Results on quasi-*Einstein metric

Savita Rani and Ram Shankar Gupta

Abstract. We study the quasi-*Einstein metric on Sasakian and (κ, μ)-manifolds. We show that on Sasakian manifolds the $*$-Ricci operator commutes with tensor field ϕ and quasi-*Einstein Sasakian metric is $*$-flat. Further, we study $(\kappa < 1, \mu)$-manifolds with quasi-*Einstein metric and obtain that such manifold is $*$-flat or locally isometric to $E^{n+1} \times S^n(4)$ or *Einstein.

M.S.C. 2010: 53C25, 53D15.
Key words: quasi-*Einstein metric; contact metric; Sasakian manifold; (κ, μ)-manifold.

1 Introduction

A Riemannian manifold (M, g) is called *Einstein with $S^* = \nu g$, where S^* denotes $*$-Ricci tensor and ν is a constant. A $*$-Ricci soliton is a generalization of *Einstein metric which is given by [13]

$$\frac{1}{2}\mathcal{L}_U g + S^* = \nu g, \tag{1.1}$$

where ν is a constant and U is the potential field. If for a smooth function F, $U = \nabla F$, then (1.1) is called gradient $*$-Ricci soliton.

The $*$-Ricci tensor on an almost contact metric manifold M is defined as follows [12]:

$$S^*(Z, V) = \frac{1}{2}\,\mathrm{trace}(X \mapsto R(Z, \phi V)\phi X),\ \forall\, Z, V, X \in TM, \tag{1.2}$$

where ϕ is a $(1, 1)$-tensor field and R is a Riemann curvature tensor.

The Einstein condition $S = \nu g$ and its generalizations have been studied extensively in contact geometry (see [8, 11, 16]). A generalization of Einstein metric emerged from the m-Bakry-Emery Ricci tensor S^F_m defined as:

$$S^F_m = S + \nabla^2 F - \frac{1}{m} dF \otimes dF, \tag{1.3}$$

Balkan Journal of Geometry and Its Applications, Vol.27, No.2, 2022, pp. 103-113.

has been studied in [4, 5], where $0 < m \leq \infty$, S is the Ricci tensor and $\nabla^2 F$ denotes the Hessian form of F.

A Riemannian manifold (M, g, F, m) is called $m-$quasi-Einstein [7] if it satisfies

$$S + HessF - \frac{1}{m} dF \otimes dF = \nu g, \tag{1.4}$$

for some $m \in \mathbb{Z}^+$. Similarly, we call (M, g, F, m), $m-$quasi-*Einstein if it satisfies

$$S^* + HessF - \frac{1}{m} dF \otimes dF = \nu g. \tag{1.5}$$

If $m = \infty$, then (1.5) gives the gradient $*$-Ricci soliton. A quasi-*Einstein metric is *Einstein if $F =$ constant. We call a quasi-*Einstein metric steady, expanding, or shrinking, respectively, when $\nu = 0$, < 0 or > 0.

Sharma [15] proved that a complete K-contact metric with gradient Ricci soliton is a compact Einstein Sasakian manifold and gradient soliton is expanding. As every Sasakian manifold is a K-contact manifold, this result is also true for Sasakian manifolds. Ghosh et al. extended this result for (κ, μ)-spaces [11]. Quasi-Einstein metrics have been studied in extent for a general manifold, and gap results and rigid properties were obtained in (cf. [4, 17]). Further, Ghosh [9] studied quasi-Einstein contact metric on Sasakian manifolds and on (κ, μ)-spaces. Recently, Chen [6] studied the quasi-Einstein metric on almost cosymplectic manifolds.

However, only very little literature is available on the study of *Einstein and its generalization. This inspired us to study quasi-*Einstein structure associated with contact metric manifolds.

2 Preliminaries and some basic results

An odd-dimensional Riemannian manifold M^{2n+1} is called an almost contact metric manifold if it admits a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g satisfying [1]

$$\phi^2 Z = -Z + \eta(Z)\xi, \quad \eta(\xi) = 1, \tag{2.1}$$

$$\eta \circ \phi = 0, \ \phi\xi = 0, \tag{2.2}$$

$$g(\phi Z, \phi V) = g(Z, V) - \eta(Z)\eta(V), \ \eta(Z) = g(Z, \xi), \tag{2.3}$$

where $Z, V \in TM$. A contact manifold is a Riemannian manifold M^{2n+1} with a global 1-form η called a contact 1-form such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M and $d\eta(Z, V) = g(Z, \phi V)$.

A contact metric manifold is called Sasakian [1] if

$$(\nabla_Z \phi)V = g(Z, V)\xi - \eta(V)Z,$$

where $Z, V \in TM$.

The Riemann curvature tensor R on a Sasakian manifold is given by [1]

$$R(Z, V)\xi = \eta(V)Z - \eta(Z)V. \tag{2.4}$$

On a K-contact manifold [1], we have

$$\nabla_Z\xi = -\phi Z, \tag{2.5}$$

$$Q\xi = 2n\xi, \tag{2.6}$$

$$R(Z,\xi)\xi = Z - \eta(Z)\xi. \tag{2.7}$$

Now, we define self-adjoint operators $h = \frac{1}{2}\mathcal{L}_\xi\phi$ and $l = -R(\xi,.)\xi$, which satisfy [1]:

$$\begin{cases} l\xi = 0 = h\xi, \quad \phi h = -h\phi, \quad g(hX,Y) = g(hY,X), \\ trh = trh\phi = 0. \end{cases} \tag{2.8}$$

Also, for any contact metric manifold [1], we have

$$\nabla_Z\xi = -\phi Z - \phi hZ, \tag{2.9}$$

$$\nabla_\xi h = -\phi h^2 - \phi l + \phi. \tag{2.10}$$

A contact metric manifold M^{2n+1} is said to be (κ,μ)-space if Riemann curvature tensor R satisfies [2]

$$R(Z,V)\xi = \kappa\{\eta(V)Z - \eta(Z)V\} + \mu\{\eta(V)hZ - \eta(Z)hV\}, \tag{2.11}$$

for some $(\kappa,\mu) \in R^2$. If $\kappa = 1$ and $h = 0$, then (κ,μ)-spaces reduce to the Sasakian manifolds. The non-Sasakian manifolds have proven to be more interesting as there exists the unit tangent sphere bundle of a flat Riemannian manifold with the usual contact metric structure as an example of non-Sasakian spaces of this type. Moreover, this type of space is invariant under D-homothetic transformations. These are the driving factor for the study of this type of manifold. Boeckx proved that non-Sasakian contact metric manifold satisfying (2.11) is completely determined locally by its dimension for the constant values of κ, μ [3].

3 Quasi-*Einstein metric on a Sasakian manifold

In this section, we study the quasi-*Einstein metric on a Sasakian manifold. On a Sasakian manifold M^{2n+1} [10]

$$Q^*Z = QZ - (2n-1)Z - \eta(Z)\xi, \tag{3.1}$$

$$r^* = r - 4n^2, \tag{3.2}$$

where Q^*, Q, r^* and r are $*$-Ricci operator, Ricci operator, $*$-scalar curvature and scalar curvature respectively

Lemma 3.1. *[14] The curvature tensor R of a Riemannian manifold M^{2n+1} with quasi-*Einstein metric satisfies*

$$\begin{aligned} R(Z,V)\nabla F &= (\nabla_V Q^*)Z - (\nabla_Z Q^*)V - \frac{\nu}{m}\{(ZF)V - (VF)Z\} \\ &+ \frac{1}{m}\{(ZF)Q^*V - (VF)Q^*Z\}, \end{aligned} \tag{3.3}$$

$$S(V,\nabla F)=\frac{1}{2}Vr^{*}+\frac{2n\nu}{m}(VF)+\frac{1}{m}\{(Q^{*}V)F-r^{*}(VF)\}, \tag{3.4}$$

for any $Z,V\in TM$, where ∇F is gradient of F.

Proof. From (1.5), we have

$$Q^{*}V+\nabla_{V}\nabla F-\frac{1}{m}(VF)\nabla F=\nu V. \tag{3.5}$$

We know that

$$R(Z,V)X=\nabla_{Z}\nabla_{V}X-\nabla_{V}\nabla_{Z}X-\nabla_{[Z,V]}X. \tag{3.6}$$

Using (3.5) in (3.6), we obtain (3.3). Further, contracting (3.3) over Z, we get (3.4). □

Lemma 3.2. *[9] If $F\in C^{\infty}$ on a contact metric manifold M such that $dF=(\xi F)\eta$, where d denotes the exterior differentiation. Then F is constant on M.*

Next,

Lemma 3.3. *On a Sasakian manifold M^{2n+1}*

$$Q^{*}\phi=\phi Q^{*}. \tag{3.7}$$

Proof. Putting $Z=\xi$ in (3.1) and using (2.6), we obtain

$$Q^{*}\xi=0. \tag{3.8}$$

Differentiating (3.8) with respect to Z, we get

$$(\nabla_{Z}Q^{*})\xi=Q^{*}\phi Z. \tag{3.9}$$

If $V,Z\in\{\xi^{\perp}\}$, then with respect to ϕ basis $\{Z_i,Z_{n+i}=\phi Z_i,\xi\}$, we have

$$\begin{aligned} g(Q^{*}\phi Z,\phi V) &= \frac{1}{2}\sum_{i=1}^{2n}g(R(\phi Z,\phi^{2}V)\phi Z_i,Z_i)+\frac{1}{2}g(R(\phi Z,\phi^{2}V)\phi\xi,\xi)\\ &= \frac{1}{2}\sum_{i=1}^{2n}g(R(V,\phi Z)\phi Z_i,Z_i), \end{aligned}$$

which gives

$$g(Q^{*}\phi Z,\phi V)=g(Q^{*}V,Z). \tag{3.10}$$

From (3.1) and using the fact that Q is symmetric, we get

$$g(Q^{*}V,Z)=g(Q^{*}Z,V). \tag{3.11}$$

From (3.10) and (3.11), we get

$$-\phi Q^{*}\phi Z=Q^{*}Z,$$

wherein multiplying with 'ϕ' on both sides and using (2.1), we find

$$Q^{*}\phi Z=\phi Q^{*}Z,$$

whereby proof is complete. □

Theorem 3.4. *Let M^{2n+1} be a Sasakian manifold, then $\mathcal{L}_\xi S^* = 0$ and $\nabla_\xi Q^* = 0$.*

Proof. It is well-known that $\mathcal{L}_\xi g = 0$ on a Sasakian manifold. Also, we have

$$\begin{aligned}
&(3.12)\ \mathcal{L}_\xi(R_{Z,\phi V}\phi X) &=& \ \mathcal{L}_\xi(\nabla_Z\nabla_{\phi V}\phi X - \nabla_{\phi V}\nabla_Z\phi X - \nabla_{[Z,\phi V]}X),\\
&(3.13)\quad R_{\mathcal{L}_\xi Z,\phi V}\phi X &=& \ \nabla_{\mathcal{L}_\xi Z}\nabla_{\phi V}\phi X - \nabla_{\phi V}\nabla_{\mathcal{L}_\xi Z}\phi X - \nabla_{[\mathcal{L}_\xi Z,\phi V]}\phi X,\\
&(3.14)\quad R_{Z,\mathcal{L}_\xi\phi V}\phi X &=& \ \nabla_Z\nabla_{\mathcal{L}_\xi\phi V}\phi X - \nabla_{\mathcal{L}_\xi\phi V}\nabla_Z\phi X - \nabla_{[Z,\mathcal{L}_\xi\phi V]}\phi X,\\
&(3.15)\quad R_{Z,\phi V}\mathcal{L}_\xi\phi X &=& \ \nabla_Z\nabla_{\phi V}(\mathcal{L}_\xi\phi X) - \nabla_{\phi V}\nabla_Z(\mathcal{L}_\xi\phi X) - \nabla_{[Z,\phi V]}\mathcal{L}_\xi\phi X,
\end{aligned}$$

$\forall\ Z, V, X \in TM$.

From (3.12)∼(3.15), we obtain

$$\mathcal{L}_\xi(R_{Z,\phi V}\phi X) = R_{\mathcal{L}_\xi Z,\phi V}\phi X + R_{Z,\mathcal{L}_\xi\phi V}\phi X + R_{Z,\phi V}\mathcal{L}_\xi\phi X,$$

which gives $(\mathcal{L}_\xi R)_{Z,\phi V}\phi X = 0$ and contracting it over X, we get $\mathcal{L}_\xi S^* = 0$. This implies

$$\mathcal{L}_\xi(g(Q^*Z, V)) - S^*(\mathcal{L}_\xi Z, V) - S^*(Z, \mathcal{L}_\xi V) = 0. \tag{3.16}$$

Simplifying (3.16), we find

$$[\xi, Q^*Z] - Q^*([\xi, Z]) = 0. \tag{3.17}$$

Using (2.5) in (3.17), we get

$$\nabla_\xi Q^* = Q^*\phi - \phi Q^*. \tag{3.18}$$

Using Lemma 3.3 in (3.18), we obtain the result. □

Theorem 3.5. *Let M^{2n+1} be a Sasakian manifold satisfying (1.5), then F is constant and quasi-***Einstein soliton is steady.*

Proof. Taking inner product of (3.3) with ξ and using (3.8) and (3.9), we get

$$g(R(Z, V)\nabla F, \xi) = -2g(Q^*\phi Z, V) + \frac{\nu}{m}\{(VF)\eta(Z) - (ZF)\eta(V)\}. \tag{3.19}$$

Putting $V = \xi$ in (3.19) and using (2.4), we get

$$(\frac{\nu}{m} - 1)((\xi F)\eta(Z) - (ZF)) = 0. \tag{3.20}$$

Now, we claim that $\nu \neq m$. Infact, if $\nu = m$, then putting $Z = \xi$ in (3.3), we get

$$\begin{aligned}
R(\xi, V)\nabla F &= (\nabla_V Q^*)\xi - (\nabla_\xi Q^*)V - \{(\xi F)V - (VF)\xi\}\\
&+ \frac{1}{m}\{(\xi F)Q^*V - (VF)Q^*\xi\}.
\end{aligned} \tag{3.21}$$

From (2.4), we obtain

$$R(\xi, V)\nabla F = (VF)\xi - (\xi F)V. \tag{3.22}$$

Using (3.22) in (3.21), we get

$$(\nabla_V Q^*)\xi - (\nabla_\xi Q^*)V + \frac{1}{m}\{(\xi F)Q^*V - (VF)Q^*\xi\} = 0. \tag{3.23}$$

Using Theorem 3.4, (3.8) and (3.9) in (3.23), we obtain

$$Q^*\phi V + \frac{\xi F}{m}Q^*V = 0. \tag{3.24}$$

Taking inner product of (3.24) with Z, we find

$$g(Q^*\phi V, Z) + \frac{\xi F}{m}g(Q^*V, Z) = 0. \tag{3.25}$$

Interchanging V and Z in (3.25), we get

$$g(Q^*\phi Z, V) + \frac{\xi F}{m}g(Q^*Z, V) = 0. \tag{3.26}$$

Subtracting (3.26) from (3.25), we obtain

$$(Q^*\phi + \phi Q^*)V = 0, \tag{3.27}$$

$\forall\ V \in TM$. Using Lemma 3.3 in (3.27), we get $\phi Q^*V = 0$. Which further using (2.1) gives $Q^*V = 0$. Hence $r^* = 0$. Using this in (3.4), we find

$$S(V, \nabla F) = 2n(VF). \tag{3.28}$$

On the other hand from (3.1), we get

$$QV = (2n-1)V + \eta(V)\xi. \tag{3.29}$$

Using (3.29) in (3.28), we obtain $VF = (\xi F)\eta(V)$. Which gives F constant by use of Lemma 3.2. Therefore, from (1.5), we obtain $\nu = 0$, that gives $m = 0$, a contradiction of the fact that $m > 0$.

Hence $\nu \neq m$ and from (3.20), we have $ZF = (\xi F)\eta(Z)$. By using Lemma 3.2 we find that

$$F = \text{constant}. \tag{3.30}$$

From (3.8), we see that

$$S^*(\xi, \xi) = 0. \tag{3.31}$$

Using (3.30) and (3.31) in (1.5), we get $\nu = 0$. Hence quasi-*Einstein soliton is steady. □

Corollary 3.6. *Let M^{2n+1} be a Sasakian manifold satisfying (1.5), then M is $*$-Ricci flat, η-Einstein and scalar curvature is constant.*

Proof. Using $\nu = 0$ and (3.30) in (1.5), we see that

$$Q^* = 0. \tag{3.32}$$

Using (3.32) in (3.1), we find

$$QZ = (2n-1)Z + \eta(Z)\xi. \tag{3.33}$$

Hence M is η-Einstein. Further, from (3.33) we obtain $r = 4n^2$. Hence scalar curvature is constant. □

Now, we give an example

Example 3.1. Consider the manifold $M^3 = \{(x,y,z) \in \mathbb{R}^3 : \ x, y \neq 0\}$ endowed with the structure $\{\phi, \xi, \eta, g\}$

$$\begin{cases} \phi(e_1) = e_2,\ \phi(e_2) = -e_1,\ \phi(e_3) = 0,\ \eta = \frac{4}{4+3x^2+3y^2}(ydx - xdy) + dz, \\ e_3 = \xi = \frac{\partial}{\partial z},\ g = \frac{1}{(1+\frac{3x^2}{4}+\frac{3y^2}{4})^2}(dx \otimes dx + dy \otimes dy) + \eta \otimes \eta, \\ e_1 = (1 + \frac{3x^2+3y^2}{4})\frac{\partial}{\partial x} - y\frac{\partial}{\partial z}, e_2 = (1 + \frac{3x^2+3y^2}{4})\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}. \end{cases} \tag{3.34}$$

We subsequently have

$$[e_l, e_3] = 0 \text{ for } l = 1,2;\ [e_1, e_2] = -\frac{3y}{2}e_1 + \frac{3x}{2}e_2 + 2e_3. \tag{3.35}$$

$$\begin{cases} \nabla_{e_1}e_1 = \frac{3y}{2}e_2, \nabla_{e_2}e_1 = -\frac{3x}{2}e_2 - e_3, \nabla_{e_3}e_1 = -e_2, \nabla_{e_1}e_2 = -\frac{3y}{2}e_1 + e_3, \\ \nabla_{e_2}e_2 = \frac{3x}{2}e_1, \nabla_{e_3}e_2 = e_1,\ \nabla_{e_1}e_3 = -e_2,\ \nabla_{e_2}e_3 = e_1,\ \nabla_{e_3}e_3 = 0. \end{cases} \tag{3.36}$$

$$\begin{cases} R(e_l, e_p)e_s = 0, l \neq p \neq s,\ l,p,s = 1,2,3,\ R(e_l, e_3)e_3 = e_l, l = 1,2, \\ R(e_l, e_3)e_l = -e_3,\ l = 1,2,\ R(e_l, e_p)e_p = 0,\ l \neq p,\ l,p = 1,2, \\ S^*(e_l, e_p) = 0,\ l,p = 1,2,3. \end{cases} \tag{3.37}$$

Further, by using (3.37) in (1.5), we infer the following system of differential equations:

(3.38)

$$\begin{cases} \alpha^2 F_{xx} - 2y\alpha F_{zx} + y^2 F_{zz} - \frac{3y\alpha}{2}F_y - \frac{3yx}{2}F_z - \frac{\alpha^2}{m}F_x^2 + \frac{2y\alpha}{m}F_xF_z \\ +\frac{3x\alpha}{2}F_x - \frac{y^2}{m}F_z^2 = \nu; \quad F_{zz} - \frac{1}{m}F_z^2 = \nu, \\ \alpha^2 F_{yy} + 2x\alpha F_{zy} + x^2 F_{zz} - \frac{3x\alpha}{2}F_x + \frac{3xy}{2}F_z + \frac{3y\alpha}{2}F_y - \frac{\alpha^2}{m}F_y^2 \\ -\frac{2x\alpha}{m}F_yF_z - \frac{x^2}{m}F_z^2 = \nu, \\ \frac{3x\alpha}{2}F_y + \alpha^2 F_{xy} - y\alpha F_{zy} + \alpha F_z + x\alpha F_{xz} - yxF_{zz} + \frac{3y\alpha}{2}F_x - \frac{3y^2}{2}F_z - F_z \\ -\frac{\alpha^2}{m}F_xF_y + \frac{y\alpha}{m}F_zF_y - \frac{x\alpha}{m}F_zF_x + \frac{xy}{m}F_z^2 = 0, \\ \alpha F_{xz} - yF_{zz} + \alpha F_y + xF_z - \frac{\alpha}{m}F_xF_z + \frac{y}{m}F_z^2 = 0, \\ \alpha F_{yz} + xF_{zz} - \alpha F_x + yF_z - \frac{\alpha}{m}F_yF_z - \frac{x}{m}F_z^2 = 0, \end{cases}$$

where $\alpha = 1 + \frac{3x^2+3y^2}{4}$ and indices denote the derivative with respect to x, y and z.

Case A: Assume that the potential function F depends only on x, so (3.38) reduces to

$$\nu = 0, \quad F_x = 0. \tag{3.39}$$

Therefore, quasi-*Einstein soliton is steady and F is constant.

Case B: Now, assume that the potential function F depends only on y, so (3.38) reduces to

$$\nu = 0, \quad F_y = 0. \tag{3.40}$$

That is, quasi-*Einstein soliton is steady and F is constant.

Case C: Further, assume that the potential function F depends only on z, so (3.38) reduces to

$$\nu = 0, \quad F_z = 0. \tag{3.41}$$

Which implies quasi-*Einstein soliton is steady and F is constant.

4 Quasi-*Einstein $(\kappa < 1, \mu)$ spaces

In this section, we study quasi-*Einstein $(\kappa < 1, \mu)$-spaces.

For a (κ, μ)-space [2]

$$h^2 = -(1-\kappa)\phi^2, \tag{4.1}$$

where $\kappa \leq 1$.

Theorem 4.1. *[3] Let $(M^{2n+1}, \xi, \eta, \phi, g)$ be a non-Sasakian $(\kappa, \mu)-$space$(\kappa < 1)$. Then its Riemann curvature tensor R is given by*

$$\begin{aligned} g(R(Z,V)Y,W) = {} & (1-\frac{\mu}{2})R_1 + R_2 + \left(\frac{1-\frac{\mu}{2}}{1-\kappa}\right)R_3 - \frac{\mu}{2}R_4 + \left(\frac{\kappa-\frac{\mu}{2}}{1-\kappa}\right)R_5 \\ & +\mu g(\phi Z,V)g(\phi Y,W) + \eta(Z)\eta(W)R_6 - \eta(Z)\eta(Y)R_7 \\ & +\eta(V)\eta(Y)R_8 - \eta(V)\eta(W)R_9, \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} R_1 &= g(V,Y)g(Z,W) - g(Z,Y)g(V,W), \\ R_2 &= g(V,Y)g(hZ,W) - g(Z,Y)g(hV,W) \\ & \quad -g(V,W)g(hZ,Y) + g(Z,W)g(hV,Y), \\ R_3 &= g(hV,Y)g(hZ,W) - g(hZ,Y)g(hV,W), \\ R_4 &= g(\phi V,Y)g(\phi Z,W) - g(\phi Z,Y)g(\phi V,W), \\ R_5 &= g(\phi hV,Y)g(\phi hZ,W) - g(\phi hV,W)g(\phi hZ,Y), \\ R_6 &= (-1+\kappa+\frac{\mu}{2})g(V,Y) + (-1+\mu)g(hV,Y), \\ R_7 &= (-1+\kappa+\frac{\mu}{2})g(V,W) + (-1+\mu)g(hV,W), \\ R_8 &= (-1+\kappa+\frac{\mu}{2})g(Z,W) + (-1+\mu)g(hZ,W), \\ R_9 &= (-1+\kappa+\frac{\mu}{2})g(Z,Y) + (-1+\mu)g(hZ,Y), \end{aligned}$$

$\forall\ Z, V, Y, W \in TM$.

Using (1.2) and (4.2) we find that

$$Q^* Z = (\kappa + n\mu)\phi^2 Z, \tag{4.3}$$

and $*$-scalar curvature is given by

$$r^* = -2n(n\mu + \kappa), \tag{4.4}$$

which is constant.

Theorem 4.2. *Let M^{2n+1} be a $(\kappa < 1, \mu)$-space satisfying (1.5). Then, M is a $*$-flat or locally isometric to $E^{n+1} \times S^n(4)$ or *Einstein.*

Proof. From (4.3), we have

$$Q^* \xi = 0. \tag{4.5}$$

Differentiating (4.5) along $Z \in TM$ and using (2.9), we have

$$(\nabla_Z Q^*)\xi = Q^* \phi Z + Q^* \phi h Z. \tag{4.6}$$

Taking inner product of (3.3) with ξ and using (4.5), we obtain

$$\begin{aligned} g(R(Z,V)\nabla F, \xi) &= g((\nabla_V Q^*)Z - (\nabla_Z Q^*)V, \xi) \\ &\quad - \frac{\nu}{m}\{(ZF)\eta(V) - (VF)\eta(Z)\}. \end{aligned} \tag{4.7}$$

Using (2.11) and (4.6) in (4.7) and then replacing Z by ϕZ and V by ϕV, we get

$$(Q^*\phi + \phi Q^*)Z - hQ^*\phi Z - \phi Q^* h Z = 0, \tag{4.8}$$

$\forall\ Z \in TM$. Further, using (2.1) and (4.3) in (4.8), we find that

$$\kappa + n\mu = 0. \tag{4.9}$$

Putting $V = \xi$ in (4.7), using (2.11), $\langle (\nabla_Z Q^*)\xi, \xi\rangle = 0$, $\langle (\nabla_\xi Q^*)Z, \xi\rangle = 0$, we get

$$h\nabla F = \sigma(\nabla F - (\xi F)\xi), \tag{4.10}$$

where $\sigma = \frac{\nu - \kappa m}{m\mu}$ is constant. Differentiating (4.10) along $Z \in TM$ and using (2.9), (3.5) and (4.10), we obtain

$$\begin{aligned} (\nabla_Z h)\nabla F - hQ^*Z - \frac{\sigma(\xi F)}{m}(ZF)\xi + \nu h Z \\ = \sigma[\nu Z - Q^* Z - (Z(\xi F))\xi + (\xi F)(\phi Z + \phi h Z)]. \end{aligned} \tag{4.11}$$

Using (4.5) in (3.5), we get

$$\xi(\xi F) - \frac{1}{m}(\xi F)^2 = \nu. \tag{4.12}$$

On the other hand, using (2.11), we get $l = \mu h - \kappa\phi^2$. Using this and (4.1) in (2.10), we find

$$\nabla_\xi h = -\mu\phi h. \tag{4.13}$$

Putting ξ in place of Z in (4.11) and using (4.12) and (4.13), we find

$$\mu h \nabla F = 0. \tag{4.14}$$

From (4.14), we have either $\mu = 0$ or $\mu \neq 0$.

Case A: Let $\mu = 0$. Then using this in (4.9), we obtain $\kappa = 0$. Thus $R(Z, V)\xi = 0$ and hence in dimension 3, M is $*$-flat and in higher dimension M is locally isometric to $E^{n+1} \times S^n(4)$.[1]

Case B: In this case, we have $h\nabla F = 0$. Using $h^2 = (\kappa - 1)\phi^2$, we get

$$0 = h^2\nabla F = (\kappa - 1)\phi^2\nabla F. \tag{4.15}$$

Since $\kappa < 1$, we have $\nabla F = (\xi F)\xi$. Hence from Lemma 3.2, we get F is constant. Consequently from (3.5), $Q^*Z = \nu Z$. Hence M is $*$-Einstein. □

Acknowledgements. The first author is thankful to GGSIP University for research fellowship F.No. GGSIPU/DRC/2021/685. This work of corresponding author is partially supported by FRGS grant for the year 2021-2022, F. No. GGSIPU/DRC/FRGS/2021/594/5.

References

[1] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, 2nd Ed., Prog. Math. 203, Birkhäuser, Boston 2010.

[2] D. E. Blair, T.Koufogiorgos and B. J. Papantoniou, *Contact metric manifolds satisfying a nullity condition*, Israel J. Math. 91, 1-3 (1995), 189-214.

[3] E. Boeckx, *A full classification of contact metric (κ, μ)-spaces*, Illinois J. Math. 44, 1 (2000), 212-219.

[4] J. Case, *On the non-existence of quasi-Einstein metrics*, Pacific J. Math. 248, 2 (2010), 227-284.

[5] J. Case, Y. Shu and G. Wei, *Rigidity of quasi-Einstein metrics*, Differential Geom. Appl. 29, 1 (2010), 93-100.

[6] X. Chen, *Quasi-Einstein structures and almost cosymplectic manifolds*, RACSAM 114, 2 (2020).(14pages)

[7] X. Cui and X. Chen, *Quasi-Einstein hypersurfaces of complex space forms*, Adv. Math. Phys. 2020 (2020), 9pp.

[8] A. Ghosh, *Ricci solitons and contact metric manifolds*, Glasg. Math. J. 55, 1 (2013), 123-130.

[9] A. Ghosh, *Quasi-Einstein contact metric manifolds*, Glasg. Math. J. 57, 3 (2015), 569-577.

[10] A. Ghosh and D. S. Patra, *$*$-Ricci soliton within the frame-work of Sasakian and (κ, μ)-contact manifold*, Int. J. Geom. Methods Mod. Phys. 15, 7 (2018). ID 1850120

[11] A. Ghosh, R. Sharma and J. T. Cho, *Contact metric manifolds with η-parallel torsion tensor*, Ann. Global Anal. Geom. 34, 3 (2008), 287-299.

[12] T. Hamada, *Real hypersurfaces of complex space forms in terms of Ricci $*$-tensor*, Tokyo J. Math. 25 (2002), 473-483.

[13] G. Kaimakamis and K. Panagiotidou, *$*$-Ricci solitons of real hypersurfaces in non-flat complex space forms*, J. Geom. Phys. 86 (2014), 408-413.

[14] S. Rani and R. S. Gupta, *Quasi-*Einstein metric on Kenmotsu manifolds*, J. Int. Acad. Phys. Sci. 25, 4 (2021), 509-516.

[15] R. Sharma, *Certain results on K-contact and (κ, μ)-contact manifolds*, J. Geom. 89, 1-2 (2008), 138-147.

[16] S. Tanno, *The topology of contact Riemannian manifolds*, Illinois J. Math. 12, 4 (1968), 700-717.

[17] L. F. Wang, *Rigid properties of quasi-Einstein metrics*, Proc. Amer. Math. Soc. 139, 10 (2011), 3679-3689.

Authors' address:

Savita Rani, Ram Shankar Gupta
University School of Basic and Applied Sciences,
Guru Gobind Singh Indraprastha University,
Sector-16C, Dwarka, New Delhi-110078, India.
E-mail: mansavi.14@gmail.com, ramshankar.gupta@gmail.com(Corresponding author)