A note on the compatibility of G_2-structures with symplectic structures

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Abstract. In this paper we study the relationship between G_2-structures and 8-dimensional symplectic structures. We introduce the notion of compatibility of these structures. It is shown that a 7-manifold with G_2 structure can be embedded into an 8-dimensional symplectic manifold and with additional conditions, this symplectic structure can be chosen compatible with G_2-structure.

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1 Introduction

In the classification of Riemannian holonomy groups due to Berger, there are two exceptional cases: G_2 and $Spin(7)$. In this paper we concern with manifolds of exceptional holonomy group G_2. The compact, simple and simply connected Lie group G_2 can be defined as the group of linear transformations of $\mathbb{R}^7$ that preserves the Euclidean metric and a vector cross product. A G_2-structure (or an almost G_2-structure) on a 7-dimensional manifold Q is a nondegenerate three form Ω on it . A G_2-structure induces a unique Riemannian metric g on Q. If furthermore $Hol(g) \subseteq G_2$, then Q is called a G_2-manifold.
The geometry of G_2-manifolds has been studied extensively in several papers ([8],[4],[5],[11]). Akbulut and Salur in [1] studied the relationship between Calabi-Yau geometry and G_2 geometry. By definition a Calabi-Yau manifold is a Kähler manifold X with $c_1(X) = 0$(of course there are some inequivalent definitions). Thus a Calabi-Yau manifold is a special symplectic manifold. On the other hand the relation between symplectic geometry and contact geometry is obvious. So it is natural to expect a connection between G_2 geometry from one hand and symplectic geometry and contact geometry from another hand. In [2] the relationship between G_2 geometry and contact geometry has been studied. The relationship between G_2 geometry and symplectic geometry emerged in [9] for the first time. In [9], by using methods of spin geometry, Fernandez and Gray showed that $T^*Q \times \mathbb{R}$ admits a closed G_2-structure,

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when Q is an oriented 3-dimensional manifold and in [7], Cho and Salur computed this G_2-structure as $\Omega = Re\Theta + \omega \wedge dt$, where Θ is a certain complex valued 3-form and ω is the standard symplectic form on T^*Q.
In this paper we investigate the connection between symplectic structures and G_2-structures. The paper is organized as follows:
In section 2 we present some preliminaries. In section 3, the compatibility of symplectic structures with G_2-structures and its relation with compatibility of contact structures and G_2-structures, will be study. In particular the following theorems will be proved.
Theorem. *Let (Q, α) be a 7-dimensional contact manifold and Ω be a G_2-structure on Q compatible with α. Then Ω is compatible with symplectic form $\omega = d(e^\theta \alpha)$ on $M = Q \times \mathbb{R}$, where θ denotes the coordinate on $\mathbb{R}$.*

Theorem. Let (Q, Ω) be a hypersurface of symplectic manifold (M, ω) and ω is compatible with Ω. If furthermore Q is of contact type then Ω is compatible with contact structure of Q.

In section 4 the existence of symplectic structures on $Q \times \mathbb{R}$ and $Q \times \mathbf{S^1}$ is discussed, when Q is a 7-manifold with G_2-structure. The main results of this section are as follows:

Theorem. Let Q be a 7-dimensional manifold with a G_2-structure Ω. Then $M = Q \times \mathbb{R}$ admits an almost symplectic structure compatible with Ω. The same statement is true for $M = Q \times \mathbf{S^1}$.

Theorem. *Let Q be a connected 7-dimensional manifold with a G_2-structure. Then $M = Q \times \mathbb{R}$ is a symplectic manifold. The same statement is true for $M = Q \times \mathbf{S^1}$, when Q is furthermore noncompact.*

Theorem. *In previous Theorem , if R is a vector field on Q such that $\iota_R \varphi$ is exact, then $Q \times \mathbb{R}$ and $Q \times \mathbf{S^1}$ admits a symplectic structure compatible with φ.*

2 Preliminaries

2.1 G_2-structures

In this section V is a finite dimensional real vector space and $\langle .,. \rangle$ is an inner product on V.

Definition 2.1. A skew symmetric bilinear map

$$V \times V \to V : (u, v) \mapsto u \times v \qquad (2.1)$$

is called a cross product if it satisfies

$$\langle u \times v, u \rangle = \langle u \times v, v \rangle = 0,$$

$$|u \times v|^2 = |u|^2 |v|^2 - \langle u, v \rangle^2$$

for all $u, v \in V$.

It is well known that if V admits a non vanishing cross product, then dimension of V is 3 or 7.

Lemma 2.1. *If $\times$ be a cross product on V, then the map $\Omega : V \times V \times V \to \mathbb{R}$, defined by*

$$\Omega(u, v, w) = \langle u \times v, w\rangle, \tag{2.2}$$

is an alternating 3*-form the so called the* **associative calibration** *of* V*.*

Definition 2.2. Let V be a finite dimensional real vector space. A 3-form $\Omega \in \Lambda^3 V^*$ is called nondegenerate if, $\iota_v\Omega = 0$ implies that $v = 0$. An inner product on V is called compatible with Ω if the map (2.1) defined by (2.2) is a cross product.

Theorem 2.2. *Let V be a* 7*-dimensional real vector space and $\Omega \in \Lambda^3 V^*$. Then:*
(i) Ω is nondegenerate if and only if it admits a compatible inner product.
(ii) The inner product in (i), if it exists, is uniquely determined by Ω.
(iii) If $\Omega_1, \Omega_2 \in \Lambda^3 V^*$ are nondegenerate, then there is an automorphism $g : V \to V$ such that $g^*\Omega_2 = \Omega_1$.
(iv) If Ω is compatible with the inner product $\langle ., .\rangle$, then there is an orientation on V such that the associated volume form $dvol \in \Lambda^7 V^*$ satisfies

$$\iota_u\Omega \wedge \iota_v\Omega \wedge \Omega = 6\langle u, v\rangle dvol \tag{2.3}$$

for all $u, v \in V$.

Example 2.3. Identify $\mathbb{R}^7$ with ImO of imaginary part of octonions. then for $u, v \in \mathbb{R}^7$

$$u \times v = imuv$$

defines a cross product with respect to the standard inner product $\langle ., .\rangle$ on $\mathbb{R}^7$. The associated calibration Ω_0 reads

$$\Omega_0 = e^{123} + e^{145} + e^{167} + e^{167} + e^{246} - e^{275} - e^{347} - e^{356} \tag{2.4}$$

where $e^{ijk} = dx_i \wedge dx_j \wedge dx_k$.

Let $(V, \langle ., .\rangle)$ be an inner product space endowed with a cross product $\times$ and Ω be it's associated calibration. The sub group of $Gl(V)$ that preserve Ω is denoted by

$$G(V, \Omega) = \{g \in Gl(V) : g^*\Omega = \Omega\}.$$

The group $G(\mathbb{R}^7, \Omega_0)$ will be denoted simply by G_2. According Theorem 2.4(iii), for an arbitrary nondegenerate 3-form Ω on a 7-dimensional vector space V, The group $G(V, \Omega)$ is isomorphic to G_2.

Definition 2.4. A nondegenerate 3-form Ω on a smooth 7-dimensional manifold Q is called a G_2-structure(or an almost G_2-structure).

Remark 2.5. By Theorem 2.4(i, iv) a G_2 structure Ω on Q induces a unique Riemannian metric and a unique orientation on Q. Thus each tangent space T_pQ of Q admits a cross product defined by (2.2).

For more information about G_2-structures we refer to [13] and [10].

2.2 Almost symplectic structures and Gromov's Theorem

Let M be a $2n$-dimensional smooth manifold. A nondegenerat two form ω on M is called an almost symplectic structure. If furthermore ω is closed, then ω is called a symplectic structure on M. It is well known that an almost symplectic manifold (M, ω) admits almost complex structures J tamed by ω,i.e, $\omega(v, Jv) > 0$ for all nonzero v in TM. The space of such almost complex structures is contractible. The following theorem, due, to Gromov, states that an almost symplectic structure is homotopic to a smplectic structure. For a proof of this theorem we refer to Theorem 7.34 of [12].

Theorem 2.3. *(Gromov's Theorem) Let M be an open $2n$ dimensional manifold. Let τ be an almost symplectic structure on M and $a \in H^2(M, \mathbb{R})$. There exists a family of almost symplectic forms τ_t on M such that $\tau_0 = \tau$ and τ_1 is a symplectic form that represents the class a.*

2.3 Almost contact structures

Let M be an $(2n+1)$ dimensional smooth manifold. An almost contact structure on M is a triple (J, R, α) consists of a field J of endomorphisms of the tangent bundle, a vector field R and a 1-form α satisfying
1) $\alpha(R) = 1$,
2) $J^2(X) = -X + \alpha(X)R$,
for all X in TM.

Let (J, R, α) be an almost contact structure on M. A Riemannian metric g on M is called a compatible metric if

$$g(Ju, Jv) = g(u, v) - \alpha(u)\alpha(v),$$

for all u, v in TM. An **almost contact metric structure** on M is a quadruple (J, R, α, g), where (J, R, α) is an almost contact symplectic structure and g is a compatible metric.
It is well known that every manifold with an almost contact structure admits a compatible metric. For more details we refer to [3].

3 Compatibility of G_2-structures and symplectic structures

In [2], two kind of compatibility of contact structures and G_2 structures on a manifold, when both of them exist, has been defined. Here we need one of them, the so called A-compatibility, which we simply call it compatible.

Definition 3.1. Let Ω be a G_2-structure on 7-dimensional manifold Q. A contact structure ξ on Q is said to be compatible with Ω if there exist a vector field R on Q, a contact form α for ξ and a nonzero function $f : Q \to \mathbb{R}$ such that $d\alpha = \iota_R \Omega$ and fR is the Reeb vector field of a contact form for ξ.

In this section we consider a hypersurface of a symplectic 8-dimensional manifold, which admits a G_2-structure. We want to know how these two structures are related.

Definition 3.2. Let (M, ω) be an eight dimensional (almost) symplectic manifold and Q be a hypersurface of M with G_2-structure Ω. The (almost) symplectic form ω is called compatible with Ω if there is a vector field $R : Q \to TQ$ satisfying

$$j^*(\omega) = \iota_R \Omega,$$

where $j : Q \hookrightarrow M$ is the inclusion map.

The following example explains the motivation of this definition.

Example 3.3. Let $(x_1, ..., x_8)$ denotes the coordinates on $\mathbb{R}^8$ and consider the symplectic form ω on $\mathbb{R}^8$ as follows:

$$\omega = dx_1 \wedge dx_8 + dx_2 \wedge dx_3 + dx_4 \wedge dx_5 + dx_6 \wedge dx_7.$$

Consider $\mathbb{R}^7$ as a hypersurface in $\mathbb{R}^8$ with coordinates $(x_1, ..., x_7)$. Let Ω_0 be the standard G_2-structure on $\mathbb{R}^7$. If $R = \frac{\partial}{\partial x_1}$, we have

$$\iota_R \Omega_0 = dx_2 \wedge dx_3 + dx_4 \wedge dx_5 + dx_6 \wedge dx_7 = j^*(\omega),$$

where $j : \mathbb{R}^7 \to \mathbb{R}^8$ is defined by $j(x_1, ..., x_7) = (x_1, ..., x_7, 0)$.

Theorem 3.1. *Let (Q, α) be a 7-dimensional contact manifold and Ω be a G_2-structure on Q compatible with α. Then Ω is compatible with symplectic form $\omega = d(e^\theta \alpha)$ on $M = Q \times \mathbb{R}$, where θ denotes the coordinate on $\mathbb{R}$.*

Proof. By assumption, there is a vector field R on Q such that $\iota_R \Omega = d\alpha$. but $d\alpha = j^*(\omega)$. □

Example 3.4. Let Q be a 3-dimensional oriented Riemannian manifold. Consider the coordinates $(x_1, x_2, x_3, y_1, y_2, y_3)$ on the cotangent bundle T^*Q. Assume $\omega = -d\lambda_{can}$ be the standard symplectic form on T^*Q, where $\lambda_{can} = \sum y_i dx_i$ is the canonical 1-form on T^*Q. If t denotes the coordinate on $\mathbb{R}$, define the 3-form Ω on T^*Q by

$$\Omega = Re(\Theta) + dt \wedge \omega,$$

where $\Theta = (dx_1 + idy_1) \wedge (dx_2 + idy_2 \wedge (dx_3 + idy_3))$ is a complex valued $(3, 0)$ form on T^*Q. In [7] it is shown that Ω is a G_2-structure on $T^*Q \times \mathbb{R}$. On the other hand it is easy to see that $\alpha = dt + \lambda_{can}$ defines a contact structure on $T^*Q \times \mathbb{R}$ with the Reeb field $\frac{\partial}{\partial t}$. This contact structure is compatible with Ω. Thus Ω is compatible with symplectic structure $\omega = d(e^\theta \alpha)$ on $M = T^*Q \times \mathbb{R}^2$.

Definition 3.5. A compact and orientable hypersurface Q of a symplectic manifold (M, ω) is called of contact type if there exists a 1-form α on Q satisfying
1) $d\alpha = j^*(\omega)$,
2) $\alpha(\xi) \neq 0$ for $0 \neq \xi \in \mathcal{L}_Q$,
where $j : Q \hookrightarrow M$ is the inclusion map and $\mathcal{L}_Q$ is the canonical line bundle of Q.

Theorem 3.2. *Let (Q, Ω) be a hypersurface of symplectic manifold (M, ω) and ω is compatible with Ω. If furthermore Q is of contact type then Ω is compatible with contact structure of Q.*

Proof. Since Q is of contact type then there exists a 1-form α on Q such that $d\alpha = j^*(\omega)$ and since ω is compatible with Ω, there is a vector field R on Q such that

$$\iota_R\Omega = j^*(\omega) = d\alpha.$$

Moreover $\iota_R d\alpha = 0$ and since the restriction of $d\alpha$ to $Ker\alpha$ is symplectic, then $\alpha(R) \neq 0$ and so fR is the Reeb field of α, where $f = \frac{1}{\alpha(R)}$. □

Theorem 3.3. *Let (M, ω) be an 8-dimensional symplectic manifold and $Q \subset M$ be a closed (i.e. compact and without boundary) hypersurface of M with a closed G_2-structure Ω. If $H^1(Q) = 0$, then ω is not compatible with Ω.*

Proof. Since $j^*(\omega)$ is closed and $H^1(Q) = 0$, then $j^*(\omega) = d\alpha$ for some 1-form α on Q. If ω is compatible with Ω, then there is a vector field R on Q such that $\iota_R\Omega = d\alpha$. Thus $g(R,R)vol_\Omega = (\iota_R\Omega) \wedge (\iota_R\Omega) \wedge \Omega$ is exact and hence $\int_Q g(R,R)vol_\Omega = 0$, which is a contradiction. □

4 G_2-structures and existence of symplectic structures

In this section we show that if Q admits a G_2-structure, then $Q \times \mathbb{R}$ and $Q \times \mathbf{S^1}$ admit a symplectic structure, and hence Q can be embedded in a symplectic manifold.

Lemma 4.1. *Let $(2n+1)$-dimensional manifold Q admits an almost contact structure. Then $Q \times \mathbb{R}$ and $Q \times \mathbf{S^1}$ admit an almost complex structure.*

Proof. Let (J, R, α) be an almost contact structure on Q and g be a Riemannian compatible metric. Let D be the sub bundle of TQ generated by R and H be the orthogonal complement of D with respect to g. Thus $TQ = H \oplus D$ an hence $T(Q \times \mathbb{R}) = H \oplus D \oplus T\mathbb{R}$. So, for $X \in T(Q \times \mathbb{R})$, X splits as $X = X_H + bR + a\frac{\partial}{\partial\theta}$, where $X_H \in H$ and θ denotes the coordinate on $\mathbb{R}$. Define the automorphism J' : $T(Q \times \mathbb{R}) \to T(Q \times \mathbb{R})$ by

$$J'(X_H + bR + a\frac{\partial}{\partial\theta}) = J(X_H) + aR - b\frac{\partial}{\partial\theta}.$$

It is easy to see that J' is an almost complex structure on $Q \times \mathbb{R}$. □

Theorem 4.2. *Let Q be a 7-dimensional manifold with a G_2-structure Ω. Then $M = Q \times \mathbb{R}$ admits an almost symplectic structure compatible with Ω. The same statement is true for $M = Q \times \mathbf{S^1}$.*

Proof. Let g_Ω and $\times_\Omega$ denotes, respectively, the Reimannian metric and cross product associative to Ω on Q. Choose a nonzero vector field R on Q with $g_\Omega(R,R) = 1$ and define the 1-form α and endomorphism $J_R : TQ \to TQ$ by

$$\alpha_R(u) = g_\Omega(R,u),$$

$$J_R(u) = R \times_\Omega u.$$

The quadruple $(J_R, R, \alpha_R, g_\Omega)$ defines an almost contact metric structure on Q. Let J be the almost complex structure induced by J_R on $Q\times\mathbb{R}$. Let θ denotes the coordinate on $\mathbb{R}$ and define the Riemannian metric g and the two form ω on $M = Q \times \mathbb{R}$ by

$$g = g_\Omega + d\theta^2,$$

$$\omega(u, v) = g(Ju, v).$$

ω is an almost symplectic structure on M and for u, v in TQ we have

$$\omega(u, v) = g(Ju, v) = g(R \times u, v) = \Omega(R, u, v).$$

Thus ω and Ω are compatible. □

Corollary 4.3. *Every connected 7-dimensional manifold with G_2-structure can be embedded in an 8-dimensional symplectic manifold.*

Proof. Let Q be a 7-dimensional manifold with G_2-structure. By Theorem 4.2, $Q\times\mathbb{R}$ and $Q \times \mathbf{S}^1$ admit an almost symplectic structure. Now Gromov's Theorem follows the assertion. □

As in Corollary 4.3 mentioned if Q admits a G_2-structure, then $Q\times\mathbb{R}$ and $Q\times\mathbf{S}^1$ (if Q is not compact) admit a symplectic structure. It seems to be an open question wether or not every G_2-structure is compatible with a symplectic structure. We could not find counterexample but also did not see how to prove it.

Definition 4.1. (see[6]) Let φ be a closed G_2-structure on Q. The vector field R on Q is called a G_2-vector field if the flow of R preserves the G_2-structure. Also R is called Rochesterian if $\iota_R\varphi$ is an exact form.

Corollary 4.4. *Let (Q, φ) is a hypersurface of (M, ω) and ω is compatible with φ. If φ is closed and $\iota_R\varphi = j^*(\omega)$, then R is a G_2-vector field.*

Corollary 4.5. *In Theorem 4.2, if R is a vector field on Q such that $\iota_R\varphi$ is exact, then $Q \times \mathbb{R}$ and $Q \times \mathbf{S^1}$ admits a symplectic structure compatible with φ.*

In [6] it is shown that there is no Rochesterian vector field on a closed 7-dimensional manifold with a closed G_2-structure. So in the Corollary 4.4, if ω is exact, then Q is assumed to be noncompact or compact without boundary.

Corollary 4.6. *In Theorem 4.2, if φ is closed and R is a G_2-vector field, then there exists a symplectic form ω on $Q \times \mathbb{R}$ such that $[\omega] = [\pi^*(\iota_R\varphi)]$, where $\pi : Q \times \mathbb{R} \to Q$ is the projection map. The same result is true for $Q \times \mathbf{S^1}$.*

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