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FABER POLYNOMIALS COEFFICIENT ESTIMATES FOR BI-UNIVALENT SAKAGUCHI TYPE FUNCTIONS

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Abstract. In this work, considering a general subclass of bi-univalent Sakaguchi type functions, we determine estimates for the general Taylor-Maclaurin coefficients of the functions in these classes. For this purpose we use the Faber polynomial expansions, and in certain cases our estimates improve some of those existing coefficient bounds.

1. Introduction

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$. We also denote by S the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $\mathbb{U}$.

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, which satisfy $f^{-1}(f(z)) = z$ for all $z \in \mathbb{U}$ and $f\left(f^{-1}(w)\right) = w$ for all $|w| < r_0(f)$, with $r_0(f) \geq \frac{1}{4}$. In fact, the inverse function $g := f^{-1}$ is given by

$$\begin{aligned} g(w) = f^{-1}(w) &= w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \ldots \\ &= w + \sum_{n=2}^{\infty} A_n w^n. \end{aligned} \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent in* $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$, and let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1).

The class of analytic bi-univalent functions was first introduced and studied by Lewin [15], where it was proved that $|a_2| < 1.51$. Netanyahu [16] proved that $|a_2| \leq \frac{4}{3}$. Brannan and Taha [4] also investigated certain subclasses of bi-univalent

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functions and found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. For a brief history and interesting examples of functions in the class Σ, see [19] In fact, the aforecited work of Srivastava et al. [19] essentially revived the the investigation of various subclasses of the bi-univalent function class Σ in recent years; it was followed by such works as those by Frasin and Aouf [8], Xu et al. [21, 22], Hayami and Owa [12].

Not much is known about the bounds on the general coefficient $|a_n|$ for $n > 3$. This is because the bi-univalency requirement makes the behaviour of the coefficients of the functions f and f^{-1} unpredictable. In this paper we use the Faber polynomial expansions for a general subclass of bi-univalent Sakaguchi type functions.

The *Faber polynomials* introduced by Faber [6] play an important role in various areas of mathematical sciences, especially in geometric function theory. The recent publications [9] and [11] applying the Faber polynomial expansions to meromorphic bi-univalent functions motivated us to apply this technique to classes of analytic bi-univalent functions.

In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions given by (1.1) using Faber polynomial expansions [10, 13, 14]. Hamidi and Jahangiri [10] considered the class of *analytic bi-close-to-convex functions*. Also, Jahangiri and Hamidi [13] studied the class defined by Frasin and Aouf [8], while Jahangiri et al. [14] investigated the class of *analytic bi-univalent functions with positive real-part derivatives*.

Motivated by the works of Bulut, we defined and studied the main properties of the following classes. We begin by finding the estimate on the coefficients $|a_n|$, $|a_2|$ and $|a_3|$ for *bi-univalent Sakaguchi type functions* in the classes $P_\Sigma(\alpha, \lambda, t)$ and $Q_\Sigma(\alpha, \lambda, t)$ respectively.

2. The Classes $P_\Sigma(\alpha, \lambda, t)$ and $Q_\Sigma(\alpha, \lambda, t)$

Definition 2.1. *For $0 \le \lambda \le 1$, $|t| \le 1$ and $t \ne 1$, a function $f \in \Sigma$ given by* (1.1) *is said to be in the class $P_\Sigma(\alpha, \lambda, t)$ if the following conditions are satisfied:*

$$\operatorname{Re} \frac{(1-t)zf'(z)}{(1-\lambda)\left[f(z)-f(tz)\right]+\lambda z\left[f'(z)-tf'(tz)\right]} > \alpha,\ z \in \mathbb{U}$$

and

$$\operatorname{Re} \frac{(1-t)wg'(w)}{(1-\lambda)\left[g(w)-g(tw)\right]+\lambda w\left[g'(w)-tg'(tw)\right]} > \alpha,\ w \in \mathbb{U}$$

where $0 \le \alpha < 1$ and $g := f^{-1}$ is defined by (1.2).

Definition 2.2. *For $0 \le \lambda \le 1$, $|t| \le 1$ and $t \ne 1$, a function $f \in \Sigma$ given by* (1.1) *is said to be in the class $Q_\Sigma(\alpha, \lambda, t)$ if the following conditions are satisfied:*

$$\operatorname{Re} \frac{(1-t)\left[\lambda z^2 f''(z)+zf'(z)\right]}{f(z)-f(tz)} > \alpha,\ z \in \mathbb{U}$$

and

$$\operatorname{Re} \frac{(1-t)\left[\lambda w^2 g''(w)+wg'(w)\right]}{g(w)-g(tw)} > \alpha,\ w \in \mathbb{U}$$

where $0 \le \alpha < 1$ and $g := f^{-1}$ is defined by (1.2).

Remarks. *1. Taking $t = 0$ and $\lambda = 0$ in Definition 2.1 and Definition 2.2, we get the well-known class $P_\Sigma(\alpha) := P_\Sigma(\alpha, 0, 0) = Q_\Sigma(\alpha, 0, 0)$ of bi-starlike functions of order α. This class consists of functions $f \in \Sigma$ satisfying $\operatorname{Re} \dfrac{zf'(z)}{f(z)} > \alpha$, $z \in \mathbb{U}$, and $\operatorname{Re} \dfrac{wg'(w)}{g(w)} > \alpha$, $w \in \mathbb{U}$, where $0 \le \alpha < 1$ and $g := f^{-1}$ is defined by* (1.2).

2. The name of Sakaguchi type functions is motivated by the papers [18] *and* [7].

3. Coefficient Estimates

Using the Faber polynomial expansion of functions $f \in \mathcal{A}$ of the form (1.1), the coefficients of its inverse map $g = f^{-1}$ may be expressed like in [3], that is

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n} (a_2, a_3, \dots, a_n) w^n, \tag{3.1}$$

where

$$\begin{aligned} K_{n-1}^{-n} (a_2, a_3, \dots, a_n) &= \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!} a_2^{n-3} a_3 \\ &+ \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 + \frac{(-n)!}{(2(-n+2))!(n-5)!} a_2^{n-5} \left[a_5 + (-n+2) a_3^2\right] \\ &+ \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} \left[a_6 + (-2n+5) a_3 a_4\right] + \sum_{j \ge 7} a_2^{n-j} v_j, \end{aligned} \tag{3.2}$$

such that v_j, with $7 \le j \le n$, is a homogenous polynomial of degree j in the variables $a_2, a_3, \dots, a_n$. In particular, the first three terms of $K_{n-1}^{-n} (a_2, a_3, \dots, a_n)$ are

$$\begin{gathered} K_1^{-2} (a_2) = -2a_2, \quad K_2^{-3} (a_2, a_3) = 3 \left(2a_2^2 - a_3\right), \\ K_3^{-4} (a_2, a_3, a_4) = -4 \left(5a_2^3 - 5a_2 a_3 + a_4\right). \end{gathered} \tag{3.3}$$

For the above formulas we used the fact that for any integer $p \in \mathbb{Z}$ the expansion of K_n^p has the form (see [2, p. 349])

$$\begin{aligned} K_n^p := K_n^p (b_1, b_2, \dots b_n) &= \frac{p!}{(p-n)!n!} b_1^n + \frac{p!}{(p-n+1)!(n-2)!} b_1^{n-2} b_2 \\ &+ \frac{p!}{(p-n+2)!(n-3)!} b_1^{n-3} b_3 + \frac{p!}{(p-n+3)!(n-4)!} b_1^{n-4} \left[b_4 + \frac{p-n+3}{2} b_2^2\right] \\ &+ \frac{p!}{(p-n+4)!(n-5)!} b_1^{n-5} \left[b_5 + (p-n+4) b_2 b_3\right] + \sum_{j \ge 6} b_1^{n-j} v_j, \end{aligned}$$

such that v_j, with $6 \le j \le n$, is a homogenous polynomial of degree j in the variables $b_1, b_2, \dots, b_n$, and the notation

$$\frac{p!}{(p-n)!n!} := \frac{(p-n+1)(p-n+2)\dots p}{n!}$$

extends to any $p \in \mathbb{Z}$.

In general, for any $p \in \mathbb{Z}$ the expansion of K_n^p has the form (see [3, p. 183])

$$K_n^p(b_1, b_2, \ldots b_n) = pb_n + \frac{p(p-1)}{2} D_n^2 + \frac{p!}{(p-3)!3!} D_n^3 + \cdots + \frac{p!}{(p-n)!n!} D_n^n, \tag{3.4}$$

where D_n^p are given by (see [20, p. 268–269])

$$D_n^p := D_n^m(b_1, b_2, \ldots, b_{n-m+1}) = \sum \frac{m!}{i_1! \ldots i_{n-m+1}!} b_1^{i_1} \ldots b_n^{i_n},$$

and the sum is taken over all non-negative integers $i_1, \ldots, i_{n-m+1}$ satisfying

$$\begin{aligned} &i_1 + i_2 + \cdots + i_{n-m+1} = m, \\ &i_1 + 2i_2 + \cdots + (n-m+1)i_{n-m+1} = n. \end{aligned}$$

It is obvious that $D_n^n(b_1, b_2, \ldots, b_n) = b_1^n$.

Consequently, for any function $f \in P_\Sigma(\alpha, \lambda, t)$ of the form (1.1), we can write

$$\frac{(1-t)\left[zf'(z)\right]}{(1-\lambda)\left[f(z) - f(tz)\right] + \lambda z\left[f'(z) - tf'(tz)\right]} = 1 + \sum_{n=2}^{\infty} F_{n-1}(a_2, a_3, \ldots, a_n) z^{n-1}, \tag{3.5}$$

where F_{n-1} is the Faber polynomial of degree $(n-1)$ and

$$\begin{aligned}
F_1 =& \left[2(1-\lambda) - u_2(1-\lambda+2\lambda t)\right] \frac{a_2}{1+\lambda t}, \\
F_2 =& \frac{1}{1+\lambda t} \Big\{ \left[3(1-\lambda) - u_3(1-\lambda+3\lambda t)\right] a_3 - F_1 \left[2\lambda + u_2(1-\lambda+2\lambda t)\right] a_2 \Big\} \\
=& \left[3(1-\lambda) - u_3(1-\lambda+3\lambda t)\right] \frac{a_3}{1+\lambda t} \\
&- \left[2(1-\lambda) - u_2(1-\lambda+2\lambda t)\right] \left[2\lambda + u_2(1-\lambda+2\lambda t)\right] \frac{a_2^2}{(1+\lambda t)^2}, \\
F_3 =& \frac{1}{1+\lambda t} \Big\{ \left[4(1-\lambda) - u_4(1-\lambda+4\lambda t)\right] a_4 - F_2 \left[2\lambda + u_2(1-\lambda+2\lambda t)\right] a_2 \\
&- F_1 \left[3\lambda + u_3(1-\lambda+3\lambda t)\right] a_3 \Big\} \\
=& \left[4(1-\lambda) - u_4(1-\lambda+4\lambda t)\right] \frac{a_4}{1+\lambda t} \\
&- \left[2(1-\lambda) - u_2(1-\lambda+2\lambda t)\right] \left[3\lambda + u_3(1-\lambda+3\lambda t)\right] \frac{a_2 a_3}{(1+\lambda t)^2} \\
&- \left[3(1-\lambda) - u_3(1-\lambda+2\lambda t)\right] \left[2\lambda + u_2(1-\lambda+2\lambda t)\right] \frac{a_2 a_3}{(1+\lambda t)^2} \\
&+ \left[2(1-\lambda) - u_2(1-\lambda+2\lambda t)\right] \left[2\lambda + u_2(1-\lambda+2\lambda t)\right]^2 \frac{a_2^3}{(1+\lambda t)^3}, \text{ etc.}
\end{aligned}$$

where

$$u_n := \frac{1-t^n}{1-t}, \ n \in \mathbb{N}. \tag{3.6}$$

In general

$$F_{n-1}\left(a_2,a_3,\ldots,a_n\right)=\frac{1}{(1+\lambda t)}\Big\{[n(1-\lambda)-u_n(1-\lambda+n\lambda t)]a_n$$
$$-F_{n-2}[2\lambda+u_2(1-\lambda+2\lambda t)]a_2-F_{n-3}[3\lambda+u_3(1-\lambda+2\lambda t)]a_3$$
$$\cdots-F_1[(n-1)\lambda+u_{n-1}(1-\lambda+(n-1)t)]a_{n-1}\Big\}.$$

Similarly, if the functions $f\in Q_\Sigma(\alpha,\lambda,t)$ has the form (1.1), we can write

$$\frac{(1-t)\left[\lambda z^2 f''(z)+zf'(z)\right]}{f(z)-f(tz)}=1+\sum_{n=2}^{\infty}F_{n-1}\left(a_2,a_3,\ldots,a_n\right)z^{n-1}, \tag{3.7}$$

where F_{n-1} is the Faber polynomial of degree $(n-1)$ and

$$\begin{aligned}
F_1&=\left[2(\lambda+1)-u_2\right]a_2,\\
F_2&=\left[3(2\lambda+1)-u_3\right]a_3-F_1u_2a_2\\
&=\left[3(2\lambda+1)-u_3\right]a_3-\left[2(\lambda+1)-u_2\right]u_2a_2^2,\\
F_3&=\left[4(3\lambda+1)-u_4\right]a_4-F_2u_2a_2-F_1u_3a_3\\
&=\left[4(3\lambda+1)-u_4\right]a_4-\left[2(\lambda+1)u_3+3(2\lambda+1)u_2-2u_2u_3\right]a_2a_3\\
&\quad+\left[2(\lambda+1)-u_2\right]u_2^2a_2^3,\ \text{etc.}
\end{aligned}$$

where u_n is given by (3.6). In general

$$F_{n-1}\left(a_2,a_3,\ldots,a_n\right)=[(n(n-1)\lambda+1)-u_n]a_n-F_{n-2}u_2a_2-F_{n-3}u_3a_3$$
$$\cdots-F_1u_{n-1}a_{n-1}.$$

In our first theorem, for some special cases, we obtained an upper bound for the coefficients $|a_n|$ of bi-univalent Sakaguchi type functions in the class $P_\Sigma(\alpha,\lambda,t)$.

Theorem 3.1. *For $0\le\lambda\le1$, $|t|\le1$ with $t\neq1$, and $0\le\alpha<1$, let the function $f\in P_\Sigma(\alpha,\lambda,t)$ be given by* (1.1)*. If $a_k=0$ for all $2\le k\le n-1$, then*

$$|a_n|\le\frac{2(1-\alpha)\left|1+\lambda t\right|}{\left|n(1-\lambda)-u_n(1-\lambda+n\lambda t)\right|},\ n\ge4.$$

Proof. For the functions $f\in P_\Sigma(\alpha,\lambda,t)$ of the form (1.1) we have the expansion (3.5), and for the inverse map $g=f^{-1}$, according to (1.2), (3.1), we obtain

$$\frac{(1-t)wg'(w)}{(1-\lambda)\left[g(w)-g(tw)\right]+\lambda w\left[g'(w)-tg'(tw)\right]}$$
$$=1+\sum_{n=2}^{\infty}F_{n-1}\left(A_2,A_3,\ldots,A_n\right)w^{n-1},\ z\in\mathbb{U}, \tag{3.8}$$

where

$$A_n=\frac{1}{n}K_{n-1}^{-n}\left(a_2,a_3,\ldots,a_n\right).$$

On the other hand, since $f\in P_\Sigma(\alpha,\lambda,t)$ and $g=f^{-1}\in P_\Sigma(\alpha,\lambda,t)$, from the Definition 2.1 there exist two analytic functions $p(z)=1+\sum_{n=1}^{\infty}c_nz^n$ and $q(w)=$

$1+\sum_{n=1}^{\infty} d_n w^n$, with $\operatorname{Re} p(z)>0$, $z\in\mathbb{U}$ and $\operatorname{Re} q(w)>0$, $w\in\mathbb{U}$, such that

$$\begin{aligned}\frac{(1-t)zf'(z)}{(1-\lambda)\left[f(z)-f(tz)\right]+\lambda z\left[f'(z)-tf'(tz)\right]}&=\alpha+(1-\alpha)p(z)\\&=1+(1-\alpha)\sum_{n=1}^{\infty}K_n^1\left(c_1,c_2,\dots,c_n\right)z^n,\ z\in\mathbb{U},\end{aligned}\tag{3.9}$$

and

$$\begin{aligned}\frac{(1-t)wg'(w)}{(1-\lambda)\left[g(w)-g(tw)\right]+\lambda w\left[g'(w)-tg'(tw)\right]}&=\alpha+(1-\alpha)q(w)\\&=1+(1-\alpha)\sum_{n=1}^{\infty}K_n^1\left(d_1,d_2,\dots,d_n\right)w^n,\ w\in\mathbb{U}.\end{aligned}\tag{3.10}$$

Comparing the corresponding coefficients of (3.5) and (3.9), we get

$$F_{n-1}\left(a_2,a_3,\dots,a_n\right)=(1-\alpha)K_{n-1}^1\left(c_1,c_2,\dots,c_{n-1}\right),\ n\geq 2,\tag{3.11}$$

and similarly, from (3.8) and (3.10) we find

$$F_{n-1}\left(A_2,A_3,\dots,A_n\right)=(1-\alpha)K_{n-1}^1\left(d_1,d_2,\dots,d_{n-1}\right),\ n\geq 2.\tag{3.12}$$

Assuming that $a_k=0$ for all $2\leq k\leq n-1$, we obtain $A_n=-a_n$, and therefore

$$\begin{aligned}\frac{n(1-\lambda)-u_n(1-\lambda+n\lambda t)}{(1+\lambda t)}\,a_n&=(1-\alpha)c_{n-1},\\-\frac{n(1-\lambda)-u_n(1-\lambda+n\lambda t)}{(1+\lambda t)}\,a_n&=(1-\alpha)d_{n-1}.\end{aligned}$$

From Carathéodory lemma (see, e.g. [5]) we have $|c_n|\leq 2$ and $|d_n|\leq 2$ for all $n\in\mathbb{N}$, and taking the absolute values of the above equalities, we obtain

$$\begin{aligned}|a_n|&=\frac{(1-\alpha)|c_{n-1}|\,|1+\lambda t|}{|n(1-\lambda)-u_n(1-\lambda+n\lambda t)|}=\frac{(1-\alpha)|d_{n-1}|\,|1+\lambda t|}{|n(1-\lambda)-u_n(1-\lambda+n\lambda t)|}\\&\leq\frac{2(1-\alpha)\,|1+\lambda t|}{|n(1-\lambda)-u_n(1-\lambda+n\lambda t)|},\end{aligned}$$

where u_n is given by (3.6), which completes the proof of our theorem. □

Theorem 3.2. *For $0\leq\lambda\leq 1$, $|t|\leq 1$ with $t\neq 1$, and $0\leq\alpha<1$, let the function $f\in Q_\Sigma(\alpha,\lambda,t)$ be given by* (1.1)*. If $a_k=0$ for all $2\leq k\leq n-1$, then*

$$|a_n|\leq\frac{2(1-\alpha)}{|n\left[(n-1)\lambda+1\right]-u_n|},\ n\geq 4,$$

where u_n is given by (3.6)*.*

Proof. For the functions $f\in Q_\Sigma(\alpha,\lambda,t)$ of the form (1.1) we have the expansion (3.7), and for the inverse map $g=f^{-1}$, according to (1.2), (3.1),we obtain

$$\begin{aligned}&\frac{(1-t)\left[\lambda w^2 g''(w)+wg'(w)\right]}{g(w)-g(tw)}\\&=1+\sum_{n=2}^{\infty}F_{n-1}\left(A_2,A_3,\dots,A_n\right)w^{n-1},\ w\in\mathbb{U},\end{aligned}\tag{3.13}$$

where

$$A_n = \frac{1}{n} K_{n-1}^{-n} (a_2, a_3, \dots, a_n).$$

On the other hand, since $f \in Q_\Sigma(\alpha, \lambda, t)$ and $g = f^{-1} \in Q_\Sigma(\alpha, \lambda, t)$, from the Definition 2.2 there exist two analytic functions $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ and $q(w) = 1 + \sum_{n=1}^{\infty} d_n w^n$, with $\operatorname{Re} p(z) > 0$, $z \in \mathbb{U}$ and $\operatorname{Re} q(w) > 0$, $w \in \mathbb{U}$, such that

$$\begin{aligned} &\frac{(1-t)\left[\lambda z^2 f''(z) + z f'(z)\right]}{f(z) - f(tz)} = \alpha + (1-\alpha) p(z) \\ &= 1 + (1-\alpha) \sum_{n=1}^{\infty} K_n^1 (c_1, c_2, \dots, c_n) z^n, \ z \in \mathbb{U}, \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} &\frac{(1-t)\left[\lambda w^2 g''(w) + w g'(w)\right]}{g(w) - g(tw)} = \alpha + (1-\alpha) q(w) \\ &= 1 + (1-\alpha) \sum_{n=1}^{\infty} K_n^1 (d_1, d_2, \dots, d_n) w^n, \ w \in \mathbb{U}. \end{aligned} \tag{3.15}$$

Comparing the corresponding coefficients of (3.7) and (3.14), we get

$$F_{n-1}(a_2, a_3, \dots, a_n) = (1-\alpha) K_{n-1}^1 (c_1, c_2, \dots, c_{n-1}), \ n \geq 2, \tag{3.16}$$

and similarly, from (3.13) and (3.15) we find

$$F_{n-1}(A_2, A_3, \dots, A_n) = (1-\alpha) K_{n-1}^1 (d_1, d_2, \dots, d_{n-1}), \ n \geq 2. \tag{3.17}$$

Assuming that $a_k = 0$ for all $2 \leq k \leq n-1$, we obtain $A_n = -a_n$, and therefore

$$\begin{aligned} \left[n\left[(n-1)\lambda + 1\right] - u_n\right] a_n &= (1-\alpha) c_{n-1}, \\ -\left[n\left[(n-1)\lambda + 1\right] - u_n\right] a_n &= (1-\alpha) d_{n-1}. \end{aligned}$$

From Carathéodory lemma (see, e.g. [5]) we have $|c_n| \leq 2$ and $|d_n| \leq 2$ for all $n \in \mathbb{N}$, and taking the absolute values of the above equalities, we obtain

$$\begin{aligned} |a_n| &= \frac{(1-\alpha)|c_{n-1}|}{|n\left[(n-1)\lambda + 1\right] - u_n|} = \frac{(1-\alpha)|d_{n-1}|}{|n\left[(n-1)\lambda + 1\right] - u_n|} \\ &\leq \frac{2(1-\alpha)}{|n\left[(n-1)\lambda + 1\right] - u_n|}, \end{aligned}$$

where u_n is given by (3.6), which completes the proof of our theorem. □

Theorem 3.3. *For* $0 \leq \lambda \leq 1$, $|t| \leq 1$, $t \neq 1$, $0 \leq \alpha < 1$, *let the function* $f \in P_\Sigma(\alpha, \lambda, t)$ *be given by* (1.1). *Then, the following inequalities hold:*

$$|a_2| \leq \begin{cases} \sqrt{\dfrac{2(1-\alpha)\,|1+\lambda t|^2}{|B|}}, & \text{for} \quad 0 \leq \alpha < \dfrac{|A|}{2\,|B|}, \\ \dfrac{2(1-\alpha)\,|1+\lambda t|}{|2(1-\lambda) - u_2(1-\lambda+2\lambda t)|}, & \text{for} \quad \dfrac{|A|}{2\,|B|} \leq \alpha < 1, \end{cases} \tag{3.18}$$

$$|a_3| \leq \begin{cases} \min\left\{\left|\dfrac{4(1-\alpha)^2(1+\lambda t)^2}{[2(1-\lambda)-u_2(1-\lambda+2\lambda t)]^2}+\dfrac{2(1-\alpha)(1+\lambda t)}{3(1-\lambda)-u_3(1-\lambda+3\lambda t)}\right|;\right. \\ \qquad\qquad\qquad\qquad \left.\dfrac{2(1-\alpha)\left|1+\lambda t\right|}{|B|}\right\}, \quad \textit{for} \quad 0\leq\lambda<1, \\ \dfrac{2(1-\alpha)\left|1+\lambda t\right|}{\left|3(1-\lambda)-u_3(1-\lambda+3\lambda t)\right|}, \quad \textit{for} \quad \lambda=1, \end{cases} \tag{3.19}$$

and

$$\left|a_3-\frac{C}{[3(1-\lambda)-u_3(1-\lambda+3\lambda t)]\,(1+\lambda t)}a_2^2\right| \leq \frac{2(1-\alpha)\left|1+\lambda t\right|}{\left|3(1-\lambda)-u_3(1-\lambda+3\lambda t)\right|},$$

where

$$\begin{aligned} A =& 2\left[3(1-\lambda)-u_3(1-\lambda+3\lambda t)\right](1+\lambda t) \\ & -\left[2(1-\lambda)-u_2(1-\lambda+2\lambda t)\right]\left[2(1+\lambda)+u_2(1-\lambda+3\lambda t)\right], \\ B =& \left[3(1-\lambda)-u_3(1-\lambda+3\lambda t)\right](1+\lambda t) \\ & -\left[2(1-\lambda)-u_2(1-\lambda+2\lambda t)\right]\left[2\lambda+u_2(1-\lambda+2\lambda t)\right], \\ C =& 2\left[3(1-\lambda)-u_3(1-\lambda+3\lambda t)\right](1+\lambda t) \\ & -\left[2(1-\lambda)-u_2(1-\lambda+2\lambda t)\right]\left[2\lambda+u_2(1-\lambda+2\lambda t)\right]. \end{aligned} \tag{3.20}$$

Proof. Setting $n=2$ and $n=3$ in (3.11) and (3.12) we get, respectively,

$$\left[2(1-\lambda)-u_2(1-\lambda+2\lambda t)\right]\frac{a_2}{1+\lambda t}=(1-\alpha)c_1, \tag{3.21}$$

$$\begin{aligned} & \left[3(1-\lambda)-u_3(1-\lambda+3\lambda t)\right]\frac{a_3}{1+\lambda t} \\ & -\left[2(1-\lambda)-u_2(1-\lambda+2\lambda t)\right]\left[2\lambda+u_2(1-\lambda+2\lambda t)\right]\frac{a_2^2}{(1+\lambda t)^2}=(1-\alpha)c_2, \end{aligned} \tag{3.22}$$

$$-\left[2(1-\lambda)-u_2(1-\lambda+2\lambda t)\right]\frac{a_2}{1+\lambda t}=(1-\alpha)d_1, \tag{3.23}$$

$$\begin{aligned} & \Big\{2\left[3(1-\lambda)-u_3(1-\lambda+3\lambda t)\right](1+\lambda t) \\ & -\left[2(1-\lambda)-u_2(1-\lambda+2\lambda t)\right]\left[2\lambda+u_2(1-\lambda+2\lambda t)\right]\Big\}\frac{a_2^2}{(1+\lambda t)^2} \\ & -\left[3(1-\lambda)-u_3(1-\lambda+3\lambda t)\right]\frac{a_3}{1+\lambda t}=(1-\alpha)d_2. \end{aligned} \tag{3.24}$$

From (3.21) and (3.23), according to Carathéodory lemma we get

$$\begin{aligned} |a_2| =& \frac{(1-\alpha)|c_1|\left|1+\lambda t\right|}{\left|2(1-\lambda)-u_2(1-\lambda+2\lambda t)\right|}=\frac{(1-\alpha)|d_1|\left|1+\lambda t\right|}{\left|2(1-\lambda)-u_2(1-\lambda+2\lambda t)\right|} \\ \leq & \frac{2(1-\alpha)\left|1+\lambda t\right|}{\left|2(1-\lambda)-u_2(1-\lambda+2\lambda t)\right|}. \end{aligned} \tag{3.25}$$

Also, from (3.22) and (3.24) we obtain

$$2B\frac{a_2^2}{(1+\lambda t)^2} = (1-\alpha)(c_2+d_2), \tag{3.26}$$

then, from Carathéodory lemma we get

$$|a_2| \le \sqrt{\frac{2(1-\alpha)\,|1+\lambda t|^2}{|B|}},$$

and combining this with inequality (3.25), we obtain the desired estimate on the coefficient $|a_2|$ as asserted in (3.18).

In order to find the bound for the coefficient $|a_3|$, subtracting (3.24) from (3.22), we get

$$[3(1-\lambda) - u_3(1-\lambda+3\lambda t)]\left(-2a_2^2 + 2a_3\right) = (1-\alpha)\,(c_2-d_2)\,(1+\lambda t),$$

or

$$a_3 = a_2^2 + \frac{(1-\alpha)\,(c_2-d_2)\,(1+\lambda t)}{2\,[3(1-\lambda) - u_3(1-\lambda+3\lambda t)]}. \tag{3.27}$$

Upon substituting the value of a_2^2 from (3.21) into (3.27), it follows that

$$a_3 = \frac{(1-\alpha)^2 c_1^2 (1+\lambda t)^2}{[2(1-\lambda) - u_2(1-\lambda+3\lambda t)]^2} + \frac{(1-\alpha)\,(c_2-d_2)\,(1+\lambda t)}{2\,[3(1-\lambda) - u_3(1-\lambda+3\lambda t)]},$$

and thus, from Carathéodory lemma we obtain that

$$|a_3| \le \frac{4(1-\alpha)^2\,|1+\lambda t|^2}{|2(1-\lambda) - u_2(1-\lambda+3\lambda t)|^2} + \frac{2(1-\alpha)\,|1+\lambda t|}{|3(1-\lambda) - u_3(1-\lambda+3\lambda t)|}. \tag{3.28}$$

On the other hand, upon substituting the value of a_2^2 from (3.26) into (3.27) it follows that

$$a_3 = \frac{(1-\alpha)(1+\lambda t)\Big\{c_2 C + d_2\big\{\,[2(1-\lambda) - u_2(1-\lambda+2\lambda t)]\,[2\lambda + u_2(1-\lambda+2\lambda t)]\,\big\}\Big\}}{2B\,[3(1-\lambda) - u_3(1-\lambda+3\lambda t)]}, \tag{3.29}$$

and consequently, by Carathéodory lemma we have

$$|a_3| \le \frac{2(1-\alpha)\,|1+\lambda t|^2}{|B|}. \tag{3.30}$$

Combining (3.28) and (3.30), we get the desired estimate on the coefficient $|a_3|$ as asserted in (3.19).

Finally, from (3.24), by using Carathéodory lemma we deduce that

$$\left|a_3 - \frac{C}{[3(1-\lambda) - u_3(1-\lambda+3\lambda t)]\,(1+\lambda t)}\,a_2^2\right| \le \frac{2(1-\alpha)\,|1+\lambda t|}{|3(1-\lambda) - u_3(1-\lambda+3\lambda t)|},$$

where A, B and C are given by (3.20). □

Theorem 3.4. *For $0 \leq \lambda \leq 1$, $|t| \leq 1$, $t \neq 1$, $0 \leq \alpha < 1$, let the function $f \in Q_{\Sigma}(\alpha, \lambda, t)$ be given by (1.1). Then, the following inequalities hold:*

$$|a_2| \leq \begin{cases} \sqrt{\dfrac{2(1-\alpha)}{\left|3(2\lambda+1) - u_3 - [2(\lambda+1) - u_2]\, u_2\right|}}, \quad \text{for} \\ \qquad 0 \leq \alpha < \left|\dfrac{2\,[3(2\lambda+1) - u_3] - [2(\lambda+1) - u_2]\,[2(\lambda+1) + u_2]}{2\big\{3(2\lambda+1) - u_3 - [2(\lambda+1) - u_2]\, u_2\big\}}\right|, \\ \dfrac{2(1-\alpha)}{|2(\lambda+1) - u_2|}, \quad \text{for} \\ \qquad \left|\dfrac{2\,[3(2\lambda+1) - u_3] - [2(\lambda+1) - u_2]\,[2(\lambda+1) + u_2]}{2\big\{3(2\lambda+1) - u_3 - [2(\lambda+1) - u_2]\, u_2\big\}}\right| \leq \alpha < 1, \end{cases} \tag{3.31}$$

$$|a_3| \leq \begin{cases} \min\left\{\left|\dfrac{4(1-\alpha)^2}{[2(1-\lambda) - u_2]^2} + \dfrac{2(1-\alpha)}{3(2\lambda+1) - u_3}\right|; \right. \\ \qquad \left.\left|\dfrac{2(1-\alpha)}{3(2\lambda+1) - u_3 - [2(\lambda+1) - u_2]\, u_2}\right|\right\}, \quad \text{for} \quad 0 \leq \lambda < 1, \\ \dfrac{2(1-\alpha)}{|3(2\lambda+1) - u_3|}, \quad \text{for} \quad \lambda = 1, \end{cases} \tag{3.32}$$

and

$$\left|a_3 - \frac{2\,[3(2\lambda+1) - u_3] - [2(\lambda+1) - u_2]\, u_2}{3(2\lambda+1) - u_3}\, a_2^2\right| \leq \frac{2(1-\alpha)}{|3(2\lambda+1) - u_3|}.$$

Proof. Setting $n = 2$ and $n = 3$ in (3.16) and (3.17) we get, respectively,

$$[2(\lambda+1) - u_2]\, a_2 = (1-\alpha)c_1, \tag{3.33}$$

$$[3(2\lambda+1) - u_3]\, a_3 - [2(\lambda+1) - u_2]\, u_2 a_2^2 = (1-\alpha)c_2, \tag{3.34}$$

$$-\,[2(\lambda+1) - u_2]\, a_2 = (1-\alpha)d_1, \tag{3.35}$$

$$\big\{2\,[3(2\lambda+1) - u_3] - [2(\lambda+1) - u_2]\, u_2\big\} a_2^2 - [3(2\lambda+1) - u_3]\, a_3 = (1-\alpha)d_2. \tag{3.36}$$

From (3.33) and (3.35), according to Carathéodory lemma, we find

$$|a_2| = \frac{(1-\alpha)|c_1|}{|2(\lambda+1) - u_2|} = \frac{(1-\alpha)|d_1|}{|2(\lambda+1) - u_2|} \leq \frac{2(1-\alpha)}{|2(\lambda+1) - u_2|}. \tag{3.37}$$

Also, from (3.34) and (3.36) we obtain

$$2\big\{3(2\lambda+1) - u_3 - [2(\lambda+1) - u_2]\, u_2\big\} a_2^2 = (1-\alpha)(c_2 + d_2), \tag{3.38}$$

then, from Carathéodory lemma we get

$$|a_2| \leq \sqrt{\frac{2(1-\alpha)}{\left|3(2\lambda+1) - u_3 - [2(\lambda+1) - u_2]\, u_2\right|}},$$

and combining this with inequality (3.37), we obtain the desired estimate on the coefficient $|a_2|$ as asserted in (3.31).

In order to find the bound for the coefficient $|a_3|$, subtracting (3.36) from (3.34), we get

$$[3(2\lambda+1)-u_3]\left(-2a_2^2+2a_3\right)=(1-\alpha)\left(c_2-d_2\right),$$

or

$$a_3=a_2^2+\frac{(1-\alpha)\left(c_2-d_2\right)}{2\left[3(2\lambda+1)-u_3\right]}. \tag{3.39}$$

Upon substituting the value of a_2^2 from (3.32) into (3.37), it follows that

$$a_3=\frac{(1-\alpha)^2c_1^2}{\left[2(\lambda+1)-u_2\right]^2}+\frac{(1-\alpha)\left(c_2-d_2\right)}{2\left[3(2\lambda+1)-u_3\right]},$$

and thus, from Carathéodory lemma we obtain that

$$|a_3|\leq\frac{4(1-\alpha)^2}{\left|2(\lambda+1)-u_2\right|^2}+\frac{2(1-\alpha)}{\left|3(2\lambda+1)-u_3\right|}. \tag{3.40}$$

On the other hand, upon substituting the value of a_2^2 from (3.38) into (3.39) it follows that

$$a_3=\frac{(1-\alpha)\Big\{c_2\Big[2\left[3(2\lambda+1)-u_3\right]-\left[2(\lambda+1)-u_2\right]u_2\Big]+d_2\left[2(\lambda+1)-u_2\right]u_2\Big\}}{2\left[3(2\lambda+1)-u_3\right]\Big\{3(2\lambda+1)-u_3-\left[2(\lambda+1)-u_2\right]u_2\Big\}},$$

and consequently, by Carathéodory lemma we have

$$|a_3|\leq\frac{2(1-\alpha)}{\left|3(2\lambda+1)-u_3-\left[2(\lambda+1)-u_2\right]u_2\right|}. \tag{3.41}$$

Combining (3.40) and (3.41), we get the desired estimate on the coefficient $|a_3|$ as asserted in (3.32).

Finally, from (3.36), by using Carathéodory lemma we deduce that

$$\left|a_3-\frac{2\left[3(2\lambda+1)-u_3\right]-\left[2(\lambda+1)-u_2\right]}{3(2\lambda+1)-u_3}a_2^2\right|\leq\frac{2(1-\alpha)}{\left|3(2\lambda+1)-u_3\right|}.$$

□

Remark. *For the special case* $t=0$ *and* $\lambda=0$*, the relations* (3.18) *and* (3.19)*, or* (3.31) *and* (3.32)*, yield that*

$$|a_2|\leq\sqrt{2(1-\alpha)}$$

and

$$|a_3|\leq 4(1-\alpha)^2+1-\alpha,$$

which are the bounds for the coefficients of the functions of the well-known class $P_\Sigma(\alpha)$*, and were previously given by S. Prema and B. Srutha Keerthi* [17].

References

[1] H. Airault, P. Malliavin, *Unitarizing probability measures for representations of Virasoro algebra*, J. Math. Pures Appl. **80** 6 (2001), 627–667.

[2] H. Airault, J. Ren, *An algebra of differential operators and generating functions on the set of univalent functions*, Bull. Sci. Math. **126** 5 (2002), 343–367.

[3] H. Airault, A. Bouali, *Differential calculus on the Faber polynomials*, Bull. Sci. Math. **130** 3 (2006), 179–222.

[4] D.A. Brannan, T.S. Taha, On some classes of bi-univalent functions, in: S.M. Mazhar, A. Hamoui, N.S. Faour (Eds.), Mathematical Analysis and Its Applications, Kuwait, February 18-21, 1985, in: KFAS Proc. Ser., vol. 3, Pergamon Press (Elsevier Science Limited), Oxford, 1988, 53–60; see also D.A. Brannan, T.S. Taha, Stud. Univ. Babeş-Bolyai Math. **31** 2 (1986), 70–77.

[5] P.L. Duren, *Univalent Functions*, Grundlehren Math. Wiss. **259**, Springer, New York (1983).

[6] G. Faber, *Über polynomische Entwickelungen*, Math. Ann. **57** 3 (1903), 389–408.

[7] B.A. Frasin, *Coefficient inequalities for certain classes of Sakaguchi type functions*, Int. J. Nonlinear Sci. **10** 2 (2010), 206–211.

[8] B.A. Frasin, M.K. Aouf, *New subclasses of bi-univalent functions*, Appl. Math. Lett. **24** 9 (2011) 1569–1573.

[9] S.G. Hamidi, S.A. Halim, J.M. Jahangiri, *Coefficient estimates for a class of meromorphic bi-univalent functions*, C. R. Acad. Sci. Paris, Ser. I **351** 9-10 (2013), 349–352.

[10] S.G. Hamidi, J.M. Jahangiri, *Faber polynomial coefficient estimates for analytic bi-close-to-convex functions*, C. R. Acad. Sci. Paris, Ser. I **352** 1 (2014), 17–20.

[11] S.G. Hamidi, T. Janani, G. Murugusundaramoorthy, J.M. Jahangiri, *Coefficient estimates for certain classes of meromorphic bi-univalent functions*, C. R. Acad. Sci. Paris, Ser. I **352** 4 (2014), 277–282.

[12] T. Hayami, S. Owa, *Coefficient bounds for bi-univalent functions*, Panamer. Math. J. **22** 4 (2012), 15–26.

[13] J.M. Jahangiri, S.G. Hamidi, *Coefficient estimates for certain classes of bi-univalent functions*, Int. J. Math. Math. Sci. (2013), Article ID 190560, 4 p.

[14] J.M. Jahangiri, S.G. Hamidi, S.A. Halim, *Coefficients of bi-univalent functions with positive real part derivatives*, Bull. Malays. Math. Soc. **37** 3 (2014), 633–640

[15] M. Lewin, *On a coefficient problem for bi-univalent functions*, Proc. Amer. Math. Soc. **18** 1 (1967), 63–68.

[16] E. Netanyahu, *The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in* $|z| < 1$, Arch. Ration. Mech. Anal. **32** (1969), 100–112.

[17] S. Prema, B.S. Keerthi, *Coefficient bounds for certain subclasses of analytic functions*, J. Math. Anal. **4** 1 (2013), 22–27.

[18] K. Sakaguchi, *On a certain univalent mapping*, J. Math. Soc. Japan **11** 1 (1959), 72–75.

[19] H.M. Srivastava, A.K. Mishra, P. Gochhayat, *Certain subclasses of analytic and bi-univalent functions*, Appl. Math. Lett. **23** (2010), 1188–1192.

[20] P.G. Todorov, *On the Faber polynomials of the univalent functions of class* Σ, J. Math. Anal. Appl. **162** 1 (1991), 268–276.

[21] Q.-H. Xu, Y.-C. Gui, H.M. Srivastava, *Coefficient estimates for a certain subclass of analytic and bi-univalent functions*, Appl. Math. Lett. **25** (2012), 990–994.

[22] Q.-H. Xu, H.-G. Xiao, H.M. Srivastava, *A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems*, Appl. Math. Comput. **218** (2012), 11461–11465.

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