamabe soliton on M, then equating (3.2) and (3.7) we obtain

$$\lambda = \tfrac{qr}{2} - \mu - p\left(\tfrac{r+4n^2\alpha^2}{2n+1}\right). \tag{3.8}$$

Thus, applying lemma 2.1, from (3.8) we state the following theorem−

Theorem 3.2.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) is concircularly flat with respect to Zamkovoy connection, then an η-Ricci-Yamabe soliton (g,ξ,λ,μ,p,q) on M is shrinking, steady or expanding according as $\frac{qr}{2}-\mu < p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$, $\frac{qr}{2}-\mu = p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$ or $\frac{qr}{2}-\mu > p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

Now from (3.8) we have, when $p=0$ then $\lambda = \frac{qr}{2}-\mu$, and when $q=0$ then $\lambda = -\mu - p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$. Thus from theorem 3.2.1 we respectively conclude the following results−

Corollary 3.2.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) is concircularly flat with respect to Zamkovoy connection, then a q-η-Yamabe soliton (g,ξ,λ,μ,q) on M is shrinking, steady or expanding according as $qr<2\mu$, $qr=2\mu$ or $qr>2\mu$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

Corollary 3.2.2. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) is concircularly flat with respect to Zamkovoy connection, then a p-η-Ricci soliton (g,ξ,λ,μ,p) on M is shrinking, steady or expanding according as $-\mu < p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$, $-\mu = p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$ or $-\mu > p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

3.3 η-Ricci-Yamabe solitons on M-projectively flat anti-invariant submanifolds

Here η-Ricci-Yamabe soliton of type (p,q), q-η-Yamabe soliton and p-η-Ricci soliton on a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) is discussed, where M is M-projectively flat with respect to Zamkovoy connection ∇^* given by the equation (2.13).

Since M is M-projectively flat with respect to ∇^*, from (1.6) we have

$$R^*(X,Y)Z = \frac{1}{4n}[S^*(Y,Z)X - S^*(X,Z)Y] + \frac{1}{4n}[g(Y,Z)Q^*X - g(X,Z)Q^*Y]$$

from which we have

$$S^*(Y,Z) = \big(\frac{r^*}{2n+1}\big)g(Y,Z)$$

which is same as the equation (3.5). Hence, proceeding similarly as the previous subsection we get the following results−

Theorem 3.3.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) is M-projectively flat with respect to Zamkovoy connection, then an η-Ricci-Yamabe soliton (g,ξ,λ,μ,p,q) on M is shrinking, steady or expanding according as $\frac{qr}{2} - \mu < p\big(\frac{r+4n^2\alpha^2}{2n+1}\big)$, $\frac{qr}{2} - \mu = p\big(\frac{r+4n^2\alpha^2}{2n+1}\big)$ or $\frac{qr}{2} - \mu > p\big(\frac{r+4n^2\alpha^2}{2n+1}\big)$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

Corollary 3.3.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) is M-projectively flat with respect to Zamkovoy connection, then a q-η-Yamabe soliton (g,ξ,λ,μ,q) on M is shrinking, steady or expanding according as $qr < 2\mu$, $qr = 2\mu$ or $qr > 2\mu$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

Corollary 3.3.2. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) is M-projectively flat with respect to Zamkovoy connection, then a p-η-Ricci soliton (g,ξ,λ,μ,p) on M is shrinking, steady or expanding according as $-\mu < p\big(\frac{r+4n^2\alpha^2}{2n+1}\big)$, $-\mu = p\big(\frac{r+4n^2\alpha^2}{2n+1}\big)$ or $-\mu > p\big(\frac{r+4n^2\alpha^2}{2n+1}\big)$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

3.4 η-Ricci-Yamabe solitons on pseudo projectively flat anti-invariant submanifolds

This subsection deals with the study of η-Ricci-Yamabe soliton of type (p,q), q-η-Yamabe soliton and p-η-Ricci soliton on a $(2n+1)$-dimensional pseudo projectively flat anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) with respect to Zamkovoy connection ∇^* given by the equation (2.13).

Since M is pseudo projectively flat with respect to ∇^*, from (1.7) we have

$$aR^*(X,Y)Z = b[S^*(X,Z)Y - S^*(Y,Z)X] + cr^*[g(X,Z)Y - g(Y,Z)X]$$

which implies that

$$(a+2nb)S^*(Y,Z) = -2cnr^*g(Y,Z).$$

Let $\xi\beta = 0$, then applying (1.8), (2.16) and (2.18) in the above equation we get

$$S(Y,Z) = \left(\tfrac{r-2n\alpha^2}{2n+1}\right)g(Y,Z) + 2n\alpha^2\eta(Y)\eta(Z)$$

which is same as the equation (3.6). Hence proceeding similarly as the subsection 3.2 we reach to the following results−

Theorem 3.4.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is pseudo projectively flat with respect to Zamkovoy connection, then an η-Ricci-Yamabe soliton $(g, \xi, \lambda, \mu, p, q)$ on M is shrinking, steady or expanding according as $\frac{qr}{2} - \mu < p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$, $\frac{qr}{2} - \mu = p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$ or $\frac{qr}{2} - \mu > p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

Corollary 3.4.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is pseudo projectively flat with respect to Zamkovoy connection, then a q-η-Yamabe soliton $(g, \xi, \lambda, \mu, q)$ on M is shrinking, steady or expanding according as $qr < 2\mu$, $qr = 2\mu$ or $qr > 2\mu$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

Corollary 3.4.2. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is pseudo projectively flat with respect to Zamkovoy connection, then a p-η-Ricci soliton $(g, \xi, \lambda, \mu, p)$ on M is shrinking, steady or expanding according as $-\mu < p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$, $-\mu = p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$ or $-\mu > p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

Now, after carefully looking into the results of the four subsections of third section we reach to the following three conclusions.

First, observing theorems 3.2.1, 3.3.1 and 3.4.1 we get the following−

Conclusion 1. If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is (i) concircularly flat, (ii) M-projectively flat or (iii) pseudo projectively flat with respect to Zamkovoy connection, then an η-Ricci-Yamabe soliton $(g, \xi, \lambda, \mu, p, q)$ on M is shrinking, steady or expanding according as $\frac{qr}{2} - \mu < p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$, $\frac{qr}{2} - \mu = p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$ or $\frac{qr}{2} - \mu > p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.

Next, observing corollaries 3.1.1, 3.2.1, 3.3.1 and 3.4.1, we have

Conclusion 2. If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is (i) Ricci flat, (ii) concircularly flat, (iii) M-projectively flat or (iv) pseudo projectively flat with respect to Zamkovoy connection, then a q-η-Yamabe soliton $(g, \xi, \lambda, \mu, q)$ on M is shrinking, steady or expanding according as $qr < 2\mu$, $qr = 2\mu$ or $qr > 2\mu$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.

At last, observing corollaries 3.2.2, 3.3.2 and 3.4.2 we reach to the third conclusion−

Conclusion 3. If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is (i) concircularly flat, (ii) M-projectively flat or (iii) pseudo projectively flat with respect to Zamkovoy connection, then a p-η-Ricci soliton $(g, \xi, \lambda, \mu, p)$ on M is shrinking, steady or expanding according as $-\mu < p\big(\frac{r+4n^2\alpha^2}{2n+1}\big)$, $-\mu = p\big(\frac{r+4n^2\alpha^2}{2n+1}\big)$ or $-\mu > p\big(\frac{r+4n^2\alpha^2}{2n+1}\big)$ respectively, provided that $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.

Finally, we give the following example in which all the results proved in this paper can be verified very easily.

Example. Unit sphere S^5 is a trans-Sasakian manifold of type (-1,0)[41]. We here state an example of an anti-invariant submanifold of S^5 from [44] as:−

Let $J = (a_{ts})$ $(t, s = 1, 2, 3, 4, 5, 6)$ be the almost complex structure of $\mathbb{C}^3$ such that $a_{2i,2i-1} = 1$, $a_{2i-1,2i} = -1$ $(i = 1, 2, 3)$ and all the other components are 0. Let $S^1(\frac{1}{\sqrt{3}}) = \{z \in \mathbb{C} : |z|^2 = \frac{1}{3}\}$. We consider $S^1(\frac{1}{\sqrt{3}}) \times S^1(\frac{1}{\sqrt{3}}) \times S^1(\frac{1}{\sqrt{3}})$ in S^5 in $\mathbb{C}^3$. The position vector X of $S^1 \times S^1 \times S^1$ in S^5 in $\mathbb{C}^3$ has components given by

$$X = \frac{1}{\sqrt{3}}(cos\ u^1, sin\ u^1, cos\ u^2, sin\ u^2, cos\ u^3, sin\ u^3),$$

where u^1, u^2, u^3 are parameters on each $S^1(\frac{1}{\sqrt{3}})$.

Let $X_i = \frac{\partial X}{\partial u^i}$, then we have

$$X_1 = \frac{1}{\sqrt{3}}(-sin\ u^1, cos\ u^1, 0, 0, 0, 0),$$

$$X_2 = \frac{1}{\sqrt{3}}(0, 0, -sin\ u^2, cos\ u^2, 0, 0),$$

$$X_3 = \frac{1}{\sqrt{3}}(0, 0, 0, 0, -sin\ u^3, cos\ u^3).$$

The vector field ξ on S^5 is given by

$$\xi = JX = \frac{1}{\sqrt{3}}(-sin\ u^1, cos\ u^1, -sin\ u^2, cos\ u^2, -sin\ u^3, cos\ u^3).$$

Since $\xi = X_1 + X_2 + X_3$, ξ is tangent to $S^1 \times S^1 \times S^1$. Also the structure tensors (ϕ, ξ, η) of S^5 satisfy

$$\phi X_i = JX_i + \eta(X_i)X,\ i = 1, 2, 3,$$

which shows that ϕX_i is normal to $S^1 \times S^1 \times S^1$ for all i. Thus $S^1 \times S^1 \times S^1$ is an anti-invariant submanifold of S^5.

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