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COMMON FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS ON METRIC SPACES WITH A DIRECTED GRAPH

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Abstract. The purpose of this article is to establish the existence of common fixed points of multivalued Θ-contraction mappings on a metric space endowed with a graph. Presented theorems are generalizations of recent fixed point theorems due to Hussain et al. [Fixed Point Theory and Applications (2015) 2015:185]. An example is also given to support our generalized result.

1. Introduction and Preliminaries

The Banach Contraction Principle is one of the cornerstones in the development of Nonlinear Analysis, in general, and metric fixed point theory, in particular. The method of successive approximation introduced by Liouville in 1837 and systematically developed by Picard in 1890 culminated in formulation of Banach Contraction Principle by Polish Mathematician Stefan Banach in 1922. This theorem provides an illustration of the unifying power of functional analytic methods and usefulness of fixed point theory in analysis. Extensions of the Banach contraction principle have been obtained either by generalizing the domain of the mapping or by extending the contractive condition on the mappings see [2, 3, 4, 13, 14, 15, 16, 17, 22, 23, 24].

Very recently, Jleli and Samet [18] introduced a new type of contraction and established some new fixed point theorems for such contraction in the context of generalized metric spaces.

Definition 1.1. *Let $\Theta : (0, \infty) \to (1, \infty)$ be a function satisfying:*

- (Θ_1) Θ is nondecreasing;
- (Θ_2) for each sequence $\{\alpha_n\} \subseteq R^+$, $\lim_{n\to\infty} \Theta(\alpha_n) = 1$ if and only if $\lim_{n\to\infty}(\alpha_n) = 0$;
- (Θ_3) there exists $0 < k < 1$ and $l \in (0, \infty]$ such that $\lim_{a\to 0^+} \frac{\Theta(\alpha)-1}{\alpha^k} = l$;

A mapping $T : X \to X$ is said to be Θ-contraction if there exist the function Θ satisfying (Θ_1)-(Θ_3) and a constant $\alpha \in (0, 1)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \neq 0 \Longrightarrow \Theta(d(Tx, Ty)) \leq [\Theta(d(x, y))]^{\alpha}. \quad (1.1)$$

Theorem 1.2. [18] *Let (X, d) be a complete metric space and $T : X \to X$ be a Θ-contraction, then T has a unique fixed point.*

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To be consistent with Samet et al. [18], we denote by Ψ the set of all functions $\Theta : (0, \infty) \to (1, \infty)$ satisfying the above conditions$(\Theta_1 - \Theta_3)$.

Hussain et al. [12] modified and extended the above family Θ of functions $\Theta : (0, \infty) \to (1, \infty)$ and proved the following fixed point theorem for Θ-contractive condition in the setting of complete metric spaces.

$(\Theta_1^{'})$ Θ is nondecreasing and $\Theta(t) = 1$ if and only if $t = 0$;
(Θ_4) $\Theta(a + b) \le \Theta(a) + \Theta(b)$ for all $a, b > 0$.

To be consistent with Hussain et al. [12], we denote by Ω the set of all functions $\Theta : (0, \infty) \to (1, \infty)$ satisfying the conditions $\Theta_1^{'}$ and $(\Theta_2 - \Theta_4)$.

Theorem 1.3. [12] *Let (X, d) be a complete metric space and $T : X \to X$ be a self-mapping. If there exists a function $\Theta \in \Omega$ and positive real numbers k_1, k_2, k_3 and k_4 with $0 \le k_1 + k_2 + k_3 + 2k_4 < 1$ such that*

$$\Theta(d(Tx, Ty)) \le [\Theta(d(x, y))]^{k_1} \cdot \Theta(d(x, Tx))]^{k_2} \cdot [\Theta(d(y, Ty))]^{k_3} \cdot [\Theta((d(x, Ty) + d(y, Tx))]^{k_4} \tag{1.2}$$

for all $x, y \in X$, then T has a unique fixed point.

For more details on $\Theta-$contractions, we refer the reader to [5, 26, 33].

One of the generalization of the domain of the mapping is partially ordered metric spaces was first investigated in 2004 by Ran and Reurings [31], and then by Nieto and Lopez [29].

To extend this concept, Jachymski [20] introduced a new approach in metric fixed point theory by replacing order structure with a graph structure on a metric space. In this way, the results obtained in ordered metric spaces are generalized (see also [19] and the reference therein); in fact, Gwodzdz-lukawska and Jachymski [11] developed the Hutchinson-Barnsley theory for finite families of mappings on a metric space endowed with a directed graph.

Consistent with Jachymski [19], let (X, d) be a metric space and Δ denotes the diagonal of $X \times X$. Let G be a directed graph such that the set $V(G)$ of its vertices coincides with X and $E(G)$ be the set of edges of the graph which contains all loops, that is, $\Delta \subseteq E(G)$. Let $E^*(G)$ denotes the set of all edges of G that are not loops i.e., $E^*(G) = E(G) - \Delta$. Also assume that the graph G has no parallel edges and, thus one can identify G with the pair $(V(G), E(G))$.

Definition 1.4. [19] *An operator $T : X \to X$ is called a Banach G-contraction or simply a G-contraction if*

(a) T preserves edges of G; for each $x, y \in X$ with $(x, y) \in E(G)$, we have $(T(x), T(y)) \in E(G)$,
(b) T decreases weights of edges of G ; there exists $\alpha \in (0, 1)$ such that for all $x, y \in X$ with $(x, y) \in E(G)$, we have $d(T(x), T(y)) \le \alpha d(x, y)$.

If x and y are vertices of G, then a (directed) path in G from x to y of length $k \in \mathbb{N}$ is a finite sequence $\{x_n\}$ ($n \in \{0, 1, 2, ..., k\}$) of vertices such that $x_0 = x$, $x_k = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i \in \{1, 2, ..., k\}$.

Notice that a graph G is connected if there is a (directed) path between any two vertices and it is weakly connected if $\widetilde{G}$ is connected, where $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Denote by G^{-1} the graph obtained from G by reversing the direction of edges. Thus,

$$E\left(G^{-1}\right) = \{(x, y) \in X \times X : (y, x) \in E\left(G\right)\}.$$

It is more convenient to treat $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric, under this convention; we have that

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}).$$

If G is such that $E(G)$ is symmetric, then for $x \in V(G)$, $[x]_G$ denotes the equivalence class of the relation R defined on $V(G)$ by the rule:

$$yRz \text{ if there is a path in } G \text{ from } y \text{ to } z.$$

If $T : X \to X$ is an operator. Set

$$X_T := \{x \in X : (x, T(x)) \in E(G)\}.$$

Jachymski [20] used the following property:

(P) : for any sequence $\{x_n\}$ in X, if $x_n \to x$ as $n \to \infty$ and $(x_n, x_{n+1}) \in E(G)$, then $(x_n, x) \in E(G)$.

Theorem 1.5. [20] *Let (X, d) be a complete metric space and let G be a directed graph such that $V(G) = X$. Let $E(G)$ and the triplet (X, d, G) has property (P) and $T : X \to X$ a G-contraction. Then the following statements hold:*

1. T has a fixed point if and only if $X_T \neq \emptyset$;
2. if $X_T \neq \emptyset$ and G is weakly connected, then T is a Picard operator;
3. for any $x \in X_T$, $T\mid_{[x]_{\widetilde{G}}}$ is a Picard operator;
4. if $X_T \subseteq E(G)$, then T is a weakly Picard operator.

For detailed discussion on Picard operators, we refer the reader to Berinde et. al [7, 8].

Latif and Beg [25] introduced a notion of $K-$ multivalued mapping as an extension of Kannan mapping to multivalued mappings. Rus [32] coined the term $R-$ multivalued mapping which is a generalization of a $K-$ multivalued mapping. Abbas and Rhoades [1] introduced the notion of a generalized $R-$ multivalued mappings, which in turn generalize $R-$ multivalued mappings, and obtained common fixed point results for such mappings.
Let (X, d) be a metric space. Denote by $P(X)$ the family of all nonempty subsets of X, by $P_{cl}(X)$ the family of all nonempty closed subset of X.
A point x in X is a fixed point of a multivalued mapping $T : X \to P(X)$ iff $x \in Tx$. The set of all fixed points of multivalued mapping T is denoted by $Fix(T)$.

Suppose that $T_1, T_2 : X \to P_{cl}(X)$. Set

$$X_{T_1,T_2} := \{x \in X : (x, u_x) \in E(G) \text{ where } u_x \in T_1(x) \cap T_2(x)\}.$$

A mapping $T : X \to P_{cl}(X)$ is said to be upper semicontinuous, if for $x_n \in X$ and $y_n \in Tx_n$ with $x_n \to x_0$ and $y_n \to y_0$, implies $y_0 \in Tx_0$.

A clique in an undirected graph $G = (V, E)$ is a subset of the vertex set $W \subset V$, such that for every two vertices in W, there exists an edge connecting the two. This is equivalent to saying that the subgraph induced by W is complete i.e., for every $x, y \in W(G)$, we have $(x, y) \in E(G)$.

The aim of this paper is to prove some common fixed point results for multivalued generalized graphic Θ-contraction mappings on a metric space endowed with a graph. Our results extend and unify various comparable results in the existing literature ([12, 21, 25]).

2. Main Results

Definition 2.1. *Let $T_1, T_2 : X \to P_{cl}(X)$ be two multivalued mappings. Suppose that for every vertex x in G and for every $u_x \in T_i(x), i \in \{1,2\}$ we have $(x, u_x) \in E(G)$. A pair (T_1, T_2) is said to be a graphic $\Theta-$contraction if there exist some $\Theta \in \Omega$ and for any $x, y \in X$ with $(x, y) \in E(G)$ and $u_x \in T_i(x)$, there exists $u_y \in T_j(y)$ for $i, j \in \{1,2\}$ with $i \neq j$ such that $(u_x, u_y) \in E^*(G)$ and*

$$\Theta(d(u_x, u_y)) \leq [\Theta(d(x,y))]^{k_1}[\Theta(d(x,u_x))]^{k_2}[\Theta(d(y,u_y))]^{k_3}[\Theta(d(x,u_y)+d(y,u_x))]^{k_4} \tag{2.1}$$

hold, where k_1, k_2, k_3 and k_4 are non negative real numbers such that $0 \leq k_1 + k_2 + k_3 + 2k_4 < 1$.

Now we state our main theorem.

Theorem 2.2. *Let (X, d) be a complete metric space endowed with a directed graph G such that $V(G) = X$ and $E(G) \supseteq \Delta$. If mappings $T_1, T_2 : X \to P_{cl}(X)$ form a graphic Θ-contraction pair, then following statement hold:*

(i). $Fix(T_1) \neq \emptyset$ or $Fix(T_2) \neq \emptyset$ if and only if $Fix(T_1) = Fix(T_2) \neq \emptyset$.
(ii). $X_{T_1,T_2} \neq \emptyset$ provided that $Fix(T_1) \cap Fix(T_2) \neq \emptyset$.
(iii). If $X_{T_1,T_2} \neq \emptyset$ and G is weakly connected, then $Fix(T_1) = Fix(T_2) \neq \emptyset$ provided that either (a) T_1 or T_2 is upper semicontinuous, or (b) Θ is continuous, T_1 or T_2 is bounded and G has property (P).
(iv). $Fix(T_1) \cap Fix(T_2)$ is a clique of $\widetilde{G}$ if and only if $Fix(T_1) \cap Fix(T_2)$ is a singleton.

Proof. To prove (i), let $x^* \in T_1(x^*)$. Assume $x^* \notin T_2(x^*)$, then since the pair (T_1, T_2) form a graphic Θ-contraction, so there exists an $x \in T_2(x^*)$ with $(x^*, x) \in E^*(G)$ such that

$$\begin{aligned}\Theta(d(x^*, x)) &\leq [\Theta(d(x^*,x^*))]^{k_1}[\Theta(d(x^*,x^*))]^{k_2}[\Theta(d(x,x^*))]^{k_3}[\Theta(d(x^*,x)+d(x^*,x^*))]^{k_4} \\ &= [\Theta(d(x,x^*))]^{k_3+k_4}\end{aligned}$$

a contradiction because $k_3 + k_4 < 1$. Hence $x^* \in T_2(x^*)$ and so $Fix(T_1) \subseteq Fix(T_2)$. Similarly, $Fix(T_2) \subseteq Fix(T_1)$ and therefore $Fix(T_1) = Fix(T_2)$. Also, if $x^* \in T_2(x^*)$, then we have $x^* \in T_1(x^*)$. The converse is straightforward.

To prove (ii), let $Fix(T_1) \cap Fix(T_2) \neq \emptyset$. Then there exists $x \in X$ such that $x \in T_1(x) \cap T_2(x)$. As $\Delta \subseteq E(G)$, we conclude that $X_{T_1,T_2} \neq \emptyset$.

To prove (iii), suppose that x_0 is an arbitrary point of X. If $x_0 \in T_1(x_0)$ or $x_0 \in T_2(x_0)$, then the proof is finished. So we assume that $x_0 \notin T_i(x_0)$ for $i \in \{1,2\}$. Now for $i, j \in \{1,2\}$ with $i \neq j$, if $x_1 \in T_i(x_0)$, then there exists $x_2 \in T_j(x_1)$ with $(x_1, x_2) \in E^*(G)$ such that

$$\begin{aligned}1 &< \Theta(d(x_1,x_2)) \leq [\Theta(d(x_0,x_1))]^{k_1}[\Theta(d(x_0,x_1))]^{k_2}[\Theta(d(x_1,x_2))]^{k_3}[\Theta(d(x_0,x_2)+d(x_1,x_1))]^{k_4} \\ &\leq [\Theta(d(x_0,x_1))]^{k_1}[\Theta(d(x_0,x_1))]^{k_2}[\Theta(d(x_1,x_2))]^{k_3}[\Theta(d(x_0,x_1)+d(x_1,x_2))]^{k_4}\end{aligned}$$

Using (Θ_4), then we have

$$\begin{aligned}\Theta(d(x_1,x_2)) &\leq [\Theta(d(x_0,x_1))]^{k_1}[\Theta(d(x_0,x_1))]^{k_2}[\Theta(d(x_1,x_2))]^{k_3}[\Theta(d(x_0,x_1)]^{k_4}[\Theta(d(x_1,x_2))]^{k_4} \\ &= [\Theta(d(x_0,x_1))]^{k_1+k_2+k_4}[\Theta(d(x_1,x_2))]^{k_3+k_4}.\end{aligned}$$

Therefore, we write

$$\begin{aligned} 1 &< \Theta\left(d(x_1,x_2)\right) \leq [\Theta(d(x_0,x_1))]^{\frac{k_1+k_2+k_4}{1-k_3-k_4}} \\ &= [\Theta(d(x_0,x_1))]^{\lambda}. \end{aligned} \tag{2.2}$$

Similarly, for the point x_2 in $T_j\left(x_1\right)$, there exists $x_3 \in T_i(x_2)$ with $(x_2,x_3) \in E^*\left(G\right)$ such that

$$\begin{aligned} 1 &< \Theta\left(d(x_2,x_3)\right) \leq [\Theta(d(x_1,x_2))]^{k_1}[\Theta(d(x_1,x_2))]^{k_2}[\Theta(d(x_2,x_3))]^{k_3}[\Theta(d(x_1,x_3)+d(x_2,x_2))]^{k_4} \\ &\leq [\Theta(d(x_1,x_2))]^{k_1}[\Theta(d(x_1,x_2))]^{k_2}[\Theta(d(x_2,x_3))]^{k_3}[\Theta(d(x_1,x_2)+d(x_2,x_3))]^{k_4} \end{aligned}$$

Using (Θ_4), then we have

$$\begin{aligned} 1 &< \Theta\left(d(x_2,x_3)\right) \leq [\Theta(d(x_1,x_2))]^{k_1}[\Theta(d(x_1,x_2))]^{k_2}[\Theta(d(x_2,x_3))]^{k_3}[\Theta(d(x_1,x_2)]^{k_4}[\Theta(d(x_2,x_3))]^{k_4} \\ &= [\Theta(d(x_1,x_2))]^{k_1+k_2+k_4}[\Theta(d(x_2,x_3))]^{k_3+k_4}. \end{aligned}$$

Therefore, we write

$$\begin{aligned} 1 &< \Theta\left(d(x_2,x_3)\right) \leq [\Theta(d(x_1,x_2))]^{\frac{k_1+k_2+k_4}{1-k_3-k_4}} \\ &= [\Theta(d(x_1,x_2))]^{\lambda}. \end{aligned} \tag{2.3}$$

Continuing this way, for $x_{2n} \in T_j(x_{2n-1})$, there exist $x_{2n+1} \in T_i\left(x_{2n}\right)$ with $(x_{2n},x_{2n+1}) \in E^*\left(G\right)$ such that

$$1 < \Theta\left(d(x_{2n},x_{2n+1})\right) \leq \left[\Theta\left(d(x_{2n-1},x_{2n})\right)\right]^{\lambda}. \tag{2.4}$$

In a similar manner, for $x_{2n+1} \in T_j(x_{2n})$, there exist $x_{2n+2} \in T_i\left(x_{2n+1}\right)$ such that for $(x_{2n+1},x_{2n+2}) \in E^*\left(G\right)$ implies

$$1 < \Theta\left(d(x_{2n+1},x_{2n+2})\right) \leq \left[\Theta\left(d(x_{2n},x_{2n+1})\right)\right]^{\lambda}. \tag{2.5}$$

Hence from (2.4) and (2.5), we obtain a sequence $\{x_n\}$ in X such that for $x_n \in T_j(x_{n-1})$, there exist $x_{n+1} \in T_i\left(x_n\right)$ with $(x_n,x_{n+1}) \in E^*\left(G\right)$ and it satisfies

$$1 < \Theta\left(d(x_n,x_{n+1})\right) \leq \left[\Theta\left(d(x_{n-1},x_n)\right)\right]^{\lambda}. \tag{2.6}$$

Therefore

$$\begin{aligned} 1 &< \Theta\left(d(x_n,x_{n+1})\right) \leq \left[\Theta\left(d(x_{n-1},x_n)\right)\right]^{\lambda} \\ &\leq \left[\Theta\left(d(x_{n-2},x_{n-1})\right)\right]^{\lambda^2} \\ &\ . \\ &\ . \\ &\ . \\ &\leq \left[\Theta\left(d(x_0,x_1)\right)\right]^{\lambda^n} \end{aligned}$$

From (2.7), we obtain $\lim_{n\to\infty}\Theta\left(d(x_n,x_{n+1})\right) = 1$ that together with (Θ_2) gives

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{2.7}$$

From the condition (Θ_3), there exist $0 < h < 1$ and $l \in (0,\infty]$ such that

$$\lim_{n\to\infty}\frac{\Theta(d(x_n,x_{n+1}))-1}{d(x_n,x_{n+1})^h} = l.$$

Suppose that $l < \infty$. In this case, let $B = \frac{l}{2} > 0$. From the definition of the limit, there exists $n_1 \in \mathbb{N}$ such that

$$\left|\frac{\Theta(d(x_n, x_{n+1})) - 1}{d(x_n, x_{n+1})^h} - l\right| \leq B$$

for all $n > n_1$. This implies that

$$\frac{\Theta(d(x_n, x_{n+1})) - 1}{d(x_n, x_{n+1})^h} \geq l - B = \frac{l}{2} = B$$

for all $n > n_1$. Then

$$nd(x_n, x_{n+1})^h \leq An[\Theta(d(x_n, x_{n+1})) - 1] \tag{2.8}$$

for all $n > n_1$, where $A = \frac{1}{B}$. Now we suppose that $l = \infty$. Let $B > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_1 \in \mathbb{N}$ such that

$$B \leq \frac{\Theta(d(x_n, x_{n+1})) - 1}{d(x_n, x_{n+1})^h}$$

for all $n > n_1$. This implies that

$$nd(x_n, x_{n+1})^h \leq An[\Theta(d(x_n, x_{n+1})) - 1]$$

for all $n > n_1$, where $A = \frac{1}{B}$. Thus, in all cases, there exist $A > 0$ and $n_1 \in \mathbb{N}$ such that

$$nd(x_n, x_{n+1})^h \leq An[\Theta(d(x_n, x_{n+1})) - 1] \tag{2.9}$$

for all $n > n_1$. Thus by (2.7) and (2.10), we get

$$nd(x_n, x_{n+1})^h \leq An([(\Theta d(x_0, x_1))]^{h^n} - 1). \tag{2.10}$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} nd(x_n, x_{n+1})^h = 0.$$

Thus, there exists $n_2 \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{1/h}} \tag{2.11}$$

for all $n > n_2$. Now we prove that $\{x_n\}$ is a Cauchy sequence. For $m > n > n_2$ we have,

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{1/h}}. \tag{2.12}$$

By the convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/h}}$, we get $d(x_n, x_m) \to 0$ as $n, m \to \infty$. Therefore $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists an element $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

Now, if T_i is upper semicontinuous, then as $x_{2n} \in X$, $x_{2n+1} \in T_i(x_{2n})$ with $x_{2n} \to x^*$ and $x_{2n+1} \to x^*$ as $n \to \infty$ implies that $x^* \in T_i(x^*)$. Using (i), we get $x^* \in T_i(x^*) = T_j(x^*)$. Similarly the result hold when T_j is upper semicontinuous.

Suppose that Θ is continuous. Since x_{2n} converges to x^* as $n \to \infty$ and $(x_{2n}, x_{2n+1}) \in E(G)$, we have $(x_{2n}, x^*) \in E(G)$. For $x_{2n} \in T_j(x_{2n-1})$, there exists $u_n \in T_i(x^*)$ such that $(x_{2n}, u_n) \in E^*(G)$. As $\{u_n\}$ is bounded, $\limsup_{n\to\infty} u_n = u^*$, and $\liminf_{n\to\infty} u_n =$

u_* both exist. Assume that $u^* \neq x^*$. Since (T_1, T_2) is a graphic $\Theta-$contraction, so we have

$$\begin{aligned} 1 &< \Theta\left(d(x_{2n}, u_n)\right) \leq [\Theta(d(x_{2n-1}, x^*))]^{k_1} [\Theta(d(x_{2n-1}, x_{2n}))]^{k_2} \\ &\quad \times [\Theta(d(x^*, u_n))]^{k_3} [\Theta(d(x_{2n-1}, u_n) + d(x^*, x_{2n}))]^{k_4}. \end{aligned}$$

Using (Θ_4), then we have

$$\begin{aligned} 1 &< \Theta\left(d(x_{2n}, u_n)\right) \leq [\Theta(d(x_{2n-1}, x^*))]^{k_1} [\Theta(d(x_{2n-1}, x_{2n}))]^{k_2} \\ &\quad \times [\Theta(d(x^*, u_n))]^{k_3} [\Theta(d(x_{2n-1}, u_n)]^{k_4} [\Theta(d(x^*, x_{2n}))]^{k_4}. \end{aligned}$$

On taking $\limsup_{n\to\infty}$ and using the continuity of Θ, we get

$$\Theta\left(d(x^*, u^*)\right) \leq [\Theta(d(x^*, u^*))]^{k_3+k_4},$$

a contradiction because $k_3 + k_4 < 1$. Hence $u^* = x^*$. Similarly, taking the $\liminf_{n\to\infty}$ gives $u_* = x^*$. Since $u_n \in T_i\left(x^*\right)$ for all $n \geq 1$ and $T_i\left(x^*\right)$ is a closed set, it follows that $x^* \in T_i\left(x^*\right)$. Now from (i), we get $x^* \in T_i(x^*)$ and hence $Fix(T_1) = Fix(T_2)$. Finally to prove (iv), suppose the set $Fix\left(T_1\right) \cap Fix\left(T_2\right)$ is a clique of $\widetilde{G}$. We are to show that $Fix\left(T_1\right) \cap Fix\left(T_2\right)$ is singleton. Assume on contrary that there exist u and v such that $u, v \in Fix\left(T_1\right) \cap Fix\left(T_2\right)$ but $u \neq v$. As $(u, v) \in E^*(G)$ and (T_1, T_2) form a graphic $\Theta-$contraction, so for $(u_x, v_y) \in E^*\left(G\right)$ implies

$$\Theta\left(d(u, v)\right) \leq [\Theta(d(u, v))]^{k_1} [\Theta(d(u, u))]^{k_2} [\Theta(d(v, v))]^{k_3} [\Theta(d\left(u, v\right) + d\left(v, u\right))]^{k_4}$$

Using (Θ_4), then we have

$$\Theta\left(d(u, v)\right) \leq [\Theta(d(u, v))]^{k_1} [\Theta(d(u, u))]^{k_2} [\Theta(d(v, v))]^{k_3} [\Theta(d\left(u, v\right)]^{k_4} [\Theta(d\left(v, u\right))]^{k_4}$$

Using (Θ_1), then we have

$$\Theta\left(d(u, v)\right) \leq [\Theta(d(u, v))]^{k_1+2k_4}$$

a contradiction as $k_1 + 2k_4 < 1$. Hence $u = v$. Conversely, if $Fix(T_1) \cap Fix(T_2)$ is singleton, then it follows that $Fix(T_1) \cap Fix(T_2)$ is a clique of $\widetilde{G}$. □

For specific choices of function Θ, we obtain some significant results. First, by taking $\Theta(t) = e^{\sqrt{t}}$ in (2.1), we get the following result.

Theorem 2.3. *Let (X, d) be a complete metric space endowed with a directed graph G such that $V(G) = X$ and $E(G) \supseteq \Delta$ and $T_1, T_2 : X \to P_{cl}(X)$ be multivalued mappings. If for any $x, y \in X$ with $(x, y) \in E\left(G\right)$ and $u_x \in T_i(x)$, there exists $u_y \in T_j(y)$ for $i, j \in \{1, 2\}$ with $i \neq j$ such that $(u_x, u_y) \in E^*\left(G\right)$ and*

$$\sqrt{d(u_x, u_y)} \leq k_1\sqrt{d(x, y)} + k_2\sqrt{d(x, u_x)} + k_3\sqrt{d(y, u_y)} + k_4\sqrt{d\left(x, u_y\right) + d\left(y, u_x\right)} \tag{2.13}$$

where k_1, k_2, k_3 and k_4 are non negative real numbers such that $0 \leq k_1 + k_2 + k_3 + 2k_4 < 1$. Then following statement hold:

(i). $Fix(T_1) \neq \emptyset$ or $Fix(T_2) \neq \emptyset$ if and only if $Fix\left(T_1\right) = Fix\left(T_2\right) \neq \emptyset$.
(ii). $X_{T_1,T_2} \neq \emptyset$ provided that $Fix\left(T_1\right) \cap Fix\left(T_2\right) \neq \emptyset$.
(iii). If $X_{T_1,T_2} \neq \emptyset$ and G is weakly connected, then $Fix\left(T_1\right) = Fix\left(T_2\right) \neq \emptyset$ provided that either (a) T_1 or T_2 is upper semicontinuous, or (b) Θ is continuous, T_1 or T_2 is bounded and G has property (P).

(iv). $Fix\,(T_1) \cap Fix\,(T_2)$ is a clique of $\widetilde{G}$ if and only if $Fix\,(T_1) \cap Fix\,(T_2)$ is a singleton.

Remark 2.4. *Notice that condition (2.14) is equivalent to*

$$\begin{aligned} d(u_x, u_y) \quad \leq \quad & k_1^2 d(x,y) + k_2^2 d(x,u_x) + k_3^2 d(y,u_y) + k_4^2 (d\,(x,u_y) + d\,(y,u_x)) \\ & +2k_1k_2\sqrt{d(x,y)d(x,u_x)} + 2k_1k_3\sqrt{d(x,y)d(y,u_y)} \\ & +2k_1k_4\sqrt{d(x,y)[d\,(x,u_y) + d\,(y,u_x)]} + 2k_2k_3\sqrt{d(x,u_x)d(y,u_y)} \\ & +2k_2k_4\sqrt{d(x,u_x)[d\,(x,u_y) + d\,(y,u_x)]} + +2k_3k_4\sqrt{d(y,u_y)[d\,(x,u_y) + d\,(y,u_x)]}. \end{aligned}$$

Next, in view of Remark 2.4 , by taking $k_1 = k_4 = 0$ *in Theorem 2.3, we obtain the following result for multivalued mappings.*

Theorem 2.5. *Let* (X, d) *be a complete metric space endowed with a directed graph* G *such that* $V(G) = X$ *and* $E(G) \supseteq \Delta$ *and* $T_1, T_2 : X \to P_{cl}(X)$ *be multivalued mappings. If for any* $x, y \in X$ *with* $(x, y) \in E\,(G)$ *and* $u_x \in T_i(x)$*, there exists* $u_y \in T_j(y)$ *for* $i, j \in \{1, 2\}$ *with* $i \neq j$ *such that* $(u_x, u_y) \in E^*\,(G)$ *and*

$$d(u_x, u_y) \leq k_2^2 d(x,u_x) + k_3^2 d(y,u_y) + 2k_2k_3\sqrt{d(x,u_x)d(y,u_y)}$$

where k_2 *and* k_3 *are non negative real numbers such that* $k_2 + k_3 < 1$*, then following statement hold:*

(i). $Fix(T_1) \neq \emptyset$ or $Fix(T_2) \neq \emptyset$ if and only if $Fix\,(T_1) = Fix\,(T_2) \neq \emptyset$.
(ii). $X_{T_1,T_2} \neq \emptyset$ provided that $Fix\,(T_1) \cap Fix\,(T_2) \neq \emptyset$.
(iii). If $X_{T_1,T_2} \neq \emptyset$ and G is weakly connected, then $Fix\,(T_1) = Fix\,(T_2) \neq \emptyset$ provided that either (a) T_1 or T_2 is upper semicontinuous, or (b) Θ is continuous, T_1 or T_2 is bounded and G has property (P).
(iv). $Fix\,(T_1) \cap Fix\,(T_2)$ is a clique of $\widetilde{G}$ if and only if $Fix\,(T_1) \cap Fix\,(T_2)$ is a singleton.

On the other hand, by taking $k_1 = k_2 = k_3 = 0$ in Theorem 2.3, we obtain the following Chatterjea type result for multivalued mappings.

Corollary 2.6. *Let* (X, d) *be a complete metric space endowed with a directed graph* G *such that* $V(G) = X$ *and* $E(G) \supseteq \Delta$ *and* $T_1, T_2 : X \to P_{cl}(X)$ *be multivalued mappings. If for any* $x, y \in X$ *with* $(x, y) \in E\,(G)$ *and* $u_x \in T_i(x)$*, there exists* $u_y \in T_j(y)$ *for* $i, j \in \{1, 2\}$ *with* $i \neq j$ *such that* $(u_x, u_y) \in E^*\,(G)$ *and*

$$d(u_x, u_y) \leq k_4^2 (d\,(x,u_y) + d\,(y,u_x))$$

where k_4 *is non negative real number such* $2k_4 < 1$*, then following statement hold:*

(i). $Fix(T_1) \neq \emptyset$ or $Fix(T_2) \neq \emptyset$ if and only if $Fix\,(T_1) = Fix\,(T_2) \neq \emptyset$.
(ii). $X_{T_1,T_2} \neq \emptyset$ provided that $Fix\,(T_1) \cap Fix\,(T_2) \neq \emptyset$.
(iii). If $X_{T_1,T_2} \neq \emptyset$ and G is weakly connected, then $Fix\,(T_1) = Fix\,(T_2) \neq \emptyset$ provided that either (a) T_1 or T_2 is upper semicontinuous, or (b) Θ is continuous, T_1 or T_2 is bounded and G has property (P).
(iv). $Fix\,(T_1) \cap Fix\,(T_2)$ is a clique of $\widetilde{G}$ if and only if $Fix\,(T_1) \cap Fix\,(T_2)$ is a singleton.

From Theorem 2.3, by taking $k_4 = 0$, we obtain the extension of Reich contraction for multivalued mappings.

Corollary 2.7. *Let (X, d) be a complete metric space endowed with a directed graph G such that $V(G) = X$ and $E(G) \supseteq \Delta$ and $T_1, T_2 : X \to P_{cl}(X)$ be multivalued mappings. If for any $x, y \in X$ with $(x, y) \in E(G)$ and $u_x \in T_i(x)$, there exists $u_y \in T_j(y)$ for $i, j \in \{1, 2\}$ with $i \neq j$ such that $(u_x, u_y) \in E^*(G)$ and*

$$\begin{aligned} d(u_x, u_y) \leq\ & k_1^2 d(x, y) + k_2^2 d(x, u_x) + k_3^2 d(y, u_y) \\ & +2k_1k_2\sqrt{d(x, y)d(x, u_x)} \\ & +2k_1k_3\sqrt{d(x, y)d(y, u_y)} + 2k_2k_3\sqrt{d(x, u_x)d(y, u_y)} \end{aligned}$$

where k_1, k_2 and k_3 are non negative real numbers such that $k_1 + k_2 + k_3 < 1$, then following statement hold:

(i). $Fix(T_1) \neq \emptyset$ or $Fix(T_2) \neq \emptyset$ if and only if $Fix(T_1) = Fix(T_2) \neq \emptyset$.
(ii). $X_{T_1,T_2} \neq \emptyset$ provided that $Fix(T_1) \cap Fix(T_2) \neq \emptyset$.
(iii). If $X_{T_1,T_2} \neq \emptyset$ and G is weakly connected, then $Fix(T_1) = Fix(T_2) \neq \emptyset$ provided that either (a) T_1 or T_2 is upper semicontinuous, or (b) Θ is continuous, T_1 or T_2 is bounded and G has property (P).
(iv). $Fix(T_1) \cap Fix(T_2)$ is a clique of $\widetilde{G}$ if and only if $Fix(T_1) \cap Fix(T_2)$ is a singleton.

If we take $k_2 = k_3 = k_4 = 0$ in Theorem 2.3, then we get the following result.

Corollary 2.8. *Let (X, d) be a complete metric space endowed with a directed graph G such that $V(G) = X$ and $E(G) \supseteq \Delta$ and $T_1, T_2 : X \to P_{cl}(X)$ be multivalued mappings. If for any $x, y \in X$ with $(x, y) \in E(G)$ and $u_x \in T_i(x)$, there exists $u_y \in T_j(y)$ for $i, j \in \{1, 2\}$ with $i \neq j$ such that $(u_x, u_y) \in E^*(G)$ and*

$$d(u_x, u_y) \leq k_1^2 d(x, y) \tag{2.14}$$

where k_1 is a non negative real number such that $k_1 < 1$, then following statement hold:

(i). $Fix(T_1) \neq \emptyset$ or $Fix(T_2) \neq \emptyset$ if and only if $Fix(T_1) = Fix(T_2) \neq \emptyset$.
(ii). $X_{T_1,T_2} \neq \emptyset$ provided that $Fix(T_1) \cap Fix(T_2) \neq \emptyset$.
(iii). If $X_{T_1,T_2} \neq \emptyset$ and G is weakly connected, then $Fix(T_1) = Fix(T_2) \neq \emptyset$ provided that either (a) T_1 or T_2 is upper semicontinuous, or (b) Θ is continuous, T_1 or T_2 is bounded and G has property (P).
(iv). $Fix(T_1) \cap Fix(T_2)$ is a clique of $\widetilde{G}$ if and only if $Fix(T_1) \cap Fix(T_2)$ is a singleton.

Finally, by taking $\Theta(t) = e^{\sqrt[n]{t}}$ in (2.1) we have the following Corollary.

Corollary 2.9. *Let (X, d) be a complete metric space endowed with a directed graph G such that $V(G) = X$ and $E(G) \supseteq \Delta$ and $T_1, T_2 : X \to P_{cl}(X)$ be multivalued mappings. If for any $x, y \in X$ with $(x, y) \in E(G)$ and $u_x \in T_i(x)$, there exists $u_y \in T_j(y)$ for $i, j \in \{1, 2\}$ with $i \neq j$ such that $(u_x, u_y) \in E^*(G)$ and*

$$\sqrt[n]{d(u_x, u_y)} \leq k_1\sqrt[n]{d(x, y)} + k_2\sqrt[n]{d(x, u_x)} + k_3\sqrt[n]{d(y, u_y)} + k_4\sqrt[n]{d(x, u_y) + d(y, u_x)}$$

where k_1, k_2, k_3 and k_4 are non negative real numbers such that $k_1+k_2+k_3+2k_4 < 1$, then following statement hold:

(i). $Fix(T_1) \neq \emptyset$ or $Fix(T_2) \neq \emptyset$ if and only if $Fix\,(T_1) = Fix\,(T_2) \neq \emptyset$.
(ii). $X_{T_1,T_2} \neq \emptyset$ provided that $Fix\,(T_1) \cap Fix\,(T_2) \neq \emptyset$.
(iii). If $X_{T_1,T_2} \neq \emptyset$ and G is weakly connected, then $Fix\,(T_1) = Fix\,(T_2) \neq \emptyset$ provided that either (a) T_1 or T_2 is upper semicontinuous, or (b) Θ is continuous, T_1 or T_2 is bounded and G has property (P).
(iv). $Fix\,(T_1) \cap Fix\,(T_2)$ is a clique of $\widetilde{G}$ if and only if $Fix\,(T_1) \cap Fix\,(T_2)$ is a singleton.

Example 2.10. *Consider the sequence* $\{x_n\}$ *as follows:*

$x_n = 1 + 5 + 9 + ... + (4n - 3) = n(2n - 1)$.
Let $X = \{x_n : n \in \mathbb{N}\} = V(G)$,

$$E\,(G) = \{(x, y) : x \leq y \text{ where } x, y \in V\,(G)\}$$

and

$$E^*\,(G) = \{(x, y) : x < y \text{ where } x, y \in V\,(G)\}.$$

Let $V\,(G)$ be endowed with usual metric $d\,(x, y) = |x - y|$. Then (X, d) is a complete metric space. Define $T_1, T_2 : X \to P_{cl}(X)$ as follows:

$$T_1\,(x) = \{x_1\} \text{ for } x \in X$$

and

$$T_2\,(x) = \begin{cases} \{x_1\} & , \quad x = x_1 \\ \{x_1, x_{n-1}\} & , \quad x = x_n, \text{ for } n > 1. \end{cases}$$

If we consider the mapping $\Theta : (0, \infty) \to (1, \infty)$ defined by

$$\Theta(t) = e^{\sqrt{te^t}}.$$

We can easily show that $\Theta \in \Omega$ and the pair of mappings (T_1, T_2) is a graphic Θ-contraction. Indeed, the following holds:

$$e^{\sqrt{d(u_x,u_y)e^{d(u_x,u_y))}}} \leq e^{k_1\sqrt{d(x,y)e^{d(x,y)}}}$$

for some $k_1 \in (0, 1)$. The above condition is equivalent to

$$d(u_x, u_y)e^{d(u_x,u_y)} \leq k_1^2 d(x, y)e^{d(x,y)}.$$

So, we have to check that

$$d(u_x, u_y)e^{d(u_x,u_y)-d(x,y)} \leq k_1^2 d(x, y)$$

for some $k_1 \in (0, 1)$. For $(u_x, u_y) \in E^*\,(G)$, we consider the following cases:

Case 01. If $x = x_1, y = x_m$, for $m > 1$, then for $u_x = x_1 \in T_1\,(x)$, there exists $u_y = x_{m-1} \in T_2\,(y)$, such that

$$\begin{aligned} d(u_x, u_y)e^{d(u_x,u_y)-d(x,y)} &= (2m^2 - 5m + 2)e^{-(4m-3)} \\ &< (2m^2 - m - 1)e^{-1} \\ &= e^{-1}d\,(x, y) \end{aligned}$$

Case 02. When $x = x_n$, $y = x_m$ with $m > n > 1$, then for $u_x = x_1 \in T_1\,(x)$, there exists $u_y = x_{n-1} \in T_2\,(y)$, such that

$$\begin{aligned} d(u_x, u_y)e^{d(u_x,u_y)-d(x,y)} &= (2n^2 - 5n + 2)e^{-(2m^2-4n^2+6n-m-2)} \\ &< (2m^2 - 2n^2 - m + n)e^{-1} \\ &= e^{-1}d\,(x, y) \end{aligned}$$

Now we show that for $x, y \in X$, $u_x \in T_2(x)$; there exists $u_y \in T_1(y)$ such that $(u_x, u_y) \in E^*(G)$ and (2.15) is satisfied. For this, we consider the following cases:

Case 01. If $x = x_n$, $y = x_1$ with $n > 1$, we have for $u_x = x_{n-1} \in T_2(x)$, there exists $u_y = x_1 \in T_1(y)$, such that

$$\begin{aligned} d(u_x, u_y)e^{d(u_x,u_y)-d(x,y)} &= (2n^2 - 5n + 2)e^{-(4n-3)} \\ &< (2n^2 - n - 1)e^{-1} \\ &= e^{-1}d(x, y) \end{aligned}$$

Case 02. In case $x = x_n$, $y = x_m$ with $m > n > 1$, then for $u_x = x_{n-1} \in T_2(x)$, there exists $u_y = x_1 \in T_1(y)$, such that

$$\begin{aligned} d(u_x, u_y)e^{d(u_x,u_y)-d(x,y)} &= (2n^2 - 5n + 2)e^{-(2m^2-4n^2+6n-m-2)} \\ &< (2m^2 - 2n^2 - m + n)e^{-1} \\ &= e^{-1}d(x, y) \end{aligned}$$

Thus for all x, y in $V(G)$, (2.15) is satisfied. Hence all the conditions of Corollary 2.8 are satisfied. Moreover, $x_1 = 1$ is the common fixed point of T_1 and T_2 with $Fix(T_1) = Fix(T_2)$.

Competing interests

The authors declare that they have no competing interests.

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