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SOME RESULTS ON IMPLICIT MULTISTEP FIXED POINT ITERATIVE SCHEMES FOR CONTRACTIVE-LIKE OPERATORS IN CONVEX METRIC SPACES

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Abstract. In this paper, we establish and prove strong convergence, T-stability, convergence rate and data dependence results for multistep fixed point iterative schemes using a class of contractive-like operators in convex metric spaces. Our results show that the proposed implicit multistep schemes have better convergence rate than the well-known explicit multistep schemes and multistep SP-iterative schemes. This is shown by analytical processes and validated with numerical examples. Several known results in the literature are embedded in these present results.

1. Introduction

The Picard iteration defined by: For $x_0 \in X$

$$x_n = Tx_{n-1},\ n \geq 1 \tag{1.1}$$

was first considered by Banach (1922) for a self-map T in a complete metric space (X, d) satisfying

$$d(Tx, Ty) \leq cd(x, y) \tag{1.2}$$

(called strict contraction), for all $x, y \in X$ and some $c \in (0, 1)$. When the Banach's contractive condition (1.2) is weaker, then the Picard iteration will no longer converge to a fixed point, hence, other iterative procedures are considered to approximate fixed point of weaker conditions.

Mann (1953) defined a more general iteration in a Banach space E setting satisfying Lipschitz pseudocontraction operator. The Mann iteration is given as: For $x_0 \in E$

$$x_n = (1-\alpha_n)x_{n-1} + \alpha_n Tx_{n-1},\ n \geq 1 \tag{1.3}$$

where $\alpha_n \subset [0, 1]$. By letting $\alpha_n = 1$ in (1.3) yields Picard scheme (1.1).
Ishikawa (1974) defined another iteration as: For $x_0 \in E$

$$\begin{aligned} x_n &= (1-\alpha_n)x_{n-1} + \alpha_n Ty_{n-1} \\ y_{n-1} &= (1-\beta_n)x_{n-1} + \beta_n Tx_{n-1},\ n \geq 1 \end{aligned} \tag{1.4}$$

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where $\alpha_n, \beta_n \subset [0,1]$. The Ishikawa iteration is a double Mann iteration and has better convergence rate than Mann iteration.
The three-step iteration was defined by Noor (2000) as: For $x_0 \in E$

$$\begin{aligned} x_n &= (1-\alpha_n)x_{n-1} + \alpha_n T y_{n-1} \\ y_{n-1} &= (1-\beta_n)x_{n-1} + \beta_n T z_{n-1} \\ z_{n-1} &= (1-\gamma_n)x_{n-1} + \gamma_n T x_{n-1},\ n \geq 1 \end{aligned} \tag{1.5}$$

where $\alpha_n, \beta_n, \gamma_n \in [0,1]$ with $\sum \alpha_n = \infty$. The Noor iteration is more general than Mann and Ishikawa iterative schemes.

Rhoades and Soltuz (2004) defined a multistep iterative scheme in a normed linear space as: For $x_0 \in E$

$$\begin{aligned} x_n &= (1-\alpha_n)x_{n-1} + \alpha_n T y_{n-1}^{(1)} \\ y_{n-1}^{(l)} &= (1-\beta_n^{(l)})x_{n-1} + \beta_n^{(l)} T y_{n-1}^{(l+1)},\ l = 1,2,\ldots,k-2 \\ y_{n-1}^{(k-1)} &= (1-\beta_n)x_{n-1} + \beta_n T x_{n-1},\ k \geq 2,\ n \geq 1 \end{aligned} \tag{1.6}$$

where $\alpha_n, \beta_n^{(l)} \in [0,1]$, for $l = 1,2,\ldots,k-1$ with $\sum \alpha_n = \infty$. The scheme (1.6) generalized the Noor iteration (1.5), Ishikawa iteration (1.4) and Mann iteration (1.3), in particular, if $k = 3$ in (1.6), we recover the form of (1.5); if $k = 2$, we have (1.4); on putting $k = 2$ and $\beta_n^{(l)} = 0$ for each l, we have (1.3).
Another iterative scheme, called the Thianwan scheme, was introduced by Thianwan (2009) as: For $x_0 \in E$,

$$\begin{aligned} x_n &= (1-\alpha_n)y_{n-1} + \alpha_n T y_{n-1} \\ y_{n-1} &= (1-\beta_n)x_{n-1} + \beta_n T x_{n-1},\ n \geq 1 \end{aligned} \tag{1.7}$$

where $\alpha_n, \beta_n \subseteq [0,1]$ with $\sum \alpha_n = \infty$. The three steps of (1.7) called SP-iteration was introduced by Phuengrattana and Suntai (2011) and was defined by, for $x_0 \in E$,

$$\begin{aligned} x_n &= (1-\alpha_n)y_{n-1}^1 + \alpha_n T y_{n-1}^1 \\ y_{n-1}^1 &= (1-\beta_n)y_{n-1}^2 + \beta_n T y_{n-1}^2 \\ y_n^2 &= (1-\gamma_n)x_n + \gamma_n T x_n,\ n \geq 1 \end{aligned} \tag{1.8}$$

where $\alpha_n, \beta_n, \gamma \subseteq [0,1]$ with $\sum \alpha_n = \infty$. The two step of Thianwan scheme is easily seen from (1.8) when $\gamma_n = 0$.
In an attempt to generalize both (1.7) and (1.8), Gürsoy et al. (2013) introduced a multistep-SP scheme in a Banach space as: For $x_0 \in E$,

$$\begin{aligned} x_n &= (1-\alpha_n)y_{n-1}^1 + \alpha_n T y_{n-1}^1 \\ y_{n-1}^l &= (1-\beta_n^l)y_{n-1}^{l+1} + \beta_n^l T y_{n-1}^{l+1},\ l = 1,2,3,\ldots,k-2 \\ y_{n-1}^{k-1} &= (1-\beta_n^{k-1})x_{n-1} + \beta_n^{k-1} T x_{n-1},\ k \geq 2,\ n \geq 1 \end{aligned} \tag{1.9}$$

where $\alpha_n, \beta_n^l, \subseteq [0,1]$, $l = 1,2,\ldots,k-1$ with $\sum \alpha_n = \infty$.
Numerous results have been proved for the strong and weak convergence of the aforementioned schemes for the fixed points of different types of contractive-like operators in various spaces [Reich (1979), Ciric (1999), Berinde (2004), Chugh and Kumar (2011), Xue and Zhang (2013)]. Their stability results have been discussed in Osilike (1995), Berinde (2002), Imoru and Olatinwo (2003), Olatinwo (2011) and Berinde (2011). The results concerning the data dependence of these schemes have been established and proved by [Soltuz (2008), Chugh and Kumar (2012) and

Gürsoy et al (2013)].
Implicit schemes of fixed point have received low attention and few works have been done regarding its stability, convergence rate and data dependence results. The implicit schemes are mostly employed to reduce the computational cost when the explicit or SP-iterative schemes are heavy-handed. For some recent results on implicit schemes of fixed points, see [Ciric et al. (2008), Xue and Zhang (2013), Chugh et al. (2015)].

2. Preliminaries

We begin this section with the following definitions which are useful to our results.

Definition 2.1 (Takahashi, 1970). *Let (X, d) be a metric space. A mapping $W : X \times X \times [0,1] \to X$ is called a convex structure on X if for all $x, y \in X$ and $\lambda \in [0,1]$*

$$d(q, W(x, y, \lambda)) \leq \lambda d(q, x) + (1-\lambda) d(q, y) \tag{2.1}$$

holds for all $q \in X$. The metric space (X, d) together with a convex structure W is called a convex metric space.

See also Reich and Shafrir (1990).

Definition 2.2 (Guay et al (1982)). *A convex metric space (X, d, W) is said to satisfy Property (I) if for all $x, y, p \in X$ and $\lambda \in [0,1]$,*

$$d(W(x, p, \lambda), W(y, p, \lambda)) \leq \lambda d(x, y) \tag{2.2}$$

Property (I) is always satisfied in a normed linear space.
Obviously, every normed space $(X, \|.\|)$ and their subsets are strongly convex metric spaces with W defined by $W(x, q, \lambda) = \lambda x + (1-\lambda) q$ for all $x, q \in X$ and $\lambda \in [0,1]$. But not every convex metric space is embedded in normed space.

Definition 2.3 (Takahashi, 1970). *A nonempty subset K of a convex metric space (X, d, W) is said to be convex if $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in [0,1]$.*

Definition 2.4. *A nonempty subset K is said to be p-starshaped, where $p \in K$, provided $W(x, p, \lambda) \in K$ for all $x \in K$ and $\lambda \in [0,1]$ i.e. the segment $[p, x] = \{W(x, p, \lambda) : 0 \leq \lambda \leq 1\}$ joining p to x is contained in K for all $x \in K$.*
K is said to be starshaped if it is p-starshaped for some $p \in K$.

Clearly, each convex set is starshaped but not conversely.
The concept of T-stability is defined as follow:

Definition 2.5 (Olatinwo, 2011). *Let (X, d, W) be a convex metric space and $T : X \to X$ a self-mapping. Suppose that $F_T = \{p \in X : Tp = p\}$ is the set of fixed points of T.*
Let $\{x_n\}_{n=0}^{\infty} \subset X$ be the sequence generated by an iterative procedure involving T which is defined by

$$x_{n+1} = f_{T,\alpha_n}^{x_n}, \ n \geq 0 \tag{2.3}$$

where $x_0 \in X$ is the initial approximation and $f_{T,\alpha_n}^{x_n}$ is some function having convex structure such that $\alpha_n \in [0,1]$. Suppose that $\{x_n\}$ converges to a fixed point p of T. Let $\{y_n\}_{n=0}^{\infty} \subset X$ and set $\epsilon_n = d\left(y_{n+1}, f_{T,\alpha_n}^{y_n}\right)$, $n = 0, 1, 2, \ldots$. Then, the iterative procedure (2.15) is said to be T-stable or stable with respect to T if and only if $\lim_{n\to\infty} \epsilon_n = 0$ implies $\lim_{n\to\infty} y_n = p$.

Definition 2.6 (Berinde, 2002)**.** *Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two nonnegative real sequences which converge to a and b, respectively. Let*

$$l = \lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|}$$

1. *if $l = 0$, then $\{a_n\}_{n=0}^{\infty}$ converges to a faster than $\{b_n\}_{n=0}^{\infty}$ to b.*
2. *if $0 < l < \infty$, then both $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ have the same convergence rate.*
3. *if $l = \infty$, then $\{b_n\}_{n=0}^{\infty}$ converges to b faster than $\{a_n\}_{n=0}^{\infty}$ to a.*

Let $\{x_n\}$ and $\{y_n\}$ be two iterative sequences converging to the same fixed point z of T such that

$$d(x_n, z) \leq a_n \text{ and } d(y_n, z) \leq b_n,\ n \geq 1$$

where a_n and b_n are sequences of positive real numbers (converging to zero). In view of Definition 2.6, if a_n converges faster than b_n, then we say that the sequence x_n converges faster than the sequence y_n.

Definition 2.7 (Gursoy et al., 2013)**.** *Let S and T be two operators on a metric space X. One says S is approximate operator of T if, for all $x \in X$ and for a real number $\epsilon > 0$, one has $d(Tx, Sx) \leq \epsilon$.*

One of the most generalized Banach operator (1.2) used by several authors is the one proved by Zamfirescu operator [Zamfirescu (1972)]. The Zamfirescu operator is stated as:

Let X be a complete metric space and T be a self map of X. The operator T is Zamfirescu operator if for each pair of points $x, y \in X$, at least one of the following is true:

$$\begin{aligned} Z_1:&\ d(Tx, Ty) \leq a d(x, y) \\ Z_2:&\ d(Tx, Ty) \leq b\left[d(x, Tx) + d(y, Ty)\right] \\ Z_3:&\ d(Tx, Ty) \leq c\left[d(x, Ty) + d(y, Tx)\right] \end{aligned} \tag{2.4}$$

where a, b and c are non-negative constants satisfying $a \in [0, 1)$, $b, c \leq \frac{1}{2}$.
An equivalent form of (2.4) is

$$d(Tx, Ty) \leq a \max\left\{d(x, y), \frac{1}{2}\left[d(x, Tx) + d(y, Ty)\right], \frac{1}{2}\left[d(x, Ty) + d(y, Tx)\right]\right\} \tag{2.5}$$

for $x, y \in X$ and $a \in [0, 1)$. See also [Reich (1971)]

Berinde (2002) observed that the condition (2.5) implies

$$d(Tx, Ty) \leq 2hd(x, Tx) + hd(x, y) \tag{2.6}$$

where $h = \max\left\{a, \frac{a}{2-a}\right\}$.

Rhoades (1993) used a more general contractive condition than (2.5): For $x, y \in X$, there exists $a \in [0, 1)$ such that

$$d(Tx, Ty) \leq a \max\left\{d(x, y), \frac{1}{2}\left[d(x, Tx) + d(y, Ty)\right], d(x, Ty), d(y, Tx)\right\} \tag{2.7}$$

Osilike (1995) extended and generalized the contractive condition (2.7): For $x, y \in X$, there exists $a \in [0, 1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq Ld(x, Tx) + ad(x, y) \tag{2.8}$$

Imoru and Olatinwo (2003) employed a much more general class of operators T than (2.8) satisfying the following contractive conditions

$$d(Tx, Ty) \leq ad(x, y) + \varphi\left(d(x, Tx)\right) \text{ for } x, y \in X \tag{2.9}$$

where $a \in [0, 1)$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone increasing function with $\varphi(0) = 0$. The following Lemmas will be helpful.

Lemma 2.8 (Berinde, 2004). *Let δ be a real number such that $0 \leq \delta < 1$ and $\{\epsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \epsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying*

$$u_{n+1} \leq \delta u_n + \epsilon_n, \ n = 0, 1, 2, \ldots$$

we have $\lim_{n\to\infty} u_n = 0$.

Lemma 2.9 (Soltuz S. M., 2008). *Let $\{a_n\}_{n=0}^{\infty}$ be a nonnegative sequence for which there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, one has the following inequality:*

$$a_{n+1} \leq (1 - r_n)a_n + r_n t_n,$$

where $r_n \in (0, 1)$, for all $n \in \mathbb{N}$, $\sum_{n=1}^{\infty} r_n = \infty$, and $t_n \geq 0$ for $n \in \mathbb{N}$. Then,

$$0 \leq \lim_{n\to\infty} \sup a_n \leq \lim_{n\to\infty} \sup t_n$$

3. Main Results

In this section, we present our main results.

3.1. **Analytical Results.** Let K be a nonempty closed subset of (X, d, W) and $x_0 \in K$, we introduce an implicit multistep scheme, called WR-iterative scheme, in a convex metric space as:

$$\begin{aligned} x_n &= W(x_{n-1}^{(1)}, Tx_n, \alpha_n) \\ x_{n-1}^{(1)} &= W(x_{n-1}^{(2)}, Tx_{n-1}^{(1)}, \beta_n^{(1)}) \\ x_{n-1}^{(l)} &= W(x_{n-1}^{(l+1)}, Tx_{n-1}^{(l)}, \beta_n^{(l)}), l = 2, 3, \ldots, k-2 \\ x_{n-1}^{(k-1)} &= W(x_{n-1}, Tx_{n-1}^{(k-1)}, \beta_n^{(k-1)}), \ k \geq 2, \ n \geq 1 \end{aligned} \tag{3.1}$$

where $\{\alpha_n\}_n^{\infty}$, $\left\{\beta_n^{(l)}\right\}_n^{\infty}$ are real positive sequences in $[0, 1]$ with $\sum(1 - \alpha_n) = \infty$. Let $k = 3$ in (3.1), we obtain the implicit Noor iteration [Chugh et al., 2015] in convex metric spaces given by:

$$\begin{aligned} x_n =& W(x_{n-1}^{(1)}, Tx_n, \alpha_n) \\ x_{n-1}^{(1)} =& W(x_{n-1}^{(2)}, Tx_{n-1}^{(1)}, \beta_n^{(1)}) \\ x_{n-1}^{(2)} =& W(x_{n-1}, Tx_{n-1}^{(2)}, \beta_n^{(2)}) \end{aligned} \tag{3.2}$$

An equivalent form of (3.2) in a linear vector space is

$$\begin{aligned} x_n &= \alpha_n x_{n-1}^{(1)} + (1 - \alpha_n)Tx_n \\ x_{n-1}^{(1)} &= \beta_n^{(1)} x_{n-1}^{(2)} + (1 - \beta_n^{(1)})Tx_{n-1}^{(1)} \\ x_{n-1}^{(2)} &= \beta_n^{(2)} x_{n-1} + (1 - \beta_n^{(2)})Tx_{n-1}^{(2)} \end{aligned} \tag{3.3}$$

Putting $k = 2$ and $\beta_n^{(l)} = 1$ for $l = 2, 3, \ldots, k-1$ in (3.1) resulted to Implicit Ishikawa iteration [Xue and Zhang (2013)] in convex metric spaces and it is given as:

$$\begin{aligned} x_n &= W(x_{n-1}^{(1)}, Tx_n, \alpha_n) \\ x_{n-1}^{(1)} &= W(x_{n-1}, Tx_{n-1}^{(1)}, \beta_n^{(1)}) \end{aligned} \tag{3.4}$$

We get the Implicit Mann iteration [Ciric et al. (2008), Xue and Zhang (2013)] when $\beta_n^{(l)} = 1$, for all l, in (3.1) as:

$$x_n = W(x_{n-1}, Tx_n, \alpha_n) \tag{3.5}$$

Throughout, an operator T shall be assumed fixed, and a fixed point $p \in F(T)$ for the contractive-like operator (2.9) is unique.

Theorem 3.1. *Let (X, d, W) be a convex metric space and K a nonempty closed subset of (X, d, W). Assume T is self map of K satisfying the contractive-like operator (2.9) with $F(T) \neq \phi$. Then, for $x_0 \in K$, the sequence $\{x_n\}$ defined by (3.1) with $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$ converges strongly to the fixed point $p \in F(T)$.*

Proof

Let $x_0 \in K$ and $p \in F(T)$, then using (2.9) and (3.1) we have

$$\begin{aligned} d(x_n, p) = d\left(W(x_{n-1}^{(1)}, Tx_n, \alpha_n), p\right) &\le \alpha_n d(x_{n-1}^{(1)}, p) + (1-\alpha_n) d(Tx_n, p) \\ &\le \alpha_n d(x_{n-1}^{(1)}, p) + (1-\alpha_n) a d(x_n, p) \\ &\quad + (1-\alpha_n)\varphi\left(d(Tp, p)\right) \\ &= \alpha_n d(x_{n-1}^{(1)}, p) + (1-\alpha_n) a d(x_n, p) \end{aligned}$$

This implies

$$d(x_n, p) \le \frac{\alpha_n}{1-(1-\alpha_n)a} d(x_{n-1}^{(1)}, p) \tag{3.6}$$

Also, from (3.1), we have

$$\begin{aligned} d(x_{n-1}^{(1)}, p) = d\left(W(x_{n-1}^{(2)}, Tx_{n-1}^{(1)}, \beta_n^{(1)}), p\right) &\le \beta_n^{(1)} d(x_{n-1}^{(2)}, p) + (1-\beta_n^{(1)}) d(Tx_{n-1}^{(1)}, p) \\ &\le \beta_n^{(1)} d(x_{n-1}^{(2)}, p) + (1-\beta_n^{(1)}) a d(x_{n-1}^{(1)}, p) \\ &\quad + (1-\beta_n^{(1)})\varphi\left(d(Tp, p)\right) \\ &= \beta_n^{(1)} d(x_{n-1}^{(2)}, p) + (1-\beta_n^{(1)}) a d(x_{n-1}^{(1)}, p) \end{aligned}$$

This becomes

$$d(x_{n-1}^{(1)}, p) \le \frac{\beta_n^{(1)}}{1-(1-\beta_n^{(1)})a} d(x_{n-1}^{(2)}, p) \tag{3.7}$$

Also, from (3.1)

$$\begin{aligned} d(x_{n-1}^{(2)}, p) = d\left(W(x_{n-1}^{(3)}, Tx_{n-1}^{(2)}, \beta_n^{(2)}), p\right) &\le \beta_n^{(2)} d(x_{n-1}^{(3)}, p) + (1-\beta_n^{(2)}) d(Tx_{n-1}^{(2)}, p) \\ &\le \beta_n^{(2)} d(x_{n-1}^{(3)}, p) + (1-\beta_n^{(2)}) a d(x_{n-1}^{(2)}, p) \\ &\quad + (1-\beta_n^{(2)})\varphi\left(d(Tp, p)\right) \\ &= \beta_n^{(2)} d(x_{n-1}^{(3)}, p) + (1-\beta_n^{(2)}) a d(x_{n-1}^{(2)}, p) \end{aligned}$$

which implies

$$d(x_{n-1}^{(2)}, p) \leq \frac{\beta_n^{(2)}}{1-(1-\beta_n^{(2)})a} d(x_{n-1}^{(3)}, p) \tag{3.8}$$

Continue in this manner up to $k-1$ in (3.1), we have

$$\begin{aligned} d(x_{n-1}^{(k-1)}, p) = d\left(W(x_{n-1}, Tx_{n-1}, \beta_n^{(k-1)}), p\right) &\leq \beta_n^{(k-1)} d(x_{n-1}, p) + (1-\beta_n^{(k-1)}) d(Tx_{n-1}, p) \\ &\leq \beta_n^{(k-1)} d(x_{n-1}, p) + (1-\beta_n^{(k-1)}) a d(x_{n-1}, p) \\ &\quad + (1-\beta_n^{(k-1)}) \varphi\left(d(Tp, p)\right) \\ &= \beta_n^{(k-1)} d(x_{n-1}, p) + (1-\beta_n^{(k-1)}) a d(x_{n-1}, p) \end{aligned}$$

Implying that

$$d(x_{n-1}^{(k-1)}, p) \leq \frac{\beta_n^{(k-1)}}{1-(1-\beta_n^{(k-1)})a} d(x_{n-1}, p) \tag{3.9}$$

By substituting (3.7) up to (3.9) into (3.6) becomes

$$\begin{aligned} d(x_n, p) \leq \left(\frac{\alpha_n}{1-(1-\alpha_n)a}\right) \left(\frac{\beta_n^{(1)}}{1-(1-\beta_n^{(1)})a}\right) \left(\frac{\beta_n^{(2)}}{1-(1-\beta_n^{(2)})a}\right) \cdots \\ \times \left(\frac{\beta_n^{(k-1)}}{1-(1-\beta_n^{(k-1)})a}\right) d(x_{n-1}, p) \end{aligned} \tag{3.10}$$

Let $Q_n = \dfrac{\alpha_n}{1-(1-\alpha_n)a}$, then

$$1 - Q_n = 1 - \frac{\alpha_n}{1-(1-\alpha_n)a} = \frac{1-(\alpha_n + (1-\alpha_n)a)}{1-(1-\alpha_n)a} \geq 1 - (\alpha_n + (1-\alpha_n)a)$$

This implies

$$Q_n \leq \alpha_n + (1-\alpha_n)a = \sum_{j=0}^{1} \alpha_{n,j} a^j < \sum_{j=0}^{1} \alpha_{n,j} = 1 \tag{3.11}$$

where $\alpha_n = \alpha_{n,0}$ and $(1-\alpha_n) = \alpha_{n,1}$.
Similarly, we can show that

$$\begin{aligned} \frac{\beta_n^{(1)}}{1-(1-\beta_n^{(1)})a} &\leq \beta_n^{(1)} + (1-\beta_n^{(1)})a = \sum_{j=0}^{1} \beta_{n,j}^{(1)} a^j < \sum_{j=0}^{1} \beta_{n,j}^{(k-1)} = 1 \\ \frac{\beta_n^{(2)}}{1-(1-\beta_n^{(2)})a} &\leq \beta_n^{(2)} + (1-\beta_n^{(2)})a = \sum_{j=0}^{1} \beta_{n,j}^{(2)} a^j < \sum_{j=0}^{1} \beta_{n,j}^{(2)} = 1 \\ &\vdots \\ \frac{\beta_n^{(k-1)}}{1-(1-\beta_n^{(k-1)})a} &\leq \beta_n^{(k-1)} + (1-\beta_n^{(k-1)})a = \sum_{j=0}^{1} \beta_{n,j}^{(k-1)} a^j < \sum_{j=0}^{1} \beta_{n,j}^{(k-1)} = 1 \end{aligned} \tag{3.12}$$

By applying (3.11) and (3.12) in (3.10), we have

$$\begin{aligned}
d(x_n, p) &\leq (\alpha_n + (1-\alpha_n)a)\, d(x_{n-1}, p) = [1-(1-\alpha_n)(1-a)]\, d(x_{n-1}, p) \\
&\leq [1-(1-\alpha_n)(1-a)]\,[1-(1-\alpha_{n-1})(1-a)]\, d(x_{n-2}, p) \\
&\cdots \\
&\leq \prod_{r=1}^{n} [1-(1-\alpha_r)(1-a)] d(x_0, p) \\
&\leq \mathrm{e}^{-(1-a)\sum_{r=1}^{n}(1-\alpha_r)} d(x_0, p)
\end{aligned}$$

Clearly, $d(x_0, p)$ is fixed and as $n \to \infty$, $\sum_{r=1}^{\infty}(1-\alpha_r) = \infty$.
Hence, $\lim_{n\to\infty} d(x_n, p) = 0$.
Therefore, the implicit multistep scheme (3.1) converges strongly to $p \in F(T)$.

Corollary 3.2. *Let (X, d, W) be a convex metric space and K a nonempty closed subset of (X, d, W). Let T be a self map of K satisfying the contractive-like operator (2.9) with $F(T) \neq \phi$. For $x_0 \in K$ and $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. Then, the sequence $\{x_n\}$ defined by*

i *(3.2) converge strongly to the fixed point of T.*
ii *(3.4) converge strongly to the fixed point of T.*
iii *(3.5) converge strongly to the fixed point of T.*

Proof
The proof of Corollary 1 is immediate from Theorem 3.1 by putting $\beta_n^{(l)} = 1$ for each $l = 1, 2, \ldots, k-1$, $k = 2$, $k = 3$ in (3.1).

Remark. *Corollary 1(iii) above is Theorem 9 in Chugh et al. (2015).*

Theorem 3.3. *Let K be a nonempty closed subset of a convex metric space (X, d, W). Let $T : K \to K$ be a map satisfying the contractive-like operator (2.9) with $F(T) \neq \phi$. Then, for $x_0 \in K$, the sequence $\{x_n\}$ defined by (1.9) with $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$ converges strongly to the fixed point $p \in F(T)$.*

Proof
Using (1.9) and (2.9), we have, for $x_0 \in K$ and $p \in F(T)$

$$\begin{aligned}
d(x_n, p) = d\left(W(x_{n-1}^{(1)}, Tx_{n-1}^{(1)}, \alpha_n), p\right) &\leq \alpha_n d(x_{n-1}^{(1)}, p) + (1-\alpha_n) d(Tx_{n-1}^{(1)}, p) \\
&\leq \alpha_n d(x_{n-1}^{(1)}, p) + (1-\alpha_n) a d(x_{n-1}^{(1)}, p) \\
&\quad + (1-\alpha_n)\varphi\left(d(Tp, p)\right) \\
&= [\alpha_n + (1-\alpha_n)a]\, d(x_{n-1}^{(1)}, p)
\end{aligned} \tag{3.13}$$

with

$$\begin{aligned}
d(x_{n-1}^{(1)}, p) = d\left(W(x_{n-1}^{(2)}, Tx_{n-1}^{(2)}, \beta_n^{(1)}), p\right) &\leq \beta_n^{(1)} d(x_{n-1}^{(2)}, p) + (1-\beta_n^{(1)}) d(Tx_{n-1}^{(2)}, p) \\
&\leq \beta_n^{(1)} d(x_{n-1}^{(1)}, p) + (1-\beta_n^{(1)}) a d(x_{n-1}^{(1)}, p) \\
&= \left[\beta_n^{(1)} + (1-\beta_n^{(1)})a\right] d(x_{n-1}^{(2)}, p)
\end{aligned} \tag{3.14}$$

$$d(x^{(2)}_{n-1},p) = d\left(W(x^{(3)}_{n-1},Tx^{(3)}_{n-1},\beta^{(2)}_n),p\right) \le \beta^{(2)}_n d(x^{(3)}_{n-1},p) + (1-\beta^{(2)}_n)d(Tx^{(3)}_{n-1},p)$$
$$\le \beta^{(2)}_n d(x^{(2)}_{n-1},p) + (1-\beta^{(2)}_n)ad(x^{(3)}_{n-1},p)$$
$$= \left[\beta^{(2)}_n + (1-\beta^{(2)}_n)a\right] d(x^{(3)}_{n-1},p) \tag{3.15}$$

… …

$$d(x^{(k-1)}_{n-1},p) = d\left(W(x_{n-1},Tx_{n-1},\beta^{(k-1)}_n),p\right) \le \beta^{(k-1)}_n d(x_{n-1},p) + (1-\beta^{(k-1)}_n)d(Tx_{n-1},p)$$
$$\le \beta^{(k-1)}_n d(x_{n-1},p) + (1-\beta^{(k-1)}_n)ad(x_{n-1},p)$$
$$= \left[\beta^{(k-1)}_n + (1-\beta^{(k-1)}_n)a\right] d(x_{n-1},p) \tag{3.16}$$

By combining inequalities (3.13), (3.14), (3.15) and (3.16), we obtain

$$\begin{aligned} d(x_n,p) \le{}& [\alpha_n + (1-\alpha_n)a]\left[\beta^{(1)}_n + (1-\beta^{(1)}_n)a\right]\left[\beta^{(2)}_n + (1-\beta^{(2)}_n)a\right]\times \\ &\cdots\times\left[\beta^{(k-1)}_n + (1-\beta^{(k-1)}_n)a\right] d(x_{n-1},p) \end{aligned} \tag{3.17}$$

Putting in mind inequalities (3.11) and (3.12), and let

$$\delta = [\alpha_n + (1-\alpha_n)a]\left[\beta^{(1)}_n + (1-\beta^{(1)}_n)a\right]\left[\beta^{(2)}_n + (1-\beta^{(2)}_n)a\right]\cdots\left[\beta^{(k-1)}_n + (1-\beta^{(k-1)}_n)a\right]$$
$$= \left(\sum_{j=0}^{1}\alpha_{n,j}a^j\right)\left(\sum_{j=0}^{1}\beta^{(1)}_{n,j}a^j\right)\left(\sum_{j=0}^{1}\beta^{(2)}_{n,j}a^j\right)\cdots\left(\sum_{j=0}^{1}\beta^{(k-1)}_{n,j}a^j\right)$$
$$< \left(\sum_{j=0}^{1}\alpha_{n,j}\right)\left(\sum_{j=0}^{1}\beta^{(1)}_{n,j}\right)\left(\sum_{j=0}^{1}\beta^{(2)}_{n,j}\right)\cdots\left(\sum_{j=0}^{1}\beta^{(k-1)}_{n,j}\right) = 1$$

with $\alpha_n = \alpha_{n,0}$, $(1-\alpha_n) = \alpha_{n,1}$, $\beta^{(l)}_n = \beta^{(l)}_{n,0}$, $(1-\beta^{(l)}_n) = \beta^{(l)}_{n,1}$ for $l \ge 1$ and $a^j \in [0,1)$ for $j = 0,1$.
By the application of Lemma 1, we have $\lim_{n\to\infty} d(x_n,p) = 0$.
Therefore, the scheme (1.9) with contractive-like operator (2.9) converges strongly to $p \in F(T)$.

Remark. *The strong convergence of (1.7) and (1.8) can be proved successfully in Theorem 3.2 by letting $\beta^{(l)}_n = 1$ for $l \ge 2$, $k = 2$ and $k = 3$ in (1.9), respectively.*

The convergence rate of the schemes (1.7), (1.8), (1.9), (3.1), (3.2), (3.4) and (3.5) are summarized in the following theorem.

Theorem 3.4. *Let K be a nonempty closed subset of a convex metric space (X,d,W) and let T be a contractive-like operator satisfying (2.9) with $F(T) \ne \phi$. Then, for $x_0 \in K$, the sequence $\{x_n\}$ defined by (3.1), a $(k-1)$th step scheme converges faster than its $(k-2)$th step scheme. Furthermore, the scheme (3.1) converges faster than the multistep SP-iteration (1.9) and for $k \ge 2$.*

Proof

Let $\{x_n\}$ be a sequence $(k-1)$th step that converges to p as $(n\to\infty)$ satisfying the estimate (3.10) with

$$d(x_n,p) \le \Gamma_{k-1} = \left(\frac{\alpha_n}{1-(1-\alpha_n)a}\right)\prod_{l=1}^{k-1}\left(\frac{\beta^{(l)}_n}{1-(1-\beta^{(l)}_n)a}\right)d(x_{n-1},p)$$

And let $\{y_n\}$ be a sequence of $(k-2)$th step that also converges to p such that

$$d(y_n,p) \leq \Gamma_{k-2} = \left(\frac{\alpha_n}{1-(1-\alpha_n)a}\right)\prod_{l=1}^{k-2}\left(\frac{\beta_n^{(l)}}{1-(1-\beta_n^{(l)})a}\right)d(y_{n-1},p)$$

By Definition 2.6, we have

$$0 \leq \frac{\Gamma_{k-1}}{\Gamma_{k-2}} = \frac{\beta_n^{(k-1)}}{1-(1-\beta_n^{(k-1)})a}\frac{d(x_{n-1},p)}{d(y_{n-1},p)} \leq [1-(1-\beta_n^{(k-1)})(1-a)]\frac{d(x_{n-1},p)}{d(y_{n-1},p)}$$
$$\leq \mathrm{e}^{-(1-a)\sum_t^n(1-\beta_t^{(k-1)})}\frac{d(x_0,p)}{d(y_0,p)} \tag{3.18}$$

Clearly $\frac{d(x_0,p)}{d(y_0,p)}$ with $d(y_0,p) \neq 0$ is fixed and as $n \to \infty$, the right hand side inequality of (3.18) diminishes. Hence, the sequence $\{x_n\}$ for $(k-1)$th step scheme (3.1) converges faster than the sequence $\{y_n\}$, its $(k-2)$th step scheme for $k \geq 2$. Furthermore, by using Definition 2.6, inequalities (3.11) and (3.12), we have

$$\begin{aligned}\Gamma_{k-1} &= \left(\frac{\alpha_n}{1-(1-\alpha_n)a}\right)\prod_{l=1}^{k-1}\left(\frac{\beta_n^{(l)}}{1-(1-\beta_n^{(l)})a}\right)\\ &\leq (\alpha_n+(1-\alpha_n)a)\prod_{l=1}^{k-1}\left(\beta_n^{(l)}+(1-\beta_n^{(l)})a\right)\\ &= \left(\sum_{j=0}^{1}\alpha_{n,j}a^j\right)\prod_{l=1}^{k-1}\left(\sum_{j=0}^{1}\beta_{n,j}^{(l)}a^j\right) \equiv \Gamma'_{k-1}\end{aligned}$$

where $\Gamma'_{k-1} \in (0,1)$ is the estimate in (3.17) of multistep SP-iteration (1.9).
It is easy to verify that

$$\frac{\Gamma_{k-1}}{\Gamma'_{k-1}} = 0$$

Therefore, the scheme (3.1) converges faster than the multistep SP-iteration (1.9) for $k \geq 2$.

Remark. *(i) The positive terms Γ_1, Γ_2 and Γ_3 are, respectively, found in the estimates of implicit Mann scheme, implicit Ishikawa scheme and implicit Noor scheme.*
(ii) The positive terms Γ'_1, Γ'_3 and Γ'_3 are found in estimates of Mann iteration, Thianwan iteration and SP-iteration, respectively.
(iii) Since SP-iteration is better than the explicit iterative scheme, the implicit iterative schemes are better than both the explicit iterations and the SP-iterative schemes.

Theorem 3.5. *Let (X,d,W) be a convex metric space and K a nonempty closed subset of (X,d,W). Suppose T is a self map of K satisfying the contractive-like operator (2.9) with $F(T) \neq \phi$. Then, for $x_0 \in K$, the sequence $\{x_n\}$ defined by (3.1) is T-stable.*

Proof

Let $\{y_n\} \in K$ be an arbitrary sequence and let $\epsilon_n = d(y_n, W(z_{n-1}^{(1)}, Ty_n, \alpha_n))$, where $z_{n-1}^{(1)} = W(z_{n-1}^{(2)}, Tz_{n-1}^{(1)}, \beta_n^{(1)})$, $z_{n-1}^{(2)} = W(z_{n-1}^{(3)}, Tz_{n-1}^{(2)}, \beta_n^{(2)})$ up to $z_{n-1}^{(q-1)} =$

$W(z_{n-1}^{(0)}, Tz_{n-1}^{(q-1)}, \beta_n^{(q-1)})$.
Suppose $\lim \epsilon_n = 0$ as $n \to \infty$ and $p \in F(T)$, then, by using (2.9) we have

$$\begin{aligned} d(y_n, p) &\leq d(y_n, W(z_{n-1}^{(1)}, Ty_n, \alpha_n)) + d(p, W(z_{n-1}^{(1)}, Ty_n, \alpha_n)) \\ &\leq \epsilon_n + \alpha_n d(z_{n-1}^{(1)}, p) + (1-\alpha_n) d(Ty_n, p) \\ &\leq \epsilon_n + \alpha_n d(z_{n-1}^{(1)}, p) + (1-\alpha_n) a d(y_n, p) \end{aligned}$$

Therefore

$$d(y_n, p) \leq \frac{\epsilon_n}{1-(1-\alpha_n)a} + \frac{\alpha_n}{1-(1-\alpha_n)a} d(z_{n-1}^{(1)}, p) \tag{3.19}$$

From inequalities (3.11) and (3.12) we have

$$d(z_{n-1}^{(1)}, p) \leq d(z_{n-1}^{(2)}, p) \leq \cdots \leq d(z_{n-1}^{(q-1)}, p) \leq d(y_{n-1}, p)$$

and that $\dfrac{\alpha_n}{1-(1-\alpha_n)a} \equiv \delta < 1$.
Then, Inequality (3.19) becomes

$$d(y_n, p) \leq \delta d(y_{n-1}, p) + \frac{\epsilon_n}{1-(1-\alpha_n)a}$$

By Lemma 1, we have

$$d(y_n, p) \to 0, \ n \to \infty$$

Conversely, suppose $d(y_n, p) \to 0$ for $p \in F(T)$, then

$$\begin{aligned} \epsilon_n &= d(y_n, W(z_{n-1}^{(1)}, Ty_n, \alpha_n)) \\ &\leq d(y_n, p) + d(p, W(z_{n-1}^{(1)}, Ty_n, \alpha_n)) \\ &\leq d(y_n, p) + \alpha_n d(z_{n-1}^{(1)}, p) + (1-\alpha_n) d(Ty_n, p) \\ &\leq (1-(1-\alpha_n)a) d(y_n, p) + \alpha_n d(y_{n-1}, p) \end{aligned}$$

By taking limit as $n \to \infty$, we have $d(y_n, p) \to 0$.
Hence

$$\lim \epsilon_n = 0$$

Therefore, the new iterative scheme (3.1) is T-stable.

Remark. *The stability results for implicit Mann scheme, implicit Ishikawa scheme and implicit Noor scheme, using contractive-like operator (2.9) are special cases of our results in Theorem 3.3 above.*

The following Theorem discusses the data dependence results of the iterative scheme (3.1).

Theorem 3.6. *Let T be a map of K into itself satisfying (2.9) and let S be an approximate operator of T. If $\{x_n\}, \{y_n\} \subset K$ are two iterative schemes associated to T, S respectively, where $\{x_n\}$ is the scheme (3.1) and*

$$\begin{aligned} & y_n = W(y_{n-1}^{(1)}, Sy_n, \alpha_n) \\ & y_{n-1}^{(1)} = W(y_{n-1}^{(2)}, Sy_{n-1}^{(1)}, \beta_n^{(1)}) \\ & \vdots \\ & y_{n-1}^{(k-1)} = W(y_{n-1}, Sy_{n-1}^{(k-1)}, \beta_n^{(k-1)}), n = 0, 1, 2, 3, \ldots \end{aligned} \tag{3.20}$$

where α_n, β_n^l, $l = 1, 2, \ldots, k-1$, *are nonnegative real sequence in* $[0,1]$ *satisfying* $\sum(1-\alpha_n) = \infty$. *Then, for* $p \in F(T)$, $q \in F(S)$ *and for any given* $\epsilon > 0$ *and* $k \in \mathbb{N}$ *we have,*

$$d(p,q) \le \frac{k\epsilon}{(1-a)^2}$$

Proof

Using the iterative schemes (3.1), (3.20), Definition 2.7 and condition (2.9), we obtain

$$\begin{aligned} d(x_n, y_n) &= d(W(x_{n-1}^{(1)}, Ty_n, \alpha_n), W(y_{n-1}^{(1)}, Sy_n, \alpha_n)) \\ &\le \alpha_n d(x_{n-1}^{(1)}, y_{n-1}^{(1)}) + (1-\alpha_n) d(Tx_n, Sy_n) \\ &\le \alpha_n d(x_{n-1}^{(1)}, y_{n-1}^{(1)}) + (1-\alpha_n)\left[d(Tx_n, Sx_n) + d(Sx_n, Sy_n)\right] \\ &\le \alpha_n d(x_{n-1}^{(1)}, y_{n-1}^{(1)}) + (1-\alpha_n)\left[\epsilon + \varphi\left(d(x_n, Sx_n)\right) + a d(x_n, y_n)\right] \end{aligned}$$

which further implies

$$d(x_n, y_n) \le \frac{\alpha_n}{1-a(1-\alpha_n)} d(x_{n-1}^{(1)}, y_{n-1}^{(1)}) + \frac{(1-\alpha_n)}{1-a(1-\alpha_n)}\left[\epsilon + \varphi\left(d(x_n, Sx_n)\right)\right], \tag{3.21}$$

with

$$d(x_{n-1}^{(1)}, y_{n-1}^{(1)}) \le \frac{\beta_n^{(1)}}{1-a(1-\beta_n^{(1)})} d(x_{n-1}^{(2)}, y_{n-1}^{(2)}) + \frac{(1-\beta_n^{(1)})}{1-a(1-\beta_n^{(1)})}\left[\epsilon + \varphi\left(d(x_{n-1}^{(1)}, Sx_{n-1}^{(1)})\right)\right], \tag{3.22}$$

$$d(x_{n-1}^{(2)}, y_{n-1}^{(2)}) \le \frac{\beta_n^{(2)}}{1-a(1-\beta_n^{(2)})} d(x_{n-1}^{(3)}, y_{n-1}^{(3)}) + \frac{(1-\beta_n^{(2)})}{1-a(1-\beta_n^{(2)})}\left[\epsilon + \varphi\left(d(x_{n-1}^{(2)}, Sx_{n-1}^{(2)})\right)\right], \tag{3.23}$$

$$d(x_{n-1}^{(3)}, y_{n-1}^{(3)}) \le \frac{\beta_n^{(3)}}{1-a(1-\beta_n^{(3)})} d(x_{n-1}^{(4)}, y_{n-1}^{(4)}) + \frac{(1-\beta_n^{(3)})}{1-a(1-\beta_n^{(3)})}\left[\epsilon + \varphi\left(d(x_{n-1}^{(3)}, Sx_{n-1}^{(3)})\right)\right], \tag{3.24}$$

$$d(x_{n-1}^{(4)}, y_{n-1}^{(4)}) \le \frac{\beta_n^{(4)}}{1-a(1-\beta_n^{(4)})} d(x_{n-1}^{(5)}, y_{n-1}^{(5)}) + \frac{(1-\beta_n^{(4)})}{1-a(1-\beta_n^{(4)})}\left[\epsilon + \varphi\left(d(x_{n-1}^{(4)}, Sx_{n-1}^{(4)})\right)\right],$$

$\cdots \cdots$

$$d(x_{n-1}^{(k-2)}, y_{n-1}^{(k-2)}) \le \frac{\beta_n^{(k-2)}}{1-a(1-\beta_n^{(k-2)})} d(x_{n-1}^{(k-1)}, y_{n-1}^{(k-1)}) + \frac{(1-\beta_n^{(k-2)})}{1-a(1-\beta_n^{(k-2)})}\left[\epsilon + \varphi\left(d(x_{n-1}^{(k-2)}, Sx_{n-1}^{(k-2)})\right)\right],$$

$$d(x_{n-1}^{(k-1)}, y_{n-1}^{(k-1)}) \le \frac{\beta_n^{(k-1)}}{1-a(1-\beta_n^{(k-1)})} d(x_{n-1}, y_{n-1}) + \frac{(1-\beta_n^{(k-1)})}{1-a(1-\beta_n^{(k-1)})}\left[\epsilon + \varphi\left(d(x_{n-1}^{(k-1)}, Sx_{n-1}^{(k-1)})\right)\right], \tag{3.25}$$

Combining inequalities (3.21) up to (3.25), we have

$$
\begin{aligned}
d(x_n, y_n) &\leq \frac{\alpha_n}{1-a(1-\alpha_n)}\left[\frac{\beta_n^{(1)}}{1-a(1-\beta_n^{(1)})} d(x_{n-1}^{(2)}, y_{n-1}^{(2)})\right.\\
&\left.+\frac{(1-\beta_n^{(1)})}{1-a(1-\beta_n^{(1)})}\left[\epsilon+\varphi\left(d(x_{n-1}^{(1)}, Sx_{n-1}^{(1)})\right)\right]\right]+\frac{(1-\alpha_n)}{1-a(1-\alpha_n)}\left[\epsilon+\varphi\left(d(x_n, Sx_n)\right)\right]\\
&\leq \frac{\alpha_n}{1-a(1-\alpha_n)}\left[\frac{\beta_n^{(1)}}{1-a(1-\beta_n^{(1)})}\left[\frac{\beta_n^{(2)}}{1-a(1-\beta_n^{(2)})} d(x_{n-1}^{(3)}, y_{n-1}^{(3)})\right.\right.\\
&\left.+\frac{(1-\beta_n^{(2)})}{1-a(1-\beta_n^{(2)})}\left[\epsilon+\varphi\left(d(x_{n-1}^{(2)}, Sx_{n-1}^{(2)})\right)\right]\right]\\
&\left.+\frac{(1-\beta_n^{(1)})}{1-a(1-\beta_n^{(1)})}\left[\epsilon+\varphi\left(d(x_{n-1}^{(1)}, Sx_{n-1}^{(1)})\right)\right]\right]+\frac{(1-\alpha_n)}{1-a(1-\alpha_n)}\left[\epsilon+\varphi\left(d(x_n, Sx_n)\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\alpha_n}{1-a(1-\alpha_n)}\left[\frac{\beta_n^{(1)}}{1-a(1-\beta_n^{(1)})}\left[\frac{\beta_n^{(2)}}{1-a(1-\beta_n^{(2)})}\left[\frac{\beta_n^{(3)}}{1-a(1-\beta_n^{(3)})} d(x_{n-1}^{(4)}, y_{n-1}^{(4)})\right.\right.\right.\\
&\left.+\frac{(1-\beta_n^{(3)})}{1-a(1-\beta_n^{(3)})}\left[\epsilon+\varphi\left(d(x_{n-1}^{(3)}, Sx_{n-1}^{(3)})\right)\right]\right]\\
&\left.+\frac{(1-\beta_n^{(2)})}{1-a(1-\beta_n^{(2)})}\left[\epsilon+\varphi\left(d(x_{n-1}^{(2)}, Sx_{n-1}^{(2)})\right)\right]\right]\\
&\left.+\frac{(1-\beta_n^{(1)})}{1-a(1-\beta_n^{(1)})}\left[\epsilon+\varphi\left(d(x_{n-1}^{(1)}, Sx_{n-1}^{(1)})\right)\right]\right]+\frac{(1-\alpha_n)}{1-a(1-\alpha_n)}\left[\epsilon+\varphi\left(d(x_n, Sx_n)\right)\right]\\
&\leq \frac{\alpha_n}{1-a(1-\alpha_n)}\left[\frac{\beta_n^{(1)}}{1-a(1-\beta_n^{(1)})}\left[\frac{\beta_n^{(2)}}{1-a(1-\beta_n^{(2)})}\left[\frac{\beta_n^{(3)}}{1-a(1-\beta_n^{(3)})}\left[\cdots\left[\frac{\beta_n^{(k-1)} d(x_{n-1}, y_{n-1})}{1-a(1-\beta_n^{(k-1)})}\right.\right.\right.\right.\right.\\
&\left.\left.+\frac{(1-\beta_n^{(k-1)})}{1-a(1-\beta_n^{(k-1)})}\left[\epsilon+\varphi\left(d(x_{n-1}^{(k-1)}, Sx_{n-1}^{(k-1)})\right)\right]\right]+\cdots\right]\\
&\left.+\frac{(1-\beta_n^{(3)})}{1-a(1-\beta_n^{(3)})}\left[\epsilon+\varphi\left(d(x_{n-1}^{(3)}, Sx_{n-1}^{(3)})\right)\right]\right]\\
&\left.+\frac{(1-\beta_n^{(2)})}{1-a(1-\beta_n^{(2)})}\left[\epsilon+\varphi\left(d(x_{n-1}^{(2)}, Sx_{n-1}^{(2)})\right)\right]\right]\\
&\left.+\frac{(1-\beta_n^{(1)})}{1-a(1-\beta_n^{(1)})}\left[\epsilon+\varphi\left(d(x_{n-1}^{(1)}, Sx_{n-1}^{(1)})\right)\right]\right]+\frac{(1-\alpha_n)}{1-a(1-\alpha_n)}\left[\epsilon+\varphi\left(d(x_n, Sx_n)\right)\right]
\end{aligned}
$$

This further gives

$$\begin{aligned}
d(x_n, y_n) \leq & \left[\frac{\alpha_n}{1-a(1-\alpha_n)}\right]\left[\frac{\beta_n^{(1)}}{1-a(1-\beta_n^{(1)})}\right]\left[\frac{\beta_n^{(2)}}{1-a(1-\beta_n^{(2)})}\right]\cdots\left[\frac{\beta_n^{(k-1)}}{1-a(1-\beta_n^{(k-1)})}\right] d(x_{n-1}, y_{n-1}) \\
& + \left[\frac{\alpha_n}{1-a(1-\alpha_n)}\right]\left[\frac{\beta_n^{(1)}}{1-a(1-\beta_n^{(1)})}\right]\cdots\left[\frac{\beta_n^{(k-1)}}{1-a(1-\beta_n^{(k-1)})}\right]\left[\epsilon + \varphi\left(d(x_{n-1}^{(k-1)}, Sx_{n-1}^{(k-1)})\right)\right] \\
& + \left[\frac{\alpha_n}{1-a(1-\alpha_n)}\right]\left[\frac{\beta_n^{(1)}}{1-a(1-\beta_n^{(1)})}\right]\cdots\left[\frac{\beta_n^{(k-2)}}{1-a(1-\beta_n^{(k-2)})}\right]\left[\epsilon + \varphi\left(d(x_{n-1}^{(k-2)}, Sx_{n-1}^{(k-2)})\right)\right] \\
& + \cdots + \left[\frac{\alpha_n}{1-a(1-\alpha_n)}\right]\left[\frac{\beta_n^{(1)}}{1-a(1-\beta_n^{(1)})}\right]\left[\epsilon + \varphi\left(d(x_{n-1}^{(1)}, Sx_{n-1}^{(1)})\right)\right] \\
& + \left[\frac{(1-\alpha_n)}{1-a(1-\alpha_n)}\right]\left[\epsilon + \varphi\left(d(x_n, Sx_n)\right)\right]
\end{aligned} \tag{3.26}$$

Using (3.11) and (3.12) in (3.26), we have

$$\begin{aligned}
d(x_n, y_n) \leq & \left[\frac{\alpha_n}{1-a(1-\alpha_n)}\right] d(x_{n-1}, y_{n-1}) + \left[\frac{\beta_n^{(k-1)}}{1-a(1-\beta_n^{(k-1)})}\right]\left[\epsilon + \varphi\left(d(x_{n-1}^{(k-1)}, Sx_{n-1}^{(k-1)})\right)\right] \\
& + \left[\frac{\beta_n^{(k-2)}}{1-a(1-\beta_n^{(k-2)})}\right]\left[\epsilon + \varphi\left(d(x_{n-1}^{(k-2)}, Sx_{n-1}^{(k-2)})\right)\right] + \cdots \\
& + \left[\frac{\beta_n^{(1)}}{1-a(1-\beta_n^{(1)})}\right]\left[\epsilon + \varphi\left(d(x_{n-1}^{(1)}, Sx_{n-1}^{(1)})\right)\right] + \left[\frac{(1-\alpha_n)}{1-a(1-\alpha_n)}\right]\left[\epsilon + \varphi\left(d(x_n, Sx_n)\right)\right] \\
& \leq \left[1-(1-a)(1-\alpha_n)\right] d(x_{n-1}, y_{n-1}) + \frac{(1-a)(1-\alpha_n)}{(1-a)(1-a)}\left[\epsilon + \varphi\left(d(x_{n-1}^{(k-1)}, Sx_{n-1}^{(k-1)})\right)\right] \\
& + \frac{(1-a)(1-\alpha_n)}{(1-a)(1-a)}\left[\epsilon + \varphi\left(d(x_{n-1}^{(k-2)}, Sx_{n-1}^{(k-2)})\right)\right] + \cdots \\
& + \frac{(1-a)(1-\alpha_n)}{(1-a)(1-a)}\left[\epsilon + \varphi\left(d(x_{n-1}^{(1)}, Sx_{n-1}^{(1)})\right)\right] + \frac{(1-a)(1-\alpha_n)}{(1-a)(1-a)}\left[\epsilon + \varphi\left(d(x_n, Sx_n)\right)\right] \\
& = \left[1-(1-a)(1-\alpha_n)\right] d(x_{n-1}, y_{n-1}) + \frac{(1-a)(1-\alpha_n)}{(1-a)(1-a)}\left[k\epsilon + \varphi\left(d(x_{n-1}^{(k-1)}, Sx_{n-1}^{(k-1)})\right) + \right. \\
& \left. + \varphi\left(d(x_{n-1}^{(k-2)}, Sx_{n-1}^{(k-2)})\right) + \cdots + \varphi\left(d(x_{n-1}^{(1)}, Sx_{n-1}^{(1)})\right) + \varphi\left(d(x_n, Sx_n)\right)\right]
\end{aligned} \tag{3.27}$$

By letting $a_n = d(x_n, y_n)$, $r_n = (1-a)(1-\alpha_n)$ and
$t_n = \dfrac{\left[k\epsilon + \sum_{j=1}^{k-1} \varphi\left(d(x_{n-1}^{(j)}, Sx_{n-1}^{(j)})\right) + \varphi\left(d(x_n, Sx_n)\right)\right]}{(1-a)^2}$.
Then, inequality (3.27) becomes

$$a_n \leq (1-r_n)a_{n-1} + r_n t_n$$

Since φ is continuous and $\varphi(0) = 0$, then

$$\sum_{j=1}^{k-1} \varphi\left(d(x_{n-1}^{(j)}, Sx_{n-1}^{(j)})\right) = \varphi\left(d(x_n, Sx_n)\right) = 0$$

Thus, by Theorem 3.1 and Lemma 2, we have

$$d(p,q) \leq \frac{k\epsilon}{(1-a)^2}$$

Remark. *If $k = 3$ in the schemes (3.1) and (3.20), the result concerning data dependence of implicit Noor iteration can be obtained easily from our result in Theorem 3.5. Also, when $k = 2$ and $\beta_n^l = 0$ in (3.1) and (3.20), the data dependence of Implicit Ishikawa and Mann iterations follows, respectively from the same Theorem.*

3.2. **Numerical Results.** Here, we shall consider two numerical examples to justify the aforementioned claim in Theorem 3.3.

Example 3.7. *The function $f : [6,8] \to [6,8]$ defined by $f(x) = \frac{x}{2} + 3$ is an increasing function with fixed point $p = 6.0000$ and initial guess $x_0 = 7$ using $1 - \alpha_n = 1 - \beta_n^l = \frac{1}{\sqrt{3n+1}}$, for $l \geq 1$.*

Example 3.8. *The function $f : [\frac{1}{2}, 2] \to [\frac{1}{2}, 2]$ defined by $f(x) = \frac{1}{x}$ is an oscillatory function with fixed point $p = 1.0000$ and initial guess $x_0 = 4$ using $1 - \alpha_n = 1 - \beta_n^l = \frac{1}{n+1}$, for $l \geq 1$.*

We compute the results using MATLAB program and the comparison of the results are listed in Tables 1 and 2.

Table 1: Rate of convergence of fixed point schemes for Example 1.

n	Mann scheme (1.3)	Implicit Mann (3.5)	Thianwan scheme (1.7)	Implicit Ishikawa (3.4)	SP-scheme (1.8)	Implicit Noor (3.2)	SP-Multistep scheme (1.9)	WR-scheme (3.1)
0	7.0000	7.0000	7.0000	7.0000	7.0000	7.0000	7.0000	7.0000
1	6.7500	6.6667	6.5625	6.4444	6.4219	6.2963	6.3164	6.1975
2	6.5167	6.3657	6.2670	6.1338	6.1380	6.0489	6.0713	6.0179
3	6.1405	6.3203	6.1156	6.0309	6.0393	6.0054	6.0134	6.0010
4	6.0897	6.1391	6.0472	6.0058	6.0102	6.0004	6.0022	6.0000
5	6.0561	6.0556	6.0184	6.0009	6.0025	6.0000	6.0003	6.0000
6	6.0345	6.0208	6.0070	6.0001	6.0006	6.0000	6.0000	6.0000
7	6.0209	6.0073	6.0026	6.0000	6.0001	6.0000	6.0000	6.0000
8	6.0125	6.0024	6.0009	6.0000	6.0000	6.0000	6.0000	6.0000
9	6.0075	6.0008	6.0003	6.0000	6.0000	6.0000	6.0000	6.0000
10	6.0044	6.0002	6.0001	6.0000	6.0000	6.0000	6.0000	6.0000
11	6.0026	6.0001	6.0000	6.0000	6.0000	6.0000	6.0000	6.0000
12	6.0015	6.0000	6.0000	6.0000	6.0000	6.0000	6.0000	6.0000
...	...	...	...	...	...	...	...	...
18	6.0001	6.0000	6.0000	6.0000	6.0000	6.0000	6.0000	6.0000
19	6.0000	6.0000	6.0000	6.0000	6.0000	6.0000	6.0000	6.0000
20	6.0000	6.0000	6.0000	6.0000	6.0000	6.0000	6.0000	6.0000

Remark. *For an increasing function $f(x) = \frac{x}{2} + 3$ with initial guess $x_0 = 7$ and $1 - \alpha_n = 1 - \beta_n^l = \frac{1}{\sqrt{3n+1}}$, the WR-scheme converges at 4 iterations while the SP-multistep scheme at 6 iterations. Also, the implicit Noor, implicit Ishikawa and implicit Mann converge in 12, 7 and 5 iterations, respectively, whereas the Mann, Thianwan and SP schemes converge in 19, 11 and 8 iterations. We can easily obtain 1 iteration in WR-scheme when $k = 6$.*

Table 2: Rate of convergence of fixed point schemes for Example 2.

n	Mann scheme (1.3)	Implicit Mann (3.5)	Thianwan scheme (1.7)	Implicit Ishikawa(3.4)	SP-scheme (1.8)	Implicit Noor (3.2)	SP-Multistep scheme (1.9)	WR-scheme (3.1)
1	2.1250	2.2247	1.2978	1.4558	1.0342	1.1592	1.0006	1.0540
2	1.0221	1.2675	1.0199	1.0190	0.9988	1.0013	1.0000	1.0000
3	0.9893	1.0388	1.0049	1.0004	1.0002	1.0000	1.0000	1.0000
4	1.0065	1.0043	1.0018	1.0000	1.0000	1.0000	1.0000	1.0000
5	0.9957	1.0004	1.0008	1.0000	1.0000	1.0000	1.0000	1.0000
6	1.0031	1.0000	1.0004	1.0000	1.0000	1.0000	1.0000	1.0000
7	0.9977	1.0000	1.0002	1.0000	1.0000	1.0000	1.0000	1.0000
8	1.0018	1.0000	1.0001	1.0000	1.0000	1.0000	1.0000	1.0000
...	...	...	...	...	...	...	...	...
28	1.0001	1.0000	1.0001	1.0000	1.0000	1.0000	1.0000	1.0000
29	0.9999	1.0000	1.0001	1.0000	1.0000	1.0000	1.0000	1.0000
30	1.0001	1.0000	1.0001	1.0000	1.0000	1.0000	1.0000	1.0000

Remark. *In Table 2, we observe from the oscillatory function* $f(x) = \frac{1}{x}$ *with initial guess* $x_0 = 7$ *and* $1 - \alpha_n = 1 - \beta_n^l = \frac{1}{n+1}$, *that the multistep SP-scheme has good starting points than the WR-scheme, nevertheless, the WR-scheme does better in the successive approximations.*

4. Conclusion

We have established and proved strong convergence, convergence rate, T-stability and data dependence results for the implicit multistep scheme of fixed point of contractive-like operators in convex metric spaces. This study concluded that the scheme (3.1) is valid and it has better convergence rate when compare with other iterative schemes such as: multistep SP-iteration, explicit multistep scheme, SP-iteration, two-step of Thianwan scheme, implicit Noor iteration, implicit Ishikawa iteration, implicit Mann iteration, Noor iteration, Ishikawa iteration, Mann iteration and many more iterative schemes of fixed point in the literature.

Competing interests
We hereby declare that we have no competing interests.

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