On weakly cyclic B symmetric spacetime

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Abstract. The object of the present paper is to investigate some geometric and physical properties of weakly cyclic B symmetric $(WCBS)_4$ spacetime under certain conditions. At first, the existence of $(WCBS)_4$ spacetime is showed by constructing a non-trivial example. Then it is shown that a $(WCBS)_4$ spacetime with harmonic Weyl tensor is a Yang Pure space or the integral curve of vector field ρ are geodesic and vector field ρ is irrotational, provided $r = \frac{b}{a}$. Moreover some geometric properties of $(WCBS)_4$ spacetime satisfying certain curvature restrictions are investigated and shown that conformally flat $(WCBS)_4$ spacetime is infinitesimally spatially isotropic relative to the unit timelike vector field ρ. Next we characterize viscous fluid, dust and perfect fluid $(WCBS)_4$ spacetimes and obtained interesting results. Finally, we showed that in a $(WCBS)_4$ spacetime with non-constant scalar curvature satisfying $divC = 0$ and fulfilling the condition $r = \frac{b}{a}$, if ρ is Killing vector then it is Weyl compatible, purely electric spacetime and its possible Petrov types are I or D.

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1 Introduction

A Lorentzian manifold is a special case of pseudo-Riemannian manifold. A pseudo-Riemannian manifold of dimension n is a smooth n-dimensional differentiable manifold equipped with a pseudo-Riemannian metric of signature (p, q) where $n = p+q$. Due to non-degeneracy of Lorentzian metric, the tangent vector can be classified into timelike, null or spacetime vector. A Lorentzian manifold has many applications especially in the field of relativity and cosmology. The casuality of the vector fields plays an important role and hence it becomes a convenient choice for researchers for the study of General Relativity. If a Lorentzian manifold admits a globally timelike vector field, it is called time oriented Lorentzian manifold, physically known as spacetime. In general, a Lorentzian manifold may not have a globally timelike vector field. For more details see [1, 8, 23, 4, 17] and references therein.

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In [6], it is showed that a generalistic spacetime with covariant constant energy momentum tensor is Ricci symmetric, that is, $\nabla S = 0$, where S is the Ricci tensor of the spacetime and ∇ denotes the covariant differentiation with respect to the metric tensor g. If however, $\nabla S \neq 0$, then such a spacetime may be called weakly Ricci symmetric [26]. De and Ghosh [9] studied weakly Ricci symmetric spacetimes and proved that if in a weakly Ricci symmetric spacetime of non-zero scalar curvature the matter distribution is perfect fluid, then the acceleration vector and the expansion scalar are zero and such a spacetime can not admit heat flux. A non-flat Riemannian or pseudo-Riemannian manifold $(M^n, g)(n > 2)$ is called weakly Ricci symmetric if the Ricci tensor S is of the form

$$\begin{aligned} (\nabla_X S)(U,V) &= A(X)S(U,V) + D(U)S(V,X) \\ &+ E(V)S(X,U), \end{aligned} \tag{1.1}$$

where A, D and E are 1-forms which are non-zero simultaneously. Such an n-dimensional Riemannian manifold is denoted by $(WRS)_n$. If $A = B = D = 0$, then the manifold reduces to a Ricci symmetric manifold.

A (0,2) symmetric tensor is a generalized Z tensor if

$$Z_{ij} = S_{ij} + \phi g_{ij}, \tag{1.2}$$

where ϕ is an arbitrary scalar function. Recently, Mantica and Molinari [19] introduced weakly Z symmetric manifolds. It is further weaken by De et al. [10] into weakly cyclic Z symmetric manifolds and it is denoted by $(WCZS)_n$. A non-flat Riemannian or pseudo-Riemannian manifold $(M^n, g)(n > 2)$ is called weakly cyclic Z symmetric if the Z tensor is non-zero and satisfies the following condition

$$\begin{aligned} &(\nabla_X Z)(U,V) + (\nabla_U Z)(V,X) + (\nabla_V Z)(X,U) \\ &= A(X)Z(U,V) + D(U)Z(V,X) + E(V)Z(X,U), \end{aligned} \tag{1.3}$$

for all vector fields X, U and V. Here, Z is the generalized Z tensor. De et al. [11] studied weakly cyclic Z symmetric spacetimes and showed that if a $(WCZS)_4$ spacetime satisfies $divC = 0$ and fulfills the condition $r = a$, then the spacetime is the Robertson-Walker spacetime. De et al. [20] introduced a new symmetric (0,2) tensor B_{ij} as

$$B_{ij} = aS_{ij} + brg_{ij}, \tag{1.4}$$

where a and b are non-zero arbitrary scalar functions and r is the scalar curvature. For $a = 1$ and $b = \frac{\phi}{r}$ the tensor reduces to generalized Z tensor. Thus generalized Z tensor is a particular case of B tensor and hence it give us a reason to study B tensor. Contracting (1.4) we get, scalar B as $B = (a + nb)r$. In [20], the authors introduced pseudo B symmetric manifold which is a generalization of pseudo Z symmetric manifold [21]. Motivated by this we introduced weakly cyclic B symmetric manifold. A non-flat Riemannain or pseudo-Riemannain manifold $(M^n, g)(n > 1)$ is called a weakly cyclic B symmetric manifold of dimension n if the B tensor is non-zero and satisfies the condition

$$\begin{aligned} &(\nabla_X B)(Y,Z) + (\nabla_Y B)(Z,X) + (\nabla_Z B)(X,Y) \\ &= A(X)B(Y,Z) + D(Y)B(Z,X) + E(Z)B(X,Y), \end{aligned} \tag{1.5}$$

where A, D and E are non-zero 1-forms. It will be denoted by $(WCBS)_n$ manifold. In [11], the authors investigated weakly cyclic Z symmetric spacetime and obtained interesting results. This inspired us to study weakly cyclic B symmetric spacetime.

The notion of quasi Einstein manifolds arose during the study of exact solutions of the Einstein's field equation as well as during the considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. Chaki and Maity [5] introduced the notion of quasi Einstein manifolds as a generalization of the Einstein manifolds. A pseudo-Riemannain manifold $(M^n, g)(n > 2)$ is said to be a quasi Einstein manifold if its Ricci curvature is non-zero and satisfies the condition

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y), \tag{1.6}$$

where α and β are real valued non-zero scalar functions on M. In [12], it is proved that a quasi Einstein manifolds can be taken as a model of perfect fluid spacetime in General Relativity. Also, the Robertson-Walker spacetimes are quasi Einstein manifolds. Thus quasi Einstein manifolds are important in theoretical physics, especially in General Relativity and cosmology.

The Weyl (or conformal curvature) tensor plays an important role in differential geometry and also in General Relativity providing curvature to the spacetime when the Ricci tensor is zero. The Weyl conformal tensor C in a Lorentzian manifold $(M^n, g)(n > 3)$ is defined by [29]

$$\begin{aligned} C(X,Y)U &= R(X,Y)U - \frac{1}{n-2}[g(Y,U)QX - g(X,U)QY \\ &+ S(Y,U)X - S(X,U)Y] \\ &+ \frac{r}{(n-1)(n-2)}[g(Y,U)X - g(X,U)Y], \end{aligned} \tag{1.7}$$

where Q is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S, that is, $g(QX,Y) = S(X,Y)$. The Lorenzian manifold of dimension $n(n > 3)$ is said to be conformally flat if the conformal curvature tensor C is identically zero. In [16], Endean studied cosmology in conformally flat spacetime.

Ahsan and Siddiqui [1] proved that a concircularly flat perfect fluid spacetime admits a conformal Killing vector field if and only if the energy-momentum tensor has a symmetry inheritance property. The concircular curvature tensor in a Lorentzian manifold $(M^n, g)(n > 3)$ is defined by

$$\begin{aligned} \widetilde{C}(X,Y)U &= R(X,Y)U \\ &+ \frac{r}{n(n-1)}[g(U,X)Y - g(Y,Z)X], \end{aligned} \tag{1.8}$$

for all vector fields X, Y, Z in M. For $n = 3$, the Weyl tensor as well as concircular curvature tensor vanishes identically. The Lorenzian manifold of dimension $n(n > 3)$ is said to be concircularly flat if the concircular curvature tensor $\widetilde{C}$ is identically zero.

The paper is organized as follows:

In Section 2, the existence of $(WCBS)_4$ spacetime is established by constructing a non-trivial example. Next in Section 3 it is shown that a $(WCBS)_4$ spacetime is quasi Einstein spacetime. Moreover conformally flat $(WCBS)_4$ spacetime and $(WCBS)_4$ spacetime with $divC = 0$ are studied and prove that a $(WCBS)_4$ spacetime satisfying

$divC = 0$ with assumption $r = \frac{b}{a}$, the integral curve of vector field ρ are geodesic and vector field ρ is irrotational or the spacetime is Yang Pure space. In the next section, we investigate some geometric and physical properties of this spacetime under certain curvature conditions. The last section deal with the application of $(WCBS)_4$ spacetime in General Relativity. We prove that if a perfect fluid $(WCBS)_4$ spacetime with vanishing scalar B obeys Einstein's field equation without cosmological constant then the spacetime is characterized by the following cases:
(i) The spacetime represents inflation and the fluid behaves as a cosmological constant. This is also termed as a phantom barrier.
(ii) The spacetime represents quintessence barrier and the fluid behaves as exotic matter.
The energy density and isotropic pressure for viscous fluid $(WCBS)_4$ spacetime are obtained and also we prove that a relativistic fluid $(WCBS)_4$ spacetime obeying Einstein's field equation with the cosmological constant admit heat flux, provided $\lambda + k\sigma \neq \frac{3B-2br}{2a}$. Finally, it is shown that in a $(WCBS)_4$ spacetime with non-constant scalar curvature satisfying $divC = 0$ and fulfilling the condition $r = \frac{b}{a}$, if ρ is Killing vector then it is Weyl compatible, purely electric spacetime and its possible Petrov types are I or D.

2 Existence of $(WCBS)_4$ spacetime

In this section, we prove the existence of the $(WCBS)_4$ spacetime by constructing a non-trivial example (see [24]). Now, we shall consider a Lorentzian metric g on the 4-dimensional real number space $\mathbb{R}^4$ by

$$ds^2 = e^{2z}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] - (dx^4)^2, \tag{2.1}$$

where $z = x^4 \neq 0$ and x^1, x^2, x^3, x^4 are the standard coordinates of $\mathbb{R}^4$. Then the non-vanishing components of the Christoffel symbol, the curvature tensor and the Ricci tensor are

$$\Gamma^4_{11} = \Gamma^4_{22} = \Gamma^4_{33} = e^{2z}, \qquad \Gamma^1_{14} = \Gamma^2_{24} = \Gamma^3_{34} = 1, \tag{2.2}$$

$$R_{1441} = R_{2442} = R_{3443} = e^{2z}, R_{1221} = R_{1331} = R_{2332} = -e^{4z}, \tag{2.3}$$

$$S_{11} = S_{22} = S_{33} = -3e^{2z}, \qquad S_{44} = 3, \tag{2.4}$$

and the components which can be obtained from this by symmetric properties. One can easily show that the scalar curvature r of the manifold is $r = -12$.
Let us choose an arbitrary scalar function as $a = e^{-z}$ and $b = e^{-2z}$. Making use of (1.4) the non-vanishing components of symmetric B tensor and their covariant derivatives are as follows

$$B_{11} = B_{22} = B_{33} = -3(e^z + 4), \qquad B_{44} = 3(e^{-z} + 4e^{-2z}, \tag{2.5}$$

$$B_{11,4} = B_{22,4} = B_{33,4} = -3, \qquad B_{44,4} = -3(e^{-z} + 8e^{-2z}). \tag{2.6}$$

Let us choose the associate 1-forms as follows:

$$A_i(x) = \begin{cases} \dfrac{1}{e^z + 4} & \text{for } i = 4 \\ 0\ , & \text{otherwise,} \end{cases} \tag{2.7}$$

$$D_i(x) = \begin{cases} \dfrac{-37e^z}{(e^z+4)^2} & \text{for } i = 4 \\ 0\ , & \text{otherwise} \end{cases} \tag{2.8}$$

and

$$E_i(x) = \begin{cases} \dfrac{-3e^{2z}-13}{(e^z+4)^2} & \text{for } i = 4 \\ 0\ , & \text{otherwise,} \end{cases} \tag{2.9}$$

at any point $x \in \mathbb{R}^4$. In consequence of (2.5), (2.6), (2.7), (2.8) and (2.9) we obtain

$$B_{11,4} + B_{14,1} + B_{14,1} = A_4 B_{11} + D_1 B_{41} + E_1 B_{14}, \tag{2.10}$$

$$B_{22,4} + B_{24,2} + B_{24,2} = A_4 B_{22} + D_2 B_{42} + E_2 B_{24}, \tag{2.11}$$

$$B_{33,4} + B_{34,3} + B_{34,3} = A_4 B_{33} + D_3 B_{43} + E_3 B_{34}, \tag{2.12}$$

$$B_{44,4} + B_{44,4} + B_{44,4} = A_4 B_{44} + D_4 B_{44} + E_4 B_{44}, \tag{2.13}$$

for all other cases (1.5) holds trivially. Therefore, this proves that the manifold $(\mathbb{R}^4, g)$ under consideration is a $(WCBS)_4$ spacetime with non-zero scalar curvature. Hence we can state that:

Theorem 2.1. *Let $(\mathbb{R}^4, g)$ be a Lorentzian manifold endowed with the metric given by*

$$ds^2 = g_{ij}dx^i dx^j = e^{2z}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] - (dx^4)^2$$

where $z = x^4 \neq 0$ and x^1, x^2, x^3, x^4 are the standard coordinates of $\mathbb{R}^4$. Then $(\mathbb{R}^4, g)$ is an $(WCBS)_4$ spacetime with non-zero scalar curvature $r = -12$.

3 $(WCBS)_4$ spacetime

A Lorentzian manifold (M^4, g) is said to be weakly cyclic B symmetric $(WCBS)_4$ spacetime if the B tensor is non-zero and satisfies

$$\begin{aligned} &(\nabla_X B)(Y,Z) + (\nabla_Y B)(Z,X) + (\nabla_Z B)(X,Y) \\ &= A(X)B(Y,Z) + D(Y)B(Z,X) + E(Z)B(X,Y), \end{aligned} \tag{3.1}$$

for all vector fields X, Y, Z in M^4. Here, 1-forms A, D and E are given by

$$A(X) = g(X, \rho_1), D(X) = g(X, \rho_2), E(X) = g(X, \rho_3),$$

where ρ_1, ρ_2, ρ_3 are timelike vector fields, that is, $g(\rho_i, \rho_i) = -1, i = 1, 2, 3$ corresponding to 1-forms A, D and E respectively.
Interchanging Y and Z in (3.1) we obtain

$$\begin{aligned} &(\nabla_X B)(Z,Y) + (\nabla_Z B)(Y,X) + (\nabla_Y B)(X,Z) = \\ &A(X)B(Z,Y) + D(Z)B(Y,X) + E(Y)B(X,Z). \end{aligned} \tag{3.2}$$

Combining (3.1) and (3.2) yields

$$[D(Y) - E(Y)]B(X, Z) = [D(Z) - E(Z)]B(X, Z). \tag{3.3}$$

Define a 1-form as $H(X) = D(X) - E(X) = g(X, \rho)$ for all vector fields X. Using this in (3.3) gives

$$H(Y)B(X, Z) = H(Z)B(X, Y). \tag{3.4}$$

Taking a frame field and contracting $X = Z = e_i$ where $\{e_i\}$ is the orthonormal basis of the tangent space at each point in spacetime we get

$$BH(Y) = B(\rho, Y). \tag{3.5}$$

Taking $Z = \rho$ in (3.4) gives

$$H(Y)[aS(X, \rho) + brH(X)] = -B(X, Y). \tag{3.6}$$

Replacing X by ρ in (1.4) in using it in (3.5) we obtain

$$aS(\rho, Y) = (a + 3b)rH(Y). \tag{3.7}$$

In regard of (3.6) and (3.7), we see that

$$S(X, Y) = \alpha g(X, Y) + \beta H(X)H(Y), \tag{3.8}$$

where $\alpha = -\frac{br}{a}$ and $\beta = -\frac{B}{a}$. Hence we can state the following:

Theorem 3.1. *A $(WCBS)_4$ spacetime is a quasi Einstein spacetime.*

Theorem 3.2. *A conformally flat $(WCBS)_4$ spacetime is infinitesimally spatially isotropic relative to the unit timelike vector field ρ.*

Proof. Suppose $(WCBS)_4$ spacetime is conformally flat. Making use of (3.8) and (1.7) in conformally flat $(WCBS)_4$ spacetime we obtain

$$\begin{aligned} R(X, Y)Z &= \frac{1}{2}\Big[-\frac{2br}{a} g(Y, Z)X - \frac{B}{a} H(Y)H(Z)X \\ &+ \frac{2br}{a} g(X, Z)Y + \frac{B}{a} H(X)H(Z)Y \\ &- \frac{B}{a} H(X)g(Y, Z)\rho + \frac{B}{a} H(Y)g(X, Z)\rho\Big] \\ &- \frac{r}{6}\Big[g(Y, Z)X - g(X, Z)Y\Big]. \end{aligned} \tag{3.9}$$

Let $\rho^{\perp}$ denote the 3-dimensional distribution in a conformally flat $(WCBS)_4$ spacetime orthogonal to ρ, then from (3.9) we get

$$R(X, Y)Z = \Big(\frac{br}{a} - \frac{r}{6}\Big)[g(Y, Z)X - g(X, Z)Y], \tag{3.10}$$

for all $X, Y, Z \in \rho^{\perp}$. Also taking $Y = Z = \rho$ in (3.9) gives

$$R(X, \rho)\rho = \frac{r}{6a}(6b + a)X, \tag{3.11}$$

for every $X \in \rho^{\perp}$.
According to Karchar [18], a Lorentzian manifold is called infinitesimal spatially isotropic relative to timelike unit vector field ρ if its curvature tensor R satisfies relations

$$R(X,Y)Z = l[g(Y,Z)X - g(X,Z)Y],$$

for all $X, Y, Z \in \rho^{\perp}$ and

$$R(X,\rho)\rho = mX,$$

for all $X \in \rho^{\perp}$, where l, m are real valued functions on the manifold. Thus in view of (3.10) and (3.11) we see that a conformally flat $(WCBS)_4$ spacetime is infinitesimal spatially isotropic relative to timelike unit vector field ρ.
This completes the proof. ☐

Theorem 3.3. *In a $(WCBS)_4$ spacetime satisfying $divC = 0$ with assumption $r = \frac{b}{a}$, the integral curve of vector field ρ are geodesic and vector field ρ is irrotational or the spacetime is Yang Pure space.*

Proof. Suppose $(WCBS)_4$ spacetime has harmonic conformal curvature, that is, $divC = 0$. Then (1.7) gives

$$\begin{aligned}(\nabla_X S)(Y,U) \quad &- \quad (\nabla_U S)(Y,X) \\ &= \frac{1}{6}[g(Y,U)dr(X) - g(X,Y)dr(U)].\end{aligned} \tag{3.12}$$

Making use of (3.8) in (3.12) we obtain

$$\begin{aligned}&\Big\{\frac{adr(U) - rda(U)}{a^2}\Big\}[bg(X,Y) + (a+4b)H(X)H(Y)] \\ &\quad + \frac{r}{a}\Big[db(U)g(X,Y) + \{da(U) + 4db(U)\}H(X)H(Y) \\ &\quad + (a+4b)\{(\nabla_U H)(X)H(Y) + (\nabla_U H)(Y)H(X)\}\Big] \\ &- \Big\{\frac{adr(X) - rda(X)}{a^2}\Big\}[bg(Y,U) + (a+4b)H(Y)H(U)] \\ &\quad - \frac{r}{a}\Big[db(X)g(Y,U) + \{da(X) + 4db(X)\}H(Y)H(U) \\ &\quad + (a+4b)\{(\nabla_X H)(Y)H(U) + (\nabla_X H)(U)H(Y)\}\Big] \\ &= \frac{1}{6}[g(Y,U)dr(X) - g(X,Y)dr(U)].\end{aligned} \tag{3.13}$$

Taking a frame field and contracting (3.13) over X and Y gives

$$\begin{aligned}&-\Big(1 + \frac{b}{a}\Big)dr(U) + \frac{br}{a^2}da(U) - \frac{(a+4b)}{a^2}\big[adr(\rho) \\ &\quad - rda(\rho)\big]H(U) - \frac{r}{a}db(U) - \frac{r}{a}\big[da(\rho)H(U) \\ &\quad + 4db(\rho)H(U)\big] - \frac{B}{a}\big[(\delta H)H(U) \\ &\quad + (\nabla_\rho H)(U)\big] = -\frac{1}{2}dr(U),\end{aligned} \tag{3.14}$$

where $(\delta H) = \Sigma_{i=1}^{n} \epsilon_i (\nabla_{e_i} H)(e_i)$. Putting $X = Y = \rho$ in (3.13) we get

$$\begin{aligned}
&\Big\{\frac{adr(U) - rda(U)}{a^2}\Big\}(a+3b) + \frac{r}{a}\big[3db(U) + da(U)\big]\\
&-\frac{b}{a^2}\big\{adr(\rho) - rda(\rho)\big\}H(U) + \frac{(a+4b)}{a^2}\big\{adr(\rho)\\
&-rda(\rho)\big\}H(U) + \frac{r}{a}\big\{adb(\rho) + da(\rho)\big\}H(U)\\
&+\frac{B}{a}(\nabla_\rho H)(U) = \frac{1}{6}\big[dr(\rho)H(U) + dr(U)\big]. \qquad (3.15)
\end{aligned}$$

Combining (3.14) and (3.15) yields

$$\begin{aligned}
&\frac{-2br}{a^2}da(U) + \frac{2r}{a}db(U) - \frac{r}{a}db(\rho)H(U)\\
&-\frac{B}{a}(\delta H)H(U) + \Big(\frac{4}{3} - \frac{b}{a}\Big)dr(U)\\
&-\Big(\frac{1}{6} + \frac{b}{a}\Big)dr(\rho)H(U) + \frac{br}{a^2}da(\rho)H(U) = 0. \qquad (3.16)
\end{aligned}$$

Replacing U by ρ in (3.16) and using it in (3.16) results in the following

$$\begin{aligned}
&\frac{2r}{a}\Big[\frac{adb(U) - bda(U)}{a}\Big] + \frac{2r}{a}\Big[\frac{adb(\rho) - bda(\rho)}{a}\Big]H(U)\\
&+\Big(\frac{4}{3} - \frac{b}{a}\Big)\big[dr(U) + dr(\rho)H(U)\big] = 0. \qquad (3.17)
\end{aligned}$$

If possible, suppose $r = \frac{b}{a}$, then

$$dr(U) = \frac{adb(U) - bda(U)}{a^2}, \qquad (3.18)$$

for any vector field U. In consequence of (3.17) and (3.18) we see that either $4a+3b = 0$ or $dr(U) = -dr(\rho)H(U)$. Considering the case when $4a + 3b = 0$, we see that $r = \frac{-4}{3}$ is a constant, and hence $dr = 0$. Using this facts in (3.12) gives

$$(\nabla_X S)(Y, U) - (\nabla_U S)(Y, X) = 0.$$

This means that $(WCBS)_4$ spacetime is a Yang Pure space [30].
Suppose $4a + 3b \neq 0$. Replacing Y by ρ in (3.13) and using $dr(U) = -dr(\rho)H(U)$ yields

$$(\nabla_X H)(U) - (\nabla_U H)(X) = 0. \qquad (3.19)$$

This means that the 1-form H is closed. Thus we get

$$g(\nabla_X \rho, U) = g(\nabla_U \rho, X)$$

for all X, U. Taking $U = \rho$ gives

$$g(\nabla_\rho \rho, X) = g(\nabla_X \rho, \rho).$$

Since $g(\nabla_X \rho, \rho) = 0$ implies $g(\nabla_\rho \rho, X) = 0$ for all X. Hence $\nabla_\rho \rho = 0$. This means that the integral curve of the vector field ρ are geodesic and vector field is irrotational. This completes the proof. □

A vector field ρ is a Killing vector if

$$g(Y, \nabla_\rho X) + g(\nabla_\rho Y, X) = 0, \tag{3.20}$$

for any vector fields X, Y. Hence we can state the following:

Corollary 3.4. *If a $(WCBS)_4$ spacetime with non-constant scalar curvature satisfies $divC = 0$ and fulfills the condition $r = \frac{b}{a}$, then the vector field ρ is a Killing vector if and only if ρ is parallel vector.*

4 Some geometrical properties of $(WCBS)_4$ spacetime

The k-nullity distribution $N(k)$ of a pseudo-Riemannian manifold M^n is defined by [27]

$$\begin{aligned} N(k) &: p \to N_p(k) \\ &= \{Z \in T_p(M) | R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\} \end{aligned}$$

for all $X, Y \in TM$, where k is some smooth function. If the generator ρ of the quasi-Einstein manifold M^n belongs to the k-nullity distribution $N(k)$ for some smooth function k, then M^n is called $N(k)$-quasi Einstein manifold [28].

According to Deszcz [2, 13, 14], for (0,4)-tensor field T if $R \cdot T$ and $Q(g,T)$ are linearly dependent, that is, $R \cdot T = L_T Q(g,T)$ holds on the set $U_T = \{x \in M : Q(g,T) \neq 0 \text{ at } x\}$, where L_T is some function on U_T. In particular, if $T = R$(resp., $S, C, \widetilde{C}$) then the manifold is called pseudosymmetric (resp., Ricci-pseudosymmetric, conformally pseudosymmetric, concircularly pseudosymmetric). De and Velimirović [8] studied spacetimes with semisymmetric Energy-Momentum tensor and showed that such a spacetime is Ricci semisymmetric.

In this section, $(WCBS)_4$ spacetime under certain curvature conditions such as Ricci-pseudosymmetric, conformal Ricci semisymmetric and concircular Ricci-pseudosymmetric are studied.

Theorem 4.1. *Every Ricci-pseudosymmetric $(WCBS)_4$ spacetime with non-vanishing scalar B is an $N(\frac{B-br}{3a})$-quasi Einstein spacetime.*

Proof. Suppose $(WCBS)_4$ spacetime is Ricci-pseudosymmetric, that is, $R \cdot S = L_S Q(g,S)$ holds on U_s and L_S is a certain function on U_S. Thus we get

$$\begin{aligned} &S(R(X,Y)U,V) + S(U,R(X,Y)V) = \\ &L_S[g(Y,U)S(X,V) - g(X,U)S(Y,V) \\ &+g(Y,V)S(U,X) - g(X,V)S(Y,U)]. \end{aligned} \tag{4.1}$$

In consequence of (3.8) in (4.1) we obtain

$$\begin{aligned} &H(R(X,Y)U)H(V) + H(U)H(R(X,Y)V) = \\ &L_S[g(Y,U)H(X)H(V) - g(X,U)H(Y)H(V) \\ &+g(Y,V)H(X)H(U) - g(X,V)H(Y)H(U)]. \end{aligned} \tag{4.2}$$

Contracting (4.2) over X and V and using (3.8) yields

$$R(\rho, Y)U = L_S g(Y, U)\rho + \Big[\frac{B}{a} - \frac{br}{a} - 4L_S\Big] H(Y)U. \tag{4.3}$$

Substituting $Y = U = \rho$ in (4.3) we get the following relation

$$L_S = \frac{(B - br)}{3a}. \tag{4.4}$$

Taking $U = \rho$ in (4.2) gives

$$R(X, Y)\rho = L_S[H(Y)X - H(X)Y]. \tag{4.5}$$

Making use of (4.4) in (4.5) we obtain

$$R(X, Y)\rho = \frac{B - br}{3a}[H(Y)X - H(X)Y]. \tag{4.6}$$

This means that the vector field ρ belongs to the $(\frac{B-br}{3a})$-nullity distribution. This completes the proof. □

If we take $L_S = 0$, then the manifold satisfies the condition $R \cdot S = 0$ and so it is Ricci semisymmetric. In this case, we see that $B = br$ implies $a = -3b$. Hence we can state the following:

Corollary 4.2. *In a Ricci semisymmetric $(WCBS)_4$ spacetime with non-vanishing scalar B the relation $a + 3b = 0$ holds.*

Theorem 4.3. *In a $(WCBS)_4$ spacetime with non-vanishing scalar B satisfying $C(X, Y) \cdot S = 0$, $\left(\frac{5B-8br}{12a}\right)$ is an eigenvalue of the Ricci operator Q.*

Proof. Proceeding similarly as in Theorem 4.1, we obtain the following relation

$$\begin{aligned} g(R(X, Y)U, \rho) &= \Big(\frac{5B - 8br}{12a}\Big)[g(Y, U)H(X) \\ &- g(X, U)H(Y)]. \end{aligned} \tag{4.7}$$

Contracting (4.7) over X and U yields

$$S(Y, \rho) = \Big(\frac{5B - 8br}{12a}\Big) g(Y, \rho), \tag{4.8}$$

i.e., $QY = \left(\frac{5B-8br}{12a}\right)Y$ for all vector field Y. Thus $\left(\frac{5B-8br}{12a}\right)$ is an eigenvalue of the Ricci operator Q. This completes the proof. □

Suppose $(WCBS)_4$ spacetime is concircularly pseudosymmetric, that is, $R \cdot \widetilde{C} = L_S Q(g, S)$. Then proceeding similarly as in Theorem 4.1 and Theorem 4.3 one can easily obtained the following:

Theorem 4.4. *In a $(WCBS)_4$ spacetime with non-vanishing scalar B satisfying $R \cdot \widetilde{C} = L_S Q(g, S)$, $\frac{3(B-br)}{5a}$ is an eigenvalue corresponding to Ricci operator Q and the timelike vector field ρ belongs to the $N(\frac{B-br}{5a})$-quasi Einstein spacetime.*

5 Application of $(WCBS)_4$ spacetime in General Relativity

The general theory of relativity postulate that the spacetime should be described as a curved manifold. The Einstein's field equation [23] relate the geometry of spacetime with the distribution of matter within it. Einstein's field equation is conferred by

$$S(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = kT(X,Y), \tag{5.1}$$

for all vector fields X, Y where S is the Ricci tensor of type (0,2), r is the scalar curvature, λ is the cosmological constant and k is the gravitational constant. Eq. (5.1) imply that the matter detrmines the geometry of spacetime and conversely that the motion of matter is determined by the metric tensor of the space which is not flat. Here, T is the energy momentum tensor which is a symmetric $(0,2)$-tensor with divergence zero.

The energy momentum tensor is said to describe a perfect fluid [23] if

$$T(X,Y) = (\sigma + p)H(X)H(Y) + pg(X,Y), \tag{5.2}$$

where σ is the energy density and p is the isotropic pressure of the fluid, H is a non-zero 1-form such that

$$g(X,\rho) = H(X),$$

for all X, ρ being the velocity vector field of the fluid which is a timelike vector, that is, $g(\rho,\rho) = H(\rho) = -1$.
Combining (3.8) and (5.1), the energy momentum tensor can be written as

$$T(X,Y) = \frac{r[a(2\lambda - 1) - 2b]}{2ak}g(X,Y) - \frac{B}{ak}H(X)H(Y), \tag{5.3}$$

Thus we can state the following:

Proposition 5.1. *A $(WCBS)_4$ spacetime satisfying Einstein's field equation with cosmological constant can be considered as a model of perfect fluid spacetime, in General Relativity.*

Inserting (5.2) in (5.1) without cosmological constant, we obtain

$$S(X,Y) = k(\sigma + p)H(X)H(Y) + (kp + \frac{r}{2})g(X,Y). \tag{5.4}$$

Comparing (3.8) and (5.4), we see that in a perfect fluid $(WCBS_4)$ spacetime the following relations hold

$$\alpha = -\frac{br}{a} = (kp + \frac{r}{2}) \text{ and } \beta = -\frac{B}{a} = k(\sigma + p). \tag{5.5}$$

Replacing X by QX in (5.4) and using (3.8) gives

$$\begin{aligned} S^2(X,Y) &= k(\sigma + p)(\alpha - \beta)H(X)H(Y) \\ &+ \frac{k}{2}(\sigma - p)[\alpha g(X,Y) + \beta H(X)H(Y)], \end{aligned} \tag{5.6}$$

where $S^2(X,Y) = S(QX,Y)$. Taking a frame field and contracting (5.6) over X and Y yields

$$||Q||^2 = k^2(\sigma^2 + 2p^2 - \sigma p). \tag{5.7}$$

Hence we can state the following:

Theorem 5.2. *If a perfect fluid $(WCBS)_4$ spacetime obeying Einstein's field equation without cosmological constant, then the square of the length of the Ricci operator is $k^2(\sigma^2 + 2p^2 - \sigma p)$.*

In view of (5.4), if perfect fluid $(WCBS)_4$ spacetime satisfies the timelike convergence condition, that is, $S(\rho,\rho) \geq 0$ then $\sigma + 3p \geq 0$, thus the spacetime obeys cosmic strong energy condition. Thus we can state

Proposition 5.3. *If a perfect fluid $(WCBS)_4$ spacetime obeying Einstein's field equation without the cosmological constant satisfies timelike convergence condition, then the spacetime obeys strong energy condition.*

In cosmology we know that when $\sigma = -p$ this lead to rapid expansion of the spacetime which is termed as inflation. Also $\sigma + p = 0$ is known as Phantom Barrier [7]. Here the fluid behaves as a cosmological constant [25]. And if $\sigma + 3p = 0$ then strong energy condition begins to violate and fluid behaves as exotic matter. This is termed as a Quintessence Barrier. Recent observations have indicated that our universe is in quintessence era [3].

In consequence of (5.5), we get $\sigma + p = -\frac{B}{ak}$ and $\sigma + 3p = -\frac{r}{k}$.
Suppose that the scalar B vanishes, it follows that either (i) $a + 4b = 0$ or (ii) $r = 0$. Now (i) $a+4b = 0$ implies $\sigma + p = 0$. Thus the spacetime represents phantom barrier. Again (ii) $r = 0$ implies $\sigma + 3p = 0$. Thus the spacetime represents quintessence barrier. Hence we can state the following:

Theorem 5.4. *If a perfect fluid $(WCBS)_4$ spacetime with vanishing scalar B obeys Einstein's field equation without cosmological constant then the spacetime is characterized by the following cases:*
(i) The spacetime represents inflation and the fluid behaves as a cosmological constant. This is also termed as a phantom barrier.
(ii) The spacetime represents quintessence barrier and the fluid behaves as exotic matter.

Next we state and proof the following:

Theorem 5.5. *A relativistic fluid $(WCBS)_4$ spacetime obeying Einstein's field equation with the cosmological constant admit heat flux, provided $\lambda + k\sigma \neq \frac{3B-2br}{2a}$.*

Proof. For a relativistic fluid matter distribution, the energy momentum tensor is as follows [15]

$$\begin{aligned} T(X,Y) &= pg(X,Y) + (\sigma + p)A(X)A(Y) \\ &+ A(X)B(Y) + B(Y)A(Y), \end{aligned} \tag{5.8}$$

where $A(X) = g(X,\rho), A(\rho) = -1, B(X) = g(X,\mu), B(\mu) > 0, g(\rho,\mu) = 0$. Here ρ is the velocity vector field and μ is the heat conduction vector field.
Making use of (5.8), the Einstein's field equation becomes

$$\begin{aligned} S(X,Y) &= (kp - \lambda + \frac{r}{2})g(X,Y) + k(\sigma + p)A(X)A(Y) \\ &+ k[A(X)B(Y) + B(X)A(Y)]. \end{aligned} \tag{5.9}$$

Inserting (3.8) in (5.9) gives

$$\begin{aligned} \Big(\alpha &- kp + \lambda - \frac{r}{2}\Big)g(X,Y) + [\beta - k(\sigma + p)]A(X)A(Y) \\ &- k[A(X)B(Y) + B(X)A(Y)] = 0. \end{aligned} \tag{5.10}$$

Replacing X by ρ in (5.10) we obtain

$$B(Y) = \frac{1}{k}\Big(\frac{br}{a} - \frac{B}{a} - k\sigma + \frac{r}{2} - \lambda\Big)A(Y). \tag{5.11}$$

Thus the spacetime admit heat flux if $\lambda + k\sigma \neq \frac{3B-2br}{2a}$. This completes the proof. □

Next we consider viscous fluid matter, under which the energy momentum tensor is of form:

$$T(X,Y) = pg(X,Y) + (\sigma + p)H(X)H(Y) + P(X,Y), \tag{5.12}$$

where P denotes the anisotropic pressure tensor of the fluid.
Combining (5.12), (5.1) and (3.8) yields

$$\begin{aligned} (\alpha - \frac{r}{2} - kp)g(X,Y) &+ [\beta - k(\sigma + p)]H(X)H(Y) \\ &= kP(X,Y). \end{aligned} \tag{5.13}$$

Replacing X and Y by ρ in (5.13) we get

$$-(\alpha - \frac{r}{2} - kp) + \beta - k(\sigma + p) = kI, \tag{5.14}$$

where $I = P(\rho,\rho)$. Contracting (5.13) over X and Y gives

$$4(\alpha - \frac{r}{2} - kp) - \beta + k(\sigma + p) = kJ, \tag{5.15}$$

where $J =$ Trace of P. Adding (5.14) and (5.15) the expression for isotropic pressure is given by

$$p = \frac{1}{k}\Big\{\lambda - \frac{br}{a} - \frac{r}{2} - \frac{k(I+J)}{3}\Big\}. \tag{5.16}$$

In consequence of (5.16) in (5.14) the expression for energy density is given by

$$\sigma = \frac{1}{k}\Big(\frac{br}{a} + \frac{r}{2} - \lambda - \frac{B}{a}\Big). \tag{5.17}$$

Thus we can state the following:

Theorem 5.6. *In a viscous fluid $(WCBS)_4$ spacetime obeying Einstein's field equation with cosmological constant the energy density and isotropic pressure are given by (5.17) and (5.16) respectively.*

For a pressureless fluid spacetime (dust), the energy momentum tensor is of form $T(X,Y) = \sigma H(X)H(Y)$. Proceeding similarly as in Theorem 5.6 one can easily obtain the follow:

Proposition 5.7. *A dust $(WCBS)_4$ spacetime obeying Einstein's field equation with cosmological constant is vacuum if and only if scalar B vanishes.*

Definition 5.1. A symmetric tensor b_{ij} is Weyl compatible if

$$b_{im}C^m_{jkl} + b_{jm}C^m_{kil} + b_{km}C^m_{ijl} = 0. \tag{5.18}$$

Now we examine the Weyl compatibility of $(WCBS)_4$ spacetime. In accordance of Corollary 3.4, suppose ρ is Killing vector field then ρ is parallel vector and hence we get

$$R(X,Y)\rho = [\nabla_X, \nabla_Y]\rho - \nabla_{[X,Y]}\rho = 0. \tag{5.19}$$

Contracting (5.19) over X and using (3.8) we see that $(a+3b)r = 0$. But, $r \neq 0$ hence $a + 3b = 0$. Making use of this in (3.8) yields

$$S(X,Y) = -\frac{1}{9}[g(X,Y) + H(X)H(Y)]. \tag{5.20}$$

In consequence of (5.20) the Weyl tensor is of form

$$\begin{aligned} C_{ijkl} = R_{ijkl} + \frac{1}{12}[g_{jk}g_{il} - g_{ik}g_{jl}] + \frac{1}{18}[g_{il}H_jH_k \\ -g_{jl}H_iH_k + g_{jk}H_iH_l - g_{ik}H_jH_l]. \end{aligned} \tag{5.21}$$

Since the generator ρ is parallel so transvecting (5.21) by H^l we get

$$H^lC_{ijkl} = \frac{1}{36}[g_{jk}H_i - g_{ik}H_j]. \tag{5.22}$$

In view of (5.22) we can obtain the following relation

$$(H_iC_{jklm} + H_jC_{kilm} + H_kC_{ijlm})H^m = 0. \tag{5.23}$$

Thus we can state the following:

Theorem 5.8. *In a $(WCBS)_4$ spacetime with non-constant scalar curvature satisfying $divC = 0$ and fulfilling the condition $r = \frac{b}{a}$, if ρ is Killing vector then the spacetime is Weyl compatible.*

In General Relativity, given a timelike vector field u with $u^iu_i = -1$, then the electric and magnetic components of Weyl tensor are defined by

$$E_{kl} = u^ju^mC_{jklm}, \tag{5.24}$$

(5.25) $$H_{kl} = \frac{1}{2}\varepsilon_{jkrs}u^j u^m C^{rs}_{lm},$$

where the components C^{rs}_{lm} is of type (2,2) of the Weyl tensor and ε_{jkrs} denotes the completely skew-symmetric Levi-Civita symbol. In [22] it is shown that on a 4-dimensional spacetime a timelike vector field is Weyl compatible if and only if the magnetic part of the Weyl tensor vanishes i.e., $H_{kl} = 0$. In regard of Theorem 5.8 and above result we obtain the following:

Proposition 5.9. *In a $(WCBS)_4$ spacetime with non-constant scalar curvature satisfying $divC = 0$ and fulfilling the condition $r = \frac{b}{a}$, if ρ is Killing vector then it is a purely electric spacetime.*

If the electric and magnetic parts of the Weyl tensor are proportional i.e., $\gamma E = \mu H$ for some scalar fields γ and μ including the case when one of them is zero, then the space is of type I, D or O. But $E_{kl} = \frac{R_{kl}}{4} \neq 0$, the Weyl tensor is non-vanishing so the space cannot be of type O. Thus we can state

Proposition 5.10. *In a $(WCBS)_4$ spacetime with non-constant scalar curvature satisfying $divC = 0$ and fulfilling the condition $r = \frac{b}{a}$, if ρ is Killing vector then the possible Petrov types are I or D.*

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