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# COEFFICIENT INEQUALITIES FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH HANKEL DETERMINANT

## (DEDICATED IN OCCASION OF THE 65-YEARS OF PROFESSOR R.K. RAINA)

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Abstract. In this paper we obtain the functional $|a_2a_4 - a_3^2|$ for the class $f \in R(\alpha)$. Also we give sharp upper bound for $|a_2a_4 - a_3^2|$. Our result extends corresponding previously known result.

## 1. Introduction and Preliminaries

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the unit disc $U = \{z : |z| < 1\}$. Let $P$ be the family of all functions $p$ analytic in $U$ for which $Re\{p(z)\} > 0$ and

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in U. \tag{1.2}$$

In 1976, Noonan and Thomas [10] defined the $q$th Hankel determinant of $f$ for $q \geq 1$ by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

Further, Fekete and Szegö [1] considered the Hankel determinant of $f \in A$ for $q = 2$ and $n = 1$, $H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}$. They made an early study for the estimates

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of $|a_3 - \mu a_2^2|$ when $a_1 = 1$ with $\mu$ real. The well known result due to them states that if $f \in A$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 4\mu - 3 & \text{if} \quad \mu \geq 1, \\ 1 + 2\ exp(\frac{-2\mu}{1-\mu}) & \text{if} \quad 0 \leq \mu \leq 1, \\ 3 - 4\mu & \text{if} \quad \mu \leq 0. \end{cases}$$

Furthermore, Hummel [3, 4] obtained sharp estimates for $|a_3 - \mu a_2^2|$ when $f$ is convex functions and also Keogh and Merkes [6] obtained sharp estimates for $|a_3 - \mu a_2^2|$ when $f$ is close-to-convex, starlike and convex in $U$.

Here we consider the Hankel determinant of $f \in A$ for $q = 2$ and $n = 2$,

$$H_2(2) \ = \ \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}.$$

In the present investigation we consider the following subclass $R(\alpha)$ of $A$ :

$$R(\alpha) = \left\{ f(z) \in A : \ Re \left\{ (1-\alpha)\frac{f(z)}{z} + \alpha f'(z) \right\} > 0, \ \alpha > 0, \ \ z \in U \right\} \tag{1.3}$$

and obtain sharp upper bound for the functional $|a_2 a_4 - a_3^2|$ of $f \in R(\alpha)$.

**Remark.** *The subclass $R(1) = R$ was studied systematically by MacGregor* [9] *who indeed referred to numerous earlier investigations involving functions whose derivative has a positive real part.*

To prove our main result, we need the following lemmas.

**Lemma 1.1.** [11] *If $p \in P$, then $|c_k| \leq 2$ for each $k$.*

**Lemma 1.2.** [2] *The power series for $p(z)$ given in (1.2) converges in $U$ to a function in $P$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, \quad n = 1, 2, 3, \ldots \tag{1.4}$$

*and $c_{-k} = \overline{c_k}$, are all nonnegative. They are strictly positive except for*

$$p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{it_k z}), \ \rho_k > 0, \ t_k \text{ real}$$

*and $t_k \neq t_j$ for $k \neq j$ in this case $D_n > 0$ for $n < m - 1$ and $D_n = 0$ for $n \geq m$.*

## 2. Main Result

Using the techniques of Libera and Zlotkiewicz [7, 8], we now prove the following theorem.

**Theorem 2.1.** *Let $\alpha > 0$. If $f \in R(\alpha)$, then*

$$|a_2 a_4 - a_3^2| \leq \frac{4}{(1 + 2\alpha)^2}. \tag{2.1}$$

*The result is sharp.*

*Proof.* Since $f \in R(\alpha)$, it follows from (1.3) that

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = p(z) \tag{2.2}$$

for some $p \in P$. Equating coefficients in (2.2), we have,

$$(1+\alpha)a_2 = c_1,\ \ (1+2\alpha)a_3 = c_2,\ \ (1+3\alpha)a_4 = c_3. \tag{2.3}$$

From (2.3), it can be established that

$$|a_2a_4 - a_3^2| = \left|\frac{c_1c_3}{(1+\alpha)(1+3\alpha)} - \frac{c_2^2}{(1+2\alpha)^2}\right|.$$

We make use of Lemma 1.2 to obtain the proper bound on $\left|\frac{c_1c_3}{(1+\alpha)(1+3\alpha)} - \frac{c_2^2}{(1+2\alpha)^2}\right|$. We may assume without restriction that $c_1 > 0$. We begin by rewriting (1.4) for the cases $n = 2$ and $n = 3$,

$$D_2 \ = \ \begin{vmatrix} 2 & c_1 & c_2 \\ c_1 & 2 & c_1 \\ \overline{c_2} & c_1 & 2 \end{vmatrix} \ = \ 8 + 2\text{Re }\{c_1^2c_2\} - 2|c_2|^2 - 4c_1^2 \geq 0, \tag{2.4}$$

which is equivalent to

$$2c_2 = c_1^2 + x(4-c_1^2) \tag{2.5}$$

for some $x$, $|x| \leq 1$. Then $D_3 \geq 0$ is equivalent to

$$|\ (4c_3 - 4c_1c_2 + c_1^3)(4-c_1^2) + c_1(2c_2 - c_1^2)^2\ | \leq 2(4-c_1^2)^2 - 2|2c_2 - c_1^2|^2 \tag{2.6}$$

and from (2.6) with (2.5), we have,

$$4c_3 \ = \ c_1^3 + 2(4-c_1^2)c_1x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z, \tag{2.7}$$

for some value of $z$, $|z| \leq 1$.

Suppose $c_1 = c$ and $c \in [0,2]$. Using (2.5) along with (2.7) we obtain

$$\begin{aligned}
&\left|\frac{c_1c_3}{(1+\alpha)(1+3\alpha)} - \frac{c_2^2}{(1+2\alpha)^2}\right| \\
&= \left|\frac{\alpha^2c^4 + 2\alpha^2c^2(4-c^2)x - (12\alpha^2 + 16\alpha + \alpha^2c^2 + 4)(4-c^2)x^2}{4(1+\alpha)(1+2\alpha)^2(1+3\alpha)} + \frac{c(4-c^2)(1-|x|^2)z}{2(1+\alpha)(1+3\alpha)}\right| \\
&\leq \frac{\alpha^2c^4}{4(1+\alpha)(1+2\alpha)^2(1+3\alpha)} + \frac{c(4-c^2)}{2(1+\alpha)(1+3\alpha)} + \frac{\alpha^2c^2(4-c^2)\rho}{2(1+\alpha)(1+2\alpha)^2(1+3\alpha)} \\
&\quad + \frac{(c-2)(4-c^2)[\alpha^2(c-6) - 8\alpha - 2]\rho^2}{4(1+\alpha)(1+2\alpha)^2(1+3\alpha)} \\
&\equiv F(\rho)
\end{aligned} \tag{2.8}$$

with $\rho = |x| \leq 1$ and $\alpha > 0$. We assume that the upper bound for (2.8) is attained at an interior point of the set $\{(\rho, c)|\ \rho \in [0,1],\ \ c \in [0,2]\}$, then

$$F'(\rho) = \frac{\alpha^2c^2(4-c^2)}{2(1+\alpha)(1+2\alpha)^2(1+3\alpha)} + \frac{(c-2)(4-c^2)[\alpha^2(c-6) - 8\alpha - 2]\rho}{2(1+\alpha)(1+2\alpha)^2(1+3\alpha)}. \tag{2.9}$$

We note that $F'(\rho) > 0$ and consequently F is increasing and $\max F(\rho) = F(1)$, which contradicts our assumption of having the maximum value at the interior of

$\rho \in [0,1]$. Now let

$$G(c) = F(1) = \frac{\alpha^2 c^4}{4(1+\alpha)(1+2\alpha)^2(1+3\alpha)} + \frac{c(4-c^2)}{2(1+\alpha)(1+3\alpha)} + \frac{\alpha^2 c^2(4-c^2)}{2(1+\alpha)(1+2\alpha)^2(1+3\alpha)} + \frac{(c-2)(4-c^2)[\alpha^2(c-6)-8\alpha-2]}{4(1+\alpha)(1+2\alpha)^2(1+3\alpha)}$$

then

$$G'(c) = \frac{-2c[\alpha^2 c^2 + 4\alpha + 1]}{(1+\alpha)(1+2\alpha)^2(1+3\alpha)} = 0 \tag{2.10}$$

therefore (2.10) implies $c = 0$, which is a contradiction. We note that

$$G''(c) = \frac{-6\alpha^2 c^2 - 8\alpha - 2}{(1+\alpha)(1+2\alpha)^2(1+3\alpha)} < 0.$$

Thus any maximum points of $G$ must be on the boundary of $c \in [0,2]$. However, $G(c) \geq G(2)$ and thus $G$ has maximum value at $c = 0$. The upper bound for (2.8) corresponds to $\rho = 1$ and $c = 0$, in which case

$$\left| \frac{c_1 c_3}{(1+\alpha)(1+3\alpha)} - \frac{c_2^2}{(1+2\alpha)^2} \right| \leq \frac{4}{(1+2\alpha)^2}, \quad \alpha > 0.$$

This completes the proof of the Theorem 2.1. □

**Remark.** *If $\alpha = 1$, then we get the corresponding functional $|a_2a_4 - a_3^2|$ for the class $f \in R(1) = R$, studied in* [5] *as in the following corollary.*

**Corollary 2.2.** *If $f \in R$, then*

$$|a_2a_4 - a_3^2| \leq \frac{4}{9}.$$

*The result is sharp.*

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