Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 12 Issue 1(2020), Pages 12-28.

EXISTENCE RESULTS FOR GENERALIZED EXPONENTIAL VECTOR VARIATIONAL-LIKE INEQUALITIES IN FUZZY ENVIRONMENT

MOHAMMAD FARID, SYED SHAKAIB IRFAN, IQBAL AHMAD AND PREETI SHUKLA

Abstract. In this paper, we study a generalized exponential fuzzy vector variational-like inequalities in Euclidean spaces. We construct an example to illustrate the main problem. We define a new class of α_g-relaxed exponentially (γ, η)-monotone mapping in fuzzy environment. We prove the existence of solutions to generalized exponential vector variational-like inequality with fuzzy mappings by using KKM-technique. Further, we give some consequences of the main result. The results presented in this paper unifies and extends some known results in this area.

1. Introduction

The theory of variational inequality has been introduced by Kinderlehrer and Stampacchia [16]. Variational inequality theory has appeared as an effective and powerful tools to study and investigate a wide class of problems arising in pure and applied sciences including elasticity, optimization, transportation and structural analysis, see for example [2, 9, 16, 23, 26].

Wu and Huang defined the concepts of relaxed $\eta - \alpha$ pseudomonotone mappings to study vector variational-like inequality problem in Banach spaces. The generalized variational-like inequalities with generalized α-monotone multifunctions study by Ceng *et al.* [5] [see for instance, [11, 19, 22]]. In 2004, Antczak [1] introduced the class of exponential (p, r)-invex functions for differentiable case [see for more details, [13, 21]]. The exponential and logarithmic functions are very important in mathematical modeling of various real-life problems, for example, in mathematical modeling of growth and decline of populations, digital circuit optimization in the field of electrical engineering. Very recently, Jayswal *et al.* [15] introduced exponential type vector variational-like inequality problems with exponential invexities.

2010 *Mathematics Subject Classification.* 47H05, 47H09, 47J20, 47J05, 49J40.
Key words and phrases. Generalized exponential vector variational-like inequality; Euclidean space; α_g-relaxed exponentially (γ, η)-monotone mapping; Affine mapping; Fuzzy upper and lower semicontinuous; set valued mapping.

Submitted May 29, 2019. Published February 10, 2020.
Communicated by Nawab Hussain.
The first three authors are gratefully to Deanship of Scientific Research, Qassim University, Saudi Arabia for technical and financial of the research project 3626-qec-2018-1-14-s, 1439 AH/2018 AD..

In 1965, Zadeh [27] introduced the concepts of fuzzy sets. The fuzzy set theory has much application in various branches of engineering and mathematical sciences including artificial intelligence, control engineering, computer science, management science etc., see [28]. The concept of variational inequalities for fuzzy mappings was introduced by Chang [6] *et al.* in 1989 and study the existence theorems. Recently several kinds of variational inequalities and complementarity problems for fuzzy mappings were studied [see for instance [17, 20, 8]]. Recently, Chang [7] *et al.* introduced and studied a new class of generalized vector variational-like inequalities in fuzzy environment and generalized vector variational inequalities in fuzzy environment.

Motivated by the work of Antczak [1], Irfan *et al.* [14], Jayswal *et al.* [15], Chang *et al.* [7], Ho *et al.* [13] and by the ongoing research in this direction, we introduce a more general problem generalized exponential type vector variational-like inequality problem with fuzzy mapping (in short, GEVVLIPFM) in Euclidean spaces and define a new kind of α_g-relaxed exponentially (γ, η)-monotone mapping. We prove the existence results of GEVVLIPFM by KKM-technique and Nadler results. The results presented in this paper extend and generalize many previously known results in this research area.

2. Preliminaries

Let Z be a nonempty set. We recall that a fuzzy set B in Z is characterized by a function $\mu_B : Z \to [0,1]$, called membership function of B, " which associates with each point $u \in Z$ a real number in the interval [0, 1], with the value of μ_B at u representing the grade of membership of u in B". Clearly, any crisp subset B of Z is fuzzy set if $\mu_B(u) = 1$, when $u \in B$ and $\mu_B(u) = 0$ otherwise. Let X be a nonempty subset of a vector space V and D be a nonempty set. A mapping $F : D \to \mathfrak{F}(X)$, where $\mathfrak{F}(X)$ be the collection of all fuzzy sets of X, is called a fuzzy mapping, and $F(u),\ u \in D$ is a fuzzy set in $\mathfrak{F}(X)$, denoted by F_u and $F_u(v),\ v \in X$ is the grade of membership of v in F_u, see for details [27].

Let $B \in \mathfrak{F}(X)$ and $\alpha \in [0,1]$, then the set

$$B_\alpha = \{u \in X : B(u) \geq \alpha\}$$

is called an α-cut set of B.

In the sequel, we assume that E_1 and E_2 as Euclidean space of dimensions m and n, K and C be nonempty subsets of E_1 and E_2 respectively.

Let K be a nonempty subset of E_1. Then, K is said to be

(i) cone if $\lambda K \subset K,\ \forall\ \lambda \geq 0$;
(ii) convex cone if $K + K \subset K$;
(iii) pointed cone if K is cone and $K \bigcap \{-K\} = \{0\}$;
(iv) proper cone if $K \neq E_2$.

Let $K : C \to 2^{E_2}$ be a closed pointed convex cone valued mapping with $\text{int}K(u) \neq \emptyset$ with apex at origin, where $\text{int}K(u)$ be a set of interior points of $K(u)$. Then, $K(u)$ induces a partial ordering in E_2 as:

(i) $v \leq_{K(u)} w \Leftrightarrow w - v \in K(u)$;
(ii) $v \nleq_{K(u)} w \Leftrightarrow w - v \notin K(u)$;

(iii) $v \leq_{\text{int}K(u)} w \Leftrightarrow w - v \in \text{int}K(u)$;
(iv) $v \nleq_{\text{int}K(u)} w \Leftrightarrow w - v \notin \text{int}K(u)$.

Let (E_2, K) be an ordered space with the ordering of E_2 defined by a set $K(u)$ and ordering relation $'' \leq_{K(u)} ''$ is a partial order. Then
(i) $v \nleq_{K(u)} w \Leftrightarrow v + s \nleq z + s$, for any $u, v, w, s \in E_2$;
(ii) $v \nleq_{K(u)} w \Leftrightarrow \lambda v \nleq \lambda w$, for any $\lambda \geq 0$.

In this paper, we introduce and study the following generalized exponential type vector variational-like inequality problem with fuzzy mapping (in short, GEVVLIPFM). Let $C \subseteq E_1$ be a nonempty subset of an Euclidean space R^n and (E_2, K) be an ordered Euclidean space induces by a closed convex pointed cone K whose apex at origin. Let $K : C \to 2^{E_2}$ be a closed convex pointed cone valued mapping with $\text{int}K(x) \neq \emptyset$. Let γ be a nonzero real number, $\eta : C \times C \to E_1,\ g : C \to C,\ F : C \times C \to E_2$ and $N : \mathfrak{F}(L(E_1, E_2)) \times \mathfrak{F}(L(E_1, E_2)) \times \mathfrak{F}(L(E_1, E_2)) \to \mathfrak{F}(L(E_1, E_2))$ be the mappings, where $L(E_1, E_2)$ be the space of all continuous linear mappings from E_1 to E_2 and $A_1, A_2, A_3 : C \to \mathfrak{F}(L(E_1, E_2))$ be the fuzzy mappings and $a_1 : E_1 \to [0, 1],\ a_2 : E_1 \to [0, 1],\ a_3 : E_1 \to [0, 1]$ be functions. Then the GEVVLIPFM is to find $u_0 \in C$ and $\bar{x} \in \tilde{A}_1(u_0) = (A_1(u_0))_{a_1(u_0)},\ \bar{y} \in \tilde{A}_2(u_0) = (A_2(u_0))_{a_2(u_0)},\ \bar{z} \in \tilde{A}_3(u_0) = (A_3(u_0))_{a_3(u_0)}$ such that

$$\langle N(\bar{x}, \bar{y}, \bar{z}), \frac{1}{\gamma}(e^{\gamma\eta(v, g(u_0))} - 1)\rangle + F(g(u_0), v) \nleq_{\text{int}K(u_0)} 0,\ \forall v \in C, \tag{2.1}$$

where $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3 : C \to 2^{L(E_1, E_2)}$ be set valued mappings.
The following example is provided to illustrate problem (2.1)

Example 2.1. *Let $E_1 = E_2 = R$, $C = [0, +\infty)$, $K(u_0) = [0, \infty)$, $\forall u_0 \in C$. Define $A_1, A_2, A_3 : C \to 2^{\mathfrak{F}(L(E_1, E_2))} \equiv 2^R$ by*

$$\mu_{A_1(u_0)}(x) = \begin{cases} \frac{1}{1+(x-1)^2}, & \text{if } u_0 \in [0, 1], \\ \frac{1}{1+u_0(x-2)^2}, & \text{if } u_0 \in (1, +\infty), \end{cases},$$

$$\mu_{A_2(u_0)}(y) = \begin{cases} \frac{1}{1+(y-1)^2}, & \text{if } u_0 \in [0, 1], \\ \frac{1}{1+u_0(y-2)^2}, & \text{if } u_0 \in (1, +\infty), \end{cases},$$

$$\mu_{A_3(u_0)}(z) = \begin{cases} \frac{1}{1+(z-1)^2}, & \text{if } u_0 \in [0, 1], \\ \frac{1}{1+u_0(z-2)^2}, & \text{if } u_0 \in (1, +\infty), \end{cases},$$

and $a_1 : C \to [0, 1]$, $a_2 : C \to [0, 1]$, $a_3 : C \to [0, 1]$ as

$$a_1(u_0) = \begin{cases} \frac{1}{2}, & \text{if } u_0 \in [0, 1], \\ \frac{1}{1+u_0}, & \text{if } u_0 \in (1, +\infty), \end{cases},$$

$$a_2(u_0) = \begin{cases} \frac{1}{2}, & \text{if } u_0 \in [0, 1], \\ \frac{1}{2+u_0}, & \text{if } u_0 \in (1, +\infty), \end{cases},$$

$$a_3(u_0) = \begin{cases} \frac{1}{2}, & \text{if } u_0 \in [0, 1], \\ \frac{1}{3+u_0}, & \text{if } u_0 \in (1, +\infty), \end{cases}.$$

For $u_0 \in [0,1]$

$$\begin{aligned}
\tilde{A}_1(u_0) = (A_1(u_0))_{a_1(u_0)} &= \{x \in R : \mu_{A_1(u_0)}(x) \geq \frac{1}{2}\} \\
&= \{x \in R : \frac{1}{1+(x-1)^2} \geq \frac{1}{2}\} = [0,2] \\
\tilde{A}_2(u_0) = (A_2(u_0))_{a_2(u_0)} &= \{y \in R : \mu_{A_2(u_0)}(y) \geq \frac{1}{2}\} \\
&= \{y \in R : \frac{1}{1+(y-1)^2} \geq \frac{1}{2}\} = [0,2] \\
\tilde{A}_3(u_0) = (A_3(u_0))_{a_3(u_0)} &= \{z \in R : \mu_{A_3(u_0)}(z) \geq \frac{1}{2}\} \\
&= \{z \in R : \frac{1}{1+(z-1)^2} \geq \frac{1}{2}\} = [0,2],
\end{aligned}$$

and for any $u_0 \in (1, +\infty)$*, we have*

$$\begin{aligned}
\tilde{A}_1(u_0) = (A_1(u_0))_{a_1(u_0)} &= \{x \in R : \mu_{A_1(u_0)}(x) \geq \frac{1}{1+u_0}\} \\
&= \{x \in R : \frac{1}{1+u_0(x-2)^2} \geq \frac{1}{1+u_0}\} \\
&= \{x \in R : (x-2)^2 \leq 1\} = [1,3] \\
\tilde{A}_2(u_0) = (A_2(u_0))_{a_2(u_0)} &= \{y \in R : \mu_{A_2(u_0)}(y) \geq \frac{1}{2+u_0}\} \\
&= \{y \in R : \frac{1}{1+u_0(y-2)^2} \geq \frac{1}{2+u_0}\} \\
&= \{y \in R : (y-2)^2 \leq 1\} = [1,3] \\
\tilde{A}_3(u_0) = (A_3(u_0))_{a_3(u_0)} &= \{z \in R : \mu_{A_3(u_0)}(z) \geq \frac{1}{3+u_0}\} \\
&= \{z \in R : \frac{1}{3+u_0(z-2)^2} \geq \frac{1}{3+u_0}\} \\
&= \{z \in R : (z-2)^2 \leq 1\} = [1,3].
\end{aligned}$$

Define $N : \mathfrak{F}(L(E_1,E_2)) \times \mathfrak{F}(L(E_1,E_2)) \times \mathfrak{F}(L(E_1,E_2)) \to \mathfrak{F}(L(E_1,E_2))$ *by*

$$N(x,y,z) = \{2x + y + z\}, \ \forall x,y,z \in \mathfrak{F}(L(E_1,E_2)) \equiv R,$$

$\eta : C \times C \to E_1 = R$ *such that*

$$\eta(u,v) = \ln(\frac{u}{2} - v + 1), \ \forall u,v \in C,$$

$g : C \to C$ *such that*

$$g(u) = \frac{u}{2}, \ \forall u \in C,$$

and $F : C \times C \to E_2 = R$ *such that*

$$F(u,v) = \frac{v}{2} - u, \ \forall u,v \in C.$$

Consider $\gamma = 1$.
Consider the following two cases:

Case 1. *If* $u_0 \in [0,1]$, $x \in \tilde{A}_1(u_0)$, $y \in \tilde{A}_2(u_0)$ *and* $z \in \tilde{A}_3(u_0)$ *then*

$$\begin{aligned}\langle N(x,y,z), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))}-1)\rangle + F(g(u_0),v) &= \langle 2x+y+z, e^{\ln(\frac{v}{2}-\frac{u_0}{2})}-1\rangle \\ &\quad + \frac{v}{2}-\frac{u_0}{2} \\ &= (2x+y+z+1)(\frac{v}{2}-\frac{u_0}{2}).\end{aligned}$$

Thus,

$$\begin{aligned}(2x+y+z+1)(\frac{v}{2}-\frac{u_0}{2}) &\geq 0 \\ \Rightarrow u_0 &\leq v, \ \forall v \in C.\end{aligned}$$

This shows that $u_0 = 0$ *is a solution of the* GEVVLIPFM(2.1).

Case 2. *If* $u_0 \in [0,1]$, $x \in \tilde{A}_1(u_0)$, $y \in \tilde{A}_2(u_0)$ *and* $z \in \tilde{A}_3(u_0)$ *then*

$$\begin{aligned}\langle N(x,y,z), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))}-1)\rangle + F(g(u_0),v) &= \langle 2x+y+z, e^{\ln(\frac{v}{2}-\frac{u_0}{2})}-1\rangle \\ &\quad + \frac{v}{2}-\frac{u_0}{2} \\ &= (2x+y+z+1)(\frac{v}{2}-\frac{u_0}{2}).\end{aligned}$$

Thus,

$$\begin{aligned}(2x+y+z+1)(\frac{v}{2}-\frac{u_0}{2}) &\geq 0 \\ \Rightarrow u_0 &\leq v, \ \forall v \in C.\end{aligned}$$

This shows that there is no solution for GEVVLIPFM(2.1) *in this case. Thus, from the case 1, we find that* GEVVLIPFM(2.1) *has a solution and a solution set is* $\{0\}$.

Let $C \subseteq E_1$ be a nonempty closed convex subset of an Euclidean space $E_1 = R^m$ and (E_2, K) be an ordered space induces by the closed convex pointed cone $K(u)$ whose apex at origin with $\text{int}K(u) \neq \emptyset$.

Lemma 2.1. [5] *Let* (E_2, K) *be an ordered space induced by the pointed closed convex cone* K *with* $\text{int}K(u) \neq \emptyset$. *Then, for any* $u, v, w \in E_2$, *the following relation hold:*

(i) $w \nleq_{\text{int}K} x \geq_K v \Rightarrow w \nleq_{\text{int}K} v$;
(ii) $w \ngeq_{\text{int}K} x \leq_K v \Rightarrow w \ngeq_{\text{int}K} v$.

Definition 2.1. *A mapping* $F : E_1 \to E_2$ *is a* $K(u)-$ *convex on* E_1 *if*

$$F(\lambda u + (1-\lambda)v) \leq_{K(u)} \lambda F(u) + (1-\lambda)F(v), \ \forall u, v \in E_1, \ \lambda \in [0,1].$$

Definition 2.2. *A mapping* $F : C \to E_2$ *is said to be completely continuous if for any sequence* $\{u_n\} \in C$, $u_n \rightharpoonup u_0$ *weakly, then* $F(u_n) \to F(u_0)$.

Definition 2.3. *Let* E_1 *and* E_2 *be two topological vector spaces,* $A : E_1 \to 2^{E_2}$ *be a set valued mapping and* $A^{-1}(v) = \{u \in E_1 : v \in A(u)\}$. *Then,*

(i) *A is said to be upper semicontinuous if for each* $u \in E_1$ *and each open set* V *in* E_2 *with* $A(u) \subset V$, *then there exists an open neighborhood* U *of* u *in* E_1 *such that* $A(u_0) \subset V$, *for each* $u_0 \in U$.
(ii) *A is said to be closed if for any set* $\{u_\alpha\} \to u$ *in* E_1 *and any net* $\{v_\alpha\}$ *in* E_2 *such that* $v_\alpha \to v$ *and* $v_\alpha \in A(u_\alpha)$, *for any* α, *we have* $v \in A(u)$.

(iii) *A is said to have a closed graph if the graph of A,* $\text{Graph}(A) = \{(u,v) \in E_1 \times E_2, v \in A(u)\}$ *is closed in $E_1 \times E_2$.*

Definition 2.4. *Let $F : C \to 2^{E_1}$ be a set valued mapping. Then F is said to be a KKM-mapping if for any $\{v_1, v_2, ..., v_n\}$ of C, we have $\text{co}\{v_1, v_2, ..., v_n\} \subset \bigcup_{i=1}^{n} F(v_i)$, where $\text{co}\{v_1, v_2, ..., v_n\}$ denotes the convex hull of $v_1, v_2, ..., v_n$.*

Lemma 2.2. [10] *Let C be a nonempty subset of a Hausdorff topological vector space E_1 and let $F : C \to 2^{E_1}$ be a KKM-mapping. If $F(v)$ is a closed in E_1 for all $v \in C$ and compact for some $v \in C$, then $\bigcap_{v\in C} F(v) \neq \emptyset$.*

Lemma 2.3. [18] *Let E be a normed vector space and H be a Hausdorff metric on the collection $CB(E)$ of all closed and bounded subsets of E, induced by a metric d in terms of $d(u,v) = \|u - v\|$, which is defined by*

$$H(X,Y) = \max\{\sup_{u\in X} \inf_{v\in Y} \|u - v\|, \sup_{v\in Y} \inf_{u\in X} \|u - v\|\},$$

for $X, Y \in CB(E)$. If X and Y are compact subset in E, then for each $u \in X$, there exists $v \in Y$ such that $\|u - v\| \leq H(X,Y)$.

Definition 2.5. *Let C be a nonempty closed convex subset of E_1, $\eta : E_1 \times E_1 \to E_1$ be a mapping and $N : \mathfrak{F}(L(E_1,E_2)) \to \mathfrak{F}(L(E_1,E_2))$ be a single valued mapping, where $L(E_1,E_2)$ be the space of all continuous linear mapping from E_1 to E_2. Suppose that $A : C \to \mathfrak{F}(L(E_1,E_2))$ be a fuzzy mapping with $(A(u))_{a(u)} \neq$ for all $u \in C$, where $a : E_1 \to [0,1]$ and $\tilde{A} : C \to 2^{L(E_1,E_2)}$ be a nonempty compact set valued mapping defined by $\tilde{A}(u) = (A(u))_{a(u)}$, then*

(i) *N is said to be η-hemicontinuous, if*

$$\lim_{t\to 0^+} \langle N(u + t(v - u)), \eta(v,u)\rangle = \langle Nu, \eta(v,u)\rangle, \ \forall u, v \in C.$$

(ii) *A is said to be H-hemicontinuous, if for any $u, v \in C$, the mapping $t \to H(A(u+t(v-u)), Au)$ is continuous at 0^+, where H is a Hausdorff metric defined on* $\text{CB}(L(E_1,E_2))$.

Definition 2.6. *A mapping $f : R^m \to R^n$ is lipschitz continuous on $D \subset R^m$ iff there is an $L \in R$ such that*

$$\|f(u) - f(v)\| \leq L\|u - v\|, \ \forall u, v \in D. \tag{2.2}$$

Definition 2.7. *A mapping $F : E_1 \to E_1$ is said to be affine if for any $u_i \in C$ and $\lambda_i \geq 0$, $(1 \leq i \leq n)$ with $\sum_{i=1}^{n} \lambda_i = 1$, we have $F(\sum_{i=1}^{n} \lambda_i u_i) = \sum_{i=1}^{n} \lambda_i F(u_i)$.*

Definition 2.8. *Let E_1 be an Euclidean space. A mapping $F : E_1 \to R$ is a lower semicontinuous at $u_0 \in E_1$ if $F(u_0) \leq \liminf_n F(u_n)$, for any sequence $\{u_n\} \subset E_1$ such that $\{u_n\}$ converges to u_0.*

Definition 2.9. *Let E_1 be an Euclidean space. A mapping $F : E_1 \to R$ is a weakly upper semicontinuous at $u_0 \in E_1$ if $F(u_0) \geq \limsup_n F(u_n)$, for any sequence $\{u_n\} \subset E_1$ such that $\{u_n\}$ converges to u_0 weakly.*

Lemma 2.4. [3] *Let S be a nonempty compact convex subset of a finite dimensional space and $T : S \to S$ be a continuous mapping. Then there exists $x \in S$ such that $Tx = x$.*

Definition 2.10. *Let E_1 and E_2 be two topological spaces and $A : E_1 \to \mathfrak{F}(E_2)$ be a fuzzy mapping. A mapping A is said to have fuzzy set valued if $A_u(v)$ is upper semi continuous on $E_1 \times E_2$ as a ordinary real function.*

Lemma 2.5. [3] *Let C be a closed subset of a topological space E_1, then characteristic function χ_C of C is an upper semi continuous real valued function.*

Lemma 2.6. [3] *Let C be a nonempty closed convex subset of a real Hausdorff topological vector space E_1, K be a nonempty closed convex subset of a real Hausdorff topological vector space E_2, and $a : E_1 \to [0,1]$ be a lower semi continuous function. Let $A : C \to \mathfrak{F}(K)$ be a fuzzy mapping with $(A(u))_{a(u)} \neq$ for all $u \in E_1$ and $\tilde{A} : C \to 2^K$ be a multifunction defined by $\tilde{A}(u) = (A(u))_{a(u)}$. If A is a closed set valued mapping, then $\tilde{A}$ is a closed multifunction.*

Definition 2.11. *A fuzzy mapping $A : C \to \mathfrak{F}(L(E_1, E_2))$ is said to be α_g-relaxed exponentially (γ, η)-monotone if for every pair of points $u, v \in C$, we have*

$$\langle Au - Av, \frac{1}{\gamma}(e^{\gamma\eta(u,g(v))} - 1)\rangle \geq_{K(u)} \alpha_g(u - v), \tag{2.3}$$

where $\alpha_g : E_1 \to E_2$ with $\alpha_g(tu) = t^q \alpha_g(u)$ for all $t > 0$ and $u \in E_1$, where $q > 1$ is a real number.

Definition 2.12. *Let $N : \mathfrak{F}(L(E_1, E_2)) \times \mathfrak{F}(L(E_1, E_2)) \times \mathfrak{F}(L(E_1, E_2)) \to \mathfrak{F}(L(E_1, E_2))$ be a single valued mappings and $a : E_1 \to [0,1]$ be function. A fuzzy mapping $A : C \to \mathfrak{F}(L(E_1, E_2))$ with compact valued is said to be α_g-relaxed exponentially (γ, η)-monotone with respect to first argument of N and g if for each pair of points $u, v, y, z \in C$, we have*

$$\langle N(x_1, y, z) - N(x_2, y, z), \frac{1}{\gamma}(e^{\gamma\eta(u,g(v))} - 1)\rangle \geq_{K(u)} \alpha_g(u - v), \tag{2.4}$$

$\forall x_1 \in (A(u))_{a(u)},\ x_2 \in (A(v))_{a(v)}$, *where $\alpha_g : E_1 \to E_2$ with $\alpha_g(tu) = t^q \alpha_g(u)$ for all $t > 0$ and $u \in E_1$, where $q > 1$ is a real number.*

Remark 2.1. *Some special cases:*

(i) *If $N(x, y, z) = N(x, y)$ then by Definition 2.12, we have for each pair of points $u, v, y \in C$,*

$$\langle N(x_1, y) - N(x_2, y), \frac{1}{\gamma}(e^{\gamma\eta(u,g(v))} - 1)\rangle \geq_{K(u)} \alpha_g(u - v), \tag{2.5}$$

$\forall x_1 \in (A(u))_{a(u)},\ x_2 \in (A(v))_{a(v)}$, *where $\alpha_g : E_1 \to E_2$ with $\alpha_g(tu) = t^q \alpha_g(u)$ for all $t > 0$ and $u \in E_1$, where $q > 1$ is a real number.*

(ii) *If $N(x, y, z) = N(x)$ then by Definition 2.12, we have for each pair of points $u, v \in C$,*

$$\langle N(x_1) - N(x_2), \frac{1}{\gamma}(e^{\gamma\eta(u,g(v))} - 1)\rangle \geq_{K(u)} \alpha_g(u - v), \tag{2.6}$$

$\forall x_1 \in (A(u))_{a(u)},\ x_2 \in (A(v))_{a(v)}$, *where $\alpha_g : E_1 \to E_2$ with $\alpha_g(tu) = t^q \alpha_g(u)$ for all $t > 0$ and $u \in E_1$, where $q > 1$ is a real number.*

3. Main Result

Theorem 3.1. *Let C be a nonempty closed convex bounded subset of a real Euclidean space E_1 and (E_2, K) an ordered Euclidean space induces by a pointed closed convex cone K. Let $K : C \to 2^{E_2}$ be a closed convex pointed cone valued mapping with $\operatorname{int}K(u) \neq \emptyset$ and $E_2 \setminus (\operatorname{int}K(u))$ be an upper semicontinuous mapping. Let $g : C \to C$ be a closed convex continuous single valued mapping and $\eta : C \times C \to E_1$ be an affine in the first argument with $\eta(u, u) = 0$ for all $u \in C$. Let $F : C \times C \to E_2$*

be a $K(u)$-convex in the second argument with the condition $F(u,u) = 0$ for all $u \in C$. Let $N : \mathfrak{F}(L(E_1,E_2)) \times \mathfrak{F}(L(E_1,E_2)) \times \mathfrak{F}(L(E_1,E_2)) \to \mathfrak{F}(L(E_1,E_2))$ be a Lipschitz continuous mapping with all arguments, $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3 : C \to 2^{L(E_1,E_2)}$ be a nonempty upper semi continuous compact valued mappings induced by fuzzy mappings $A_1, A_2, A_3 : C \to \mathfrak{F}(L(E_1,E_2))$, that is $\tilde{A}_1(u) = (A_1(u))_{a_1(u)}$, $\tilde{A}_2(u) = (A_2(u))_{a_2(u)}$, $\tilde{A}_3(u) = (A_3(u))_{a_3(u)}$ with $a_1 : E_1 \to [0,1]$, $a_2 : E_1 \to [0,1]$, $a_3 : E_1 \to [0,1]$. If $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{A}_3$ are H-hemicontinuous and α_g-relaxed exponentially (γ,η)-monotone with respect to first argument of N and g. Then the following two statements (i) and (ii) are equivalent:

(i) *there exist $u_0 \in C$ and $\bar{x} \in \tilde{A}_1(u_0) = (A_1(u_0))_{a_1(u_0)}$, $\bar{y} \in \tilde{A}_2(u_0) = (A_2(u_0))_{a_2(u_0)}$, $\bar{z} \in \tilde{A}_3(u_0) = (A_3(u_0))_{a_3(u_0)}$ such that*

$$\langle N(\bar{x},\bar{y},\bar{z}), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))} - 1)\rangle + F(g(u_0),v) \nleq_{\mathrm{int}K(u_0)} 0,\ \forall v \in C,$$

(ii) *there exists $u_0 \in C$ such that*

$$\langle N(\bar{r},\bar{s},\bar{t}), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))} - 1)\rangle + F(g(u_0),v) \nleq_{\mathrm{int}K(u_0)} \alpha_g(v-u_0),\ \forall v \in C,$$

$$\bar{r} \in \tilde{A}_1(v) = (A_1(v))_{a_1(v)},\ \bar{s} \in \tilde{A}_2(v) = (A_2(v))_{a_2(v)},\ \bar{t} \in \tilde{A}_3(v) = (A_3(v))_{a_3(v)}.$$

Proof. Let the statement (i) is true that is there exist $u_0 \in C$ and $\bar{x} \in \tilde{A}_1(u_0) = (A_1(u_0))_{a_1(u_0)}$, $\bar{y} \in \tilde{A}_2(u_0) = (A_2(u_0))_{a_2(u_0)}$, $\bar{z} \in \tilde{A}_3(u_0) = (A_3(u_0))_{a_3(u_0)}$ such that

$$\langle N(\bar{x},\bar{y},\bar{z}), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))} - 1)\rangle + F(g(u_0),v) \nleq_{\mathrm{int}K(u_0)} 0,\ \forall v \in C. \tag{3.1}$$

Since N is α_g-relaxed exponentially (γ,η)-monotone therefore $\forall v \in C$, $\bar{r} \in \tilde{A}_1(v) = (A_1(v))_{a_1(v)}$, $\bar{s} \in \tilde{A}_2(v) = (A_2(v))_{a_2(v)}$, $\bar{t} \in \tilde{A}_3(v) = (A_3(v))_{a_3(v)}$ we have

$$\begin{aligned}
&\langle N(\bar{r},\bar{s},\bar{t}) - N(\bar{x},\bar{y},\bar{z}), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))} - 1)\rangle + F(g(u_0),v)\\
&\geq_{K(u_0)} \alpha_g(v-u_0) + F(g(u_0),v)\\
&\langle N(\bar{r},\bar{s},\bar{t}), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))} - 1)\rangle + F(g(u_0),v)\\
&\geq_{K(u)} \langle N(\bar{x},\bar{y},\bar{z}), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))} - 1)\rangle\\
&+\alpha_g(v-u_0) + F(g(u_0),v)\\
&\langle N(\bar{r},\bar{s},\bar{t}), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))} - 1)\rangle + F(g(u_0),v) - \alpha_g(v-u_0)\\
&\geq_{K(u)} \langle N(\bar{x},\bar{y},\bar{z}), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))} - 1)\rangle + F(g(u_0),v).
\end{aligned} \tag{3.2}$$

From (3.1), (3.2) and Lemma 2.1, we have

$$\langle N(\bar{r},\bar{s},\bar{t}), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))} - 1)\rangle + F(g(u_0),v) \nleq_{\mathrm{int}K(u_0)} \alpha_g(v-u_0),\ \forall v \in C,$$

$$\bar{r} \in \tilde{A}_1(v) = (A_1(v))_{a_1(v)},\ \bar{s} \in \tilde{A}_2(v) = (A_2(v))_{a_2(v)},\ \bar{t} \in \tilde{A}_3(v) = (A_3(v))_{a_3(v)}.$$

Conversely, consider the statements (ii) is correct that is there exists $u_0 \in C$ such that

$$\langle N(\bar{r},\bar{s},\bar{t}), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))}-1)\rangle + F(g(u_0),v) \nleq_{\text{int}K(u_0)} \alpha_g(v-u_0), \tag{3.3}$$

$\forall v \in C,\ \bar{r} \in \tilde{A}_1(v) = (A_1(v))_{a_1(v)},\ \bar{s} \in \tilde{A}_2(v) = (A_2(v))_{a_2(v)},\ \bar{t} \in \tilde{A}_3(v) = (A_3(v))_{a_3(v)}$.

Let $v \in C$ be an arbitrary element. Consider $v_\lambda = \lambda v + (1-\lambda)u_0,\ \lambda \in (0,1]$. As C is convex, $v_\lambda \in C$. Let $\bar{r_\lambda} \in \tilde{A}_1(v_\lambda) = (A_1(v_\lambda))_{a_1(v_\lambda)},\ \bar{s_\lambda} \in \tilde{A}_2(v_\lambda) = (A_2(v_\lambda))_{a_2(v_\lambda)},\ \bar{t_\lambda} \in \tilde{A}_3(v_\lambda) = (A_3(v_\lambda))_{a_3(v_\lambda)}$, we get from (3.3)

$$\langle N(\bar{r_\lambda},\bar{s_\lambda},\bar{t_\lambda}), \frac{1}{\gamma}(e^{\gamma\eta(v_\lambda,g(u_0))}-1)\rangle + F(g(u_0),v_\lambda) \nleq_{\text{int}K(u_0)} \alpha_g(v_\lambda-u_0) = t^q\alpha_g(v-u_0). \tag{3.4}$$

Now,

$$\begin{aligned}
\langle N(\bar{r_\lambda},\bar{s_\lambda},\bar{t_\lambda}), \frac{1}{\gamma}(e^{\gamma\eta(v_\lambda,g(u_0))}-1)\rangle &+ F(g(u_0),v_\lambda)\\
&= \langle N(\bar{r_\lambda},\bar{s_\lambda},\bar{t_\lambda}),\\
&\quad \frac{1}{\gamma}(e^{\gamma\eta(\lambda v+(1-\lambda)u_0,g(u_0))}-1)\rangle\\
&\quad +F(g(u_0),\lambda v+(1-\lambda)u_0)\\
&= \langle N(\bar{r_\lambda},\bar{s_\lambda},\bar{t_\lambda}),\\
&\quad \frac{1}{\gamma}(e^{\gamma\eta\lambda(v,g(u_0))+(1-\lambda)\gamma\eta(u_0,g(u_0))}-1)\rangle\\
&\quad +\lambda F(g(u_0),v)+(1-\lambda)F(g(u_0),u_0)\\
&\leq_{K_{(u_0)}} \langle N(\bar{r_\lambda},\bar{s_\lambda},\bar{t_\lambda}), \frac{1}{\gamma}(\lambda(e^{\gamma\eta(v,g(u_0))}-1)\\
&\quad +(1-\lambda)(e^{\gamma\eta(v,g(u_0))}-1)\rangle + \lambda F(g(u_0),v)\\
&= \lambda\{\langle N(\bar{r_\lambda},\bar{s_\lambda},\bar{t_\lambda}), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))}-1)\rangle\\
&\quad +F(g(u_0),v)\}.
\end{aligned} \tag{3.5}$$

From (3.4), (3.5) and Lemma 2.1, we have

$$\langle N(\bar{r_\lambda},\bar{s_\lambda},\bar{t_\lambda}), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))}-1)\rangle + F(g(u_0),v) \nleq_{\text{int}K(u_0)} t^{q-1}\alpha_g(v-u_0). \tag{3.6}$$

Since $\tilde{A}_1(v_\lambda) = (A_1(v_\lambda))_{a_1(v_\lambda)},\ \tilde{A}_2(v_\lambda) = (A_2(v_\lambda))_{a_2(v_\lambda)},\ \tilde{A}_3(v_\lambda) = (A_3(v_\lambda))_{a_3(v_\lambda)}$ are compact, therefore by Lemma 2.3, for each fixed $\bar{r_\lambda} \in \tilde{A}_1(v_\lambda) = (A_1(v_\lambda))_{a_1(v_\lambda)}$, $\bar{s_\lambda} \in \tilde{A}_2(v_\lambda) = (A_2(v_\lambda))_{a_2(v_\lambda)},\ \bar{t_\lambda} \in \tilde{A}_3(v_\lambda) = (A_3(v_\lambda))_{a_3(v_\lambda)}$ there exists

$\bar{r'_\lambda} \in \tilde{A}_1(v'_\lambda) = (A_1(v'_\lambda))_{a_1(v'_\lambda)},\ \bar{s'_\lambda} \in \tilde{A}_2(v'_\lambda) = (A_2(v'_\lambda))_{a_2(v'_\lambda)},\ \bar{t'_\lambda} \in \tilde{A}_3(v'_\lambda) = (A_3(v'_\lambda))_{a_3(v'_\lambda)}$ such that

$$\begin{aligned}
\|r_\lambda - r'_\lambda\| &\leq H(\tilde{A}_1(v_\lambda),\tilde{A}_1(u_0)),\\
\|s_\lambda - s'_\lambda\| &\leq H(\tilde{A}_2(v_\lambda),\tilde{A}_2(u_0)),\\
\|t_\lambda - t'_\lambda\| &\leq H(\tilde{A}_3(v_\lambda),\tilde{A}_3(u_0))
\end{aligned} \tag{3.7}$$

Since $\tilde{A}_1(u_0)$, $\tilde{A}_2(u_0)$ and $\tilde{A}_3(u_0)$ are compact, therefore without loss of generality, we may assume that

$$\begin{aligned} r_\lambda &\to r_0 \in A_1 u_0 \text{ as } \lambda \to 0^+ \\ s_\lambda &\to s_0 \in A_2 u_0 \text{ as } \lambda \to 0^+ \\ t_\lambda &\to t_0 \in A_3 u_0 \text{ as } \lambda \to 0^+. \end{aligned}$$

Also, $\tilde{A}_1,\ \tilde{A}_2$ and $\tilde{A}_3$ are H-hemicontinuous, thus it follows that

$$\begin{aligned} &H(\tilde{A}_1(v_\lambda), \tilde{A}_1(u_0)) \to 0 \qquad \text{as } \lambda \to 0^+ \\ &H(\tilde{A}_2(v_\lambda), \tilde{A}_2(u_0)) \to 0 \qquad \text{as } \lambda \to 0^+ \\ &H(\tilde{A}_3(v_\lambda), \tilde{A}_3(u_0)) \to 0 \qquad \text{as } \lambda \to 0^+. \end{aligned}$$

By (3.7), we get

$$\begin{aligned} \|r_\lambda - r_0\| &\le \|r_\lambda - r_\lambda^{'}\| + \|r_\lambda^{'} - r_0\| \\ &\le H(\tilde{A}_1(v_\lambda), \tilde{A}_1(r_0)) + \|r_\lambda^{'} - r_0\| \to 0 \text{ as } \lambda \to 0^+, \end{aligned}$$

$$\begin{aligned} \|s_\lambda - v_0\| &\le \|s_\lambda - s_\lambda^{'}\| + \|s_\lambda^{'} - v_0\| \\ &\le H(\tilde{A}_2(v_\lambda), \tilde{A}_2(v_0)) + \|s_\lambda^{'} - v_0\| \to 0 \text{ as } \lambda \to 0^+, \end{aligned}$$

and

$$\begin{aligned} \|t_\lambda - t_0\| &\le \|t_\lambda - t_\lambda^{'}\| + \|t_\lambda^{'} - t_0\| \\ &\le H(\tilde{A}_3(v_\lambda), \tilde{A}_3(t_0)) + \|t_\lambda^{'} - t_0\| \to 0 \text{ as } \lambda \to 0^+. \end{aligned} \tag{3.8}$$

Since N is Lipschitz continuous with all arguments therefore we get

$$\begin{aligned} &\|\langle N(r_\lambda, s_\lambda, t_\lambda), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))} - 1)\rangle - t^{q-1}\alpha_g(v - u_0) \\ &-\langle N(r_0, s_0, t_0), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))} - 1)\rangle\| \\ &\le \|\langle N(r_\lambda, s_\lambda, t_\lambda) - N(r_0, s_0, t_0), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))} - 1)\rangle\| + \|t^{q-1}\alpha_g(v - u_0)\| \\ &\le \frac{1}{\gamma}\{\|N(r_\lambda, s_\lambda, t_\lambda) - N(r_0, s_\lambda, t_\lambda)\| + \|N(r_0, s_\lambda, t_\lambda) - N(r_0, s_0, t_\lambda)\| \\ &+\|N(r_0, s_0, t_\lambda) - N(r_0, s_0, t_0)\|\}\|e^{\gamma\eta(v,g(u_0))} - 1\| \\ &+t^{q-1}\|\alpha_g(v - u_0)\| \to 0 \text{ as } \lambda \to 0^+. \end{aligned} \tag{3.9}$$

By (3.4), we get

$$\begin{aligned} \langle N(r_\lambda, s_\lambda, t_\lambda), \frac{1}{\gamma}(e^{\gamma\eta(v_\lambda,g(u_0))} - 1)\rangle &+ F(g(u_0), v_\lambda) \\ &- t^{q-1}\alpha_g(v - u_0) \in E_2 \setminus (\text{int}K(u_0)). \end{aligned}$$

Since $E_2 \setminus (\text{int}K(u_0))$ is closed therefore from (3.9), we have

$$\begin{aligned} &\langle N(r_0, s_0, t_0), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))} - 1)\rangle + F(g(u_0), v) \qquad \in E_2 \setminus (\text{int}K(u_0)) \\ &\langle N(r_0, s_0, t_0), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))} - 1)\rangle + F(g(u_0), v) \qquad \nleq_{\text{int}K(u_0)} 0,\ \forall\, v \in K. \end{aligned}$$

□

Theorem 3.2. *Let C be a nonempty closed convex bounded subset of a real Euclidean space E_1 and (E_2, K) an ordered Euclidean space induces by a pointed closed convex cone K. Let $K : C \to 2^{E_2}$ be a closed convex pointed cone valued mapping with $\text{int}K(u) \neq \emptyset$ and $E_2 \setminus (\text{int}K(u))$ be an upper semicontinuous mapping. Let $g : C \to C$ be a closed convex and continuous single valued mapping and $\eta : C \times C \to E_1$ be an affine in the first argument with $\eta(u, u) = 0$ for all $u \in C$ and continuous in both variable. Let $F : C \times C \to E_2$ be a completely continuous in the first argument and affine in the second argument with the condition $F(g(u), u) = 0$ for all $u \in C$. Let $\alpha_g : E_1 \to E_2$ be a weakly lower semicontinuous with respect to g. Let $N : \mathfrak{F}(L(E_1, E_2)) \times \mathfrak{F}(L(E_1, E_2)) \times \mathfrak{F}(L(E_1, E_2)) \to \mathfrak{F}(L(E_1, E_2))$ be a Lipschitz continuous mapping with all arguments and $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3 : C \to 2^{L(E_1,E_2)}$ be a nonempty upper semi continuous compact valued mappings induced by fuzzy mappings $A_1, A_2, A_3 : C \to \mathfrak{F}(L(E_1, E_2))$, that is $\tilde{A}_1(u) = (A_1(u))_{a_1(u)}$, $\tilde{A}_2(u) = (A_2(u))_{a_2(u)}$, $\tilde{A}_3(u) = (A_3(u))_{a_3(u)}$ with $a_1 : E_1 \to [0, 1]$, $a_2 : E_1 \to [0, 1]$, $a_3 : E_1 \to [0, 1]$. If $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{A}_3$ are H-hemicontinuous and α_g-relaxed exponentially (γ, η)-monotone with respect to first argument of N and g. Then (2.1) is a solvable, that is there exist $u \in C$ and $x \in \tilde{A}_1(u) = (A_1(u))_{a_1(u)}$, $y \in \tilde{A}_2(u) = (A_2(u))_{a_2(u)}$, $z \in \tilde{A}_3(u) = (A_3(u))_{a_3(u)}$ such that*

$$\langle N(x, y, z), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u))} - 1)\rangle + F(g(u), v) \nleq_{\text{int}K(u)} 0, \ \forall v \in C.$$

Proof. Consider the set valued mapping $S : C \to 2^{E_1}$ such that $\forall v \in C$

$$\begin{aligned} S(v) = \{u \in C : &\langle N(x, y, z), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u))} - 1)\rangle + F(g(u), v) \nleq_{\text{int}K(u)} 0, \\ &x \in \tilde{A}_1(u) = (A_1(u))_{a_1(u)}, \ y \in \tilde{A}_2(u) = (A_2(u))_{a_2(u)}, \ z \in \tilde{A}_3(u) = (A_3(u))_{a_3(u)}\}. \end{aligned}$$

First, we claim that S is a KKM-mapping. If S is not a KKM-mapping then there exists $\{u_1, u_2, u_3, ..., u_m\} \subset C$ such that

$$co\{u_1, u_2, u_3, ..., u_m\} \nsubseteq \bigcup_{i=1}^{m} S(u_i),$$

that means there exists at least $u \in co\{u_1, u_2, u_3, ..., u_m\}$, $u = \sum_{i=1}^{m} \lambda_i u_i$, where $\lambda_i \geq 0$, $i = 1, 2, 3, ..., m$, $\sum_{i=1}^{m} \lambda_i = 1$, but $u \notin \bigcup_{i=1}^{m} S(u_i)$. From the construction of S, for any $x \in \tilde{A}_1(u) = (A_1(u))_{a_1(u)}$, $y \in \tilde{A}_2(u) = (A_2(u))_{a_2(u)}$, $z \in \tilde{A}_3(u) = (A_3(u))_{a_3(u)}$), we have

$$\langle N(x, y, z), \frac{1}{\gamma}(e^{\gamma\eta(u_i,g(u))} - 1)\rangle + F(g(u), u_i) \nleq_{\text{int}K(u)} 0, \text{ for } i = 1, 2, 3, ..., m. \quad (3.10)$$

From (3.10) and since η and F are affine in first and second argument, it follows that

$$\begin{aligned} 0 &= \langle N(x,y,z), \frac{1}{\gamma}(e^{\gamma\eta(u,g(u))}-1)\rangle + F(g(u),u) \\ &= \langle N(x,y,z), \frac{1}{\gamma}(e^{\gamma\eta(\sum_{i=1}^m \lambda_i u_i, g(u))}-1)\rangle + F(g(u),\sum_{i=1}^m \lambda_i u_i) \\ &= \langle N(x,y,z), \frac{1}{\gamma}(e^{\sum_{i=1}^m \lambda_i \gamma\eta(u_i, g(u))}-1)\rangle + \sum_{i=1}^m \lambda_i F(g(u),u_i) \\ &\leq_{K(u)} \langle N(x,y,z), \frac{1}{\gamma}(e^{\sum_{i=1}^m \lambda_i \gamma\eta(u_i, g(u))}-1)\rangle + \sum_{i=1}^m \lambda_i F(g(u),u_i) \\ &= \sum_{i=1}^m \lambda_i \{\langle N(x,y,z), \frac{1}{\gamma}(e^{\gamma\eta(u_i,g(u))}-1)\rangle + F(g(u),u_i)\} \leq_{\text{int}K(u)} 0, \end{aligned}$$

this shows that $0 \in \text{int}K(u)$, which contradicts the fact that $K(u)$ is proper. Hence S is a KKM-mapping.

Define another set valued mapping $W : C \to 2^{E_1}$ such that $\forall v \in C$

$$\begin{aligned} W(v) &= \{u \in C : \langle N(p,q,r), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u))}-1)\rangle + F(g(u),v) \\ &\not\leq_{\text{int}K(u)} \alpha_g(v-u), \end{aligned}$$

$\forall p \in \tilde{A}_1(v) = (A_1(v))_{a_1(v)},\ q \in \tilde{A}_2(v) = (A_2(v))_{a_2(v)},\ r \in \tilde{A}_3(v) = (A_3(v))_{a_3(v)})\}$.

Now, we will prove that $S(v) \subset W(v),\ \forall v \in C$.

Let $u \in S(v)$, there exists some $x \in \tilde{A}_1(u) = (A_1(u))_{a_1(u)},\ y \in \tilde{A}_2(u) = (A_2(u))_{a_2(u)},\ z \in \tilde{A}_3(u) = (A_3(u))_{a_3(u)})$, such that

$$\langle N(x,y,z), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u))}-1)\rangle + F(g(u),v) \not\leq_{\text{int}K(u)} 0. \tag{3.11}$$

Since N is a α_g-relaxed exponentially (γ,η)-monotone therefore $\forall v \in C,\ p \in \tilde{A}_1(v) = (A_1(v))_{a_1(v)},\ q \in \tilde{A}_2(v) = (A_2(v))_{a_2(v)},\ r \in \tilde{A}_3(v) = (A_3(v))_{a_3(v)})$ we have

$$\begin{aligned} \langle N(x,y,z), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u))}-1)\rangle + F(g(u),v) &\leq_{\text{int}K(u)} \langle N(p,q,r), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u))}-1)\rangle \\ &+ F(g(u),v) - \alpha_g(v-u). \end{aligned} \tag{3.12}$$

Using (3.11), (3.12) and Lemma 2.1, we have

$$\langle N(p,q,r), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u))}-1)\rangle + F(g(u),v) \not\leq_{\text{int}K(u)} \alpha_g(v-u),$$

$\forall v \in C,\ p \in \tilde{A}_1(v) = (A_1(v))_{a_1(v)},\ q \in \tilde{A}_2(v) = (A_2(v))_{a_2(v)},\ r \in \tilde{A}_3(v) = (A_3(v))_{a_3(v)})$.

Therefore $u \in W(v)$ that is $S(v) \subset W(v),\ \forall v \in C$. This implies that W is also a KKM-mapping.

We claim that for each $v \in C$, $W(v) \subset C$ is closed in the weak topology of E_1. Let us suppose that $\overline{u} \in \overline{W(v)}^w$, the weak closure of $W(v)$. Since E_1 is reflexive, there is a sequence $\{u_n\}$ in $W(v)$ such that $\{u_n\}$ converges weakly to $\overline{u} \in C$. Then, for each

$p \in \tilde{A}_1(v) = (A_1(v))_{a_1(v)},\ q \in \tilde{A}_2(v) = (A_2(v))_{a_2(v)},\ r \in \tilde{A}_3(v) = (A_3(v))_{a_3(v)})$, we have

$$\langle N(p,q,r), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_n))} - 1)\rangle + F(g(u_n), v) \nleq_{\mathrm{int}K(u_n)} \alpha_g(v - u_n)$$

$$\langle N(p,q,r), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_n))} - 1)\rangle + F(g(u_n), v) - \alpha_g(v - u_n) \in E_2 \setminus (-\mathrm{int}K(u_n)).$$

Since N and F are completely continuous and $E_2 \setminus (-\mathrm{int}K(u_n))$ is closed, α_g is weakly lower semicontinuous and b is continuous therefore the sequence

$$\{\langle N(p,q,r), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_n))} - 1)\rangle + F(g(u_n), v) - \alpha_g(v - u_n)\}$$

converges to

$$\langle N(p,q,r), \frac{1}{\gamma}(e^{\gamma\eta(v,g(\overline{u}))} - 1)\rangle + F(g(\overline{u}), v) - \alpha_g(v - \overline{u})$$

and

$$\langle N(p,q,r), \frac{1}{\gamma}(e^{\gamma\eta(v,g(\overline{u}))} - 1)\rangle + F(g(\overline{u}), v) - \alpha_g(v - \overline{u}) \in E_2 \setminus (-\mathrm{int}K(\overline{u})).$$

Therefore

$$\langle N(p,q,r), \frac{1}{\gamma}(e^{\gamma\eta(v,g(\overline{u}))} - 1)\rangle + F(g(\overline{u}), v) \nleq_{\mathrm{int}K(u_n)} \alpha_g(v - \overline{u}).$$

Thus, $\overline{u} \in W(v)$. This shows that $W(v)$ is weakly closed $\forall v \in C$.
Furthermore, E_1 is reflexive and $C \subset E_1$ is a nonempty closed convex and bounded. Therefore, C is weakly compact subset of E_1 and so $W(v)$ is also weakly compact. Therefore from Lemma 2.2 and Theorem 3.1, it follows that

$$\bigcap_{v \in C} W(v) \neq \emptyset.$$

Thus, there exists $\overline{u} \in C$ such that

$$\langle N(p,q,r), \frac{1}{\gamma}(e^{\gamma\eta(v,g(\overline{u}))} - 1)\rangle + F(g(\overline{u}), v) \nleq_{\mathrm{int}K(u_n)} \alpha_g(v - \overline{u}),$$

$\forall v \in C,\ p \in \tilde{A}_1(v) = (A_1(v))_{a_1(v)},\ q \in \tilde{A}_2(v) = (A_2(v))_{a_2(v)},\ r \in \tilde{A}_3(v) = (A_3(v))_{a_3(v)})$.

Hence from Theorem 3.1, we can conclude that there exist $\overline{u} \in C$ and $\overline{x} \in \tilde{A}_1(\overline{u}) = (A_1(\overline{u}))_{a_1(\overline{u})},\ \overline{y} \in \tilde{A}_2(\overline{u}) = (A_2(\overline{u}))_{a_2(\overline{u})},\ \overline{z} \in \tilde{A}_3(\overline{u}) = (A_3(\overline{u}))_{a_3(\overline{u})})$ such that

$$\langle N(\overline{x},\overline{y},\overline{z}), \frac{1}{\gamma}(e^{\gamma\eta(v,g(\overline{u}))} - 1)\rangle + F(g(\overline{u}), v) \nleq_{\mathrm{int}K(\overline{u})} 0,\ \forall v \in C,$$

that is (2.1) is solvable. □

Theorem 3.3. *Let C be a nonempty closed convex bounded subset of a real Euclidean space E_1 with $0 \in C$ and (E_2, K) an ordered Euclidean space induces by a pointed closed convex cone $K(u)$. Let $K : C \to 2^{E_2}$ be a closed convex pointed cone valued mapping with $\mathrm{int}K(u) \neq \emptyset$ and $E_2 \setminus (\mathrm{int}K(u))$ be an upper semicontinuous mapping. Let $g : C \to C$ be a closed convex and continuous single valued mapping and $\eta : C \times C \to E_1$ be an affine in the first argument with $\eta(u,u) = 0$ for all $u \in C$. Let $F : C \times C \to E_2$ be a completely continuous in the first argument and affine in the second argument with the condition*

$F(u,u)=0$ for all $u\in C$. Let $\alpha_g : E_1 \to E_2$ be a weakly lower semicontinuous. Let $N : \mathfrak{F}(L(E_1,E_2)) \times \mathfrak{F}(L(E_1,E_2)) \times \mathfrak{F}(L(E_1,E_2)) \to \mathfrak{F}(L^c(E_1,E_2))$ be a Lipschitz continuous mapping with all arguments, where $L^c(E_1,E_2)$ be a space of all completely continuous linear mapping from E_1 to E_2 and $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3 : C \to 2^{L(E_1,E_2)}$ be a nonempty upper semi continuous compact valued mappings induced by fuzzy mappings $A_1, A_2, A_3 : C \to \mathfrak{F}(L(E_1,E_2))$, that is $\tilde{A}_1(u) = (A_1(u))_{a_1(u)}$, $\tilde{A}_2(u) = (A_2(u))_{a_2(u)}$, $\tilde{A}_3(u) = (A_3(u))_{a_3(u)}$ with $a_1 : E_1 \to [0,1]$, $a_2 : E_1 \to [0,1]$, $a_3 : E_1 \to [0,1]$. If $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{A}_3$ are H-hemicontinuous and α_g-relaxed exponentially (γ,η)-monotone with respect to first argument of N and g. If there exists one $r>0$ such that

$$\langle N(p,q,s), \frac{1}{\gamma}(e^{\gamma\eta(g(0),v)}-1)\rangle + F(v,g(0)) \nleq_{\mathrm{int}K(0)} 0, \tag{3.13}$$

$\forall v \in C,\ p \in \tilde{A}_1(v) = (A_1(v))_{a_1(v)},\ q \in \tilde{A}_2(v) = (A_2(v))_{a_2(v)},\ s \in \tilde{A}_3(v) = (A_3(v))_{a_3(v)}$ with $\|v\| = r$.

Then (2.1) is solvable that is there exists $u\in C$ and $x \in \tilde{A}_1(u) = (A_1(u))_{a_1(u)}$, $y \in \tilde{A}_2(u) = (A_2(u))_{a_2(u)}$, $z \in \tilde{A}_3(u) = (A_3(u))_{a_3(u)}$ such that

$$\langle N(x,y,z), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u))}-1)\rangle + F(g(u),v) \nleq_{\mathrm{int}K(u)} 0,\ \forall v\in C.$$

Proof. For $r>0$, assume that $C_r = \{u \in E_1 : \|u\| \le r\}$. From Theorem 3.2, we know that (2.1) is solvable over C_r that is there exists $u_r \in C\bigcap C_r$ and $x_r \in \tilde{A}_1(u_r) = (A_1(u_r))_{a_1(u_r)}$, $y_r \in \tilde{A}_2(u_r) = (A_2(u_r))_{a_2(u_r)}$, $z_r \in \tilde{A}_3(u_r) = (A_3(u_r))_{a_3(u_r)}$ such that

$$\langle N(x_r,y_r,z_r), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_r))}-1)\rangle + F(g(u_r),v) \nleq_{\mathrm{int}K(u_r)} 0,\ \forall v\in C\bigcap C_r. \tag{3.14}$$

Putting $v=0$ in (3.14), we get

$$\langle N(x_r,y_r,z_r), \frac{1}{\gamma}(e^{\gamma\eta(0,g(u_r))}-1)\rangle + F(g(u_r),0) \nleq_{\mathrm{int}K(u_r)} 0. \tag{3.15}$$

If $\|u_r\| = r$ for all r, then it contradicts to (3.13). Hence $\|u_r\| < r$. For any $w\in C$, let us choose $\lambda \in (0,1)$ small enough such that $(1-\lambda)u_r + \lambda w \in C\bigcap C_r$. Putting $v = (1-\lambda)u_r + \lambda w$ in (3.14), we get

$$\langle N(x_r,y_r,z_r), \frac{1}{\gamma}(e^{\gamma\eta((1-\lambda)u_r+\lambda w,g(u_r))}-1)\rangle + F(g(u_r),(1-\lambda)u_r+\lambda w) \nleq_{\mathrm{int}K(u_r)} 0. \tag{3.16}$$

Since η and F are affine in the first and second variable, we have

$$\begin{aligned}
&\langle N(x_r,y_r,z_r), \frac{1}{\gamma}(e^{\gamma\eta((1-\lambda)u_r+\lambda w,g(u_r))}-1)\rangle + F(g(u_r),(1-\lambda)u_r+\lambda w)\\
&= \langle N(x_r,y_r,z_r), \frac{1}{\gamma}(e^{(1-\lambda)\gamma\eta(u_r,g(u_r))+\lambda\gamma\eta(w,g(u_r))}-1)\rangle + \lambda F(g(u_r),w)\\
&\le_K (u_r\langle N(x_r,y_r,z_r), \frac{1}{\gamma}(1-\lambda)(e^{\gamma\eta(u_r,g(u_r))-1} + \frac{1}{\gamma}\lambda e^{\gamma\eta(w,g(u_r))}-1)\rangle\\
&+\lambda F(g(u_r),w)\\
&= \lambda\{\langle N(x_r,y_r,z_r), \frac{1}{\gamma}e^{\gamma\eta(w,g(u_r))}-1)\rangle + F(g(u_r),w)\}.
\end{aligned} \tag{3.17}$$

Hence from (3.16), (3.17) and Lemma 2.1, we get

$$\langle N(x_r, y_r, z_r), \frac{1}{\gamma} e^{\gamma\eta(w,g(u_r))} - 1)\rangle + F(g(u_r), w) \nleq_{\text{int}K(u_r)} 0, \ \forall w \in C. \tag{3.18}$$

Thus, (2.1) is solvable. □

If $N(x, y, z) = N(x, y)$ and $A_3 \equiv 0$, a zero mapping, then Theorem 3.1 reduces to the following corollary:

Corollary 3.1. *Let C be a nonempty closed convex bounded subset of a real Euclidean space E_1 and (E_2, K) an ordered Euclidean space induces by a pointed closed convex cone K. Let $K : C \to 2^{E_2}$ be a closed convex pointed cone valued mapping with $\text{int}K(u) \neq \emptyset$ and $E_2 \setminus (\text{int}K(u))$ be an upper semicontinuous mapping. Let $g : C \to C$ be a closed convex continuous single valued mapping and $\eta : C \times C \to E_1$ be an affine in the first argument with $\eta(u, u) = 0$ for all $u \in C$. Let $F : C \times C \to E_2$ be a $K(u)$-convex in the second argument with the condition $F(u, u) = 0$ for all $u \in C$. Let $N : \mathfrak{F}(L(E_1, E_2)) \times \mathfrak{F}(L(E_1, E_2)) \to \mathfrak{F}(L(E_1, E_2))$ be a Lipschitz continuous mapping with all arguments, $\tilde{A}_1, \tilde{A}_2 : C \to 2^{L(E_1,E_2)}$ be a nonempty upper semi continuous compact valued mappings induced by fuzzy mappings $A_1, A_2 : C \to \mathfrak{F}(L(E_1, E_2))$, that is $\tilde{A}_1(u) = (A_1(u))_{a_1(u)}$, $\tilde{A}_2(u) = (A_2(u))_{a_2(u)}$ with $a_1 : E_1 \to [0, 1]$, $a_2 : E_1 \to [0, 1]$. If $\tilde{A}_1$, $\tilde{A}_2$ are H-hemicontinuous and α_g-relaxed exponentially (γ, η)-monotone with respect to first argument of N and g. Then the following two statements (i) and (ii) are equivalent:*

(i) *there exist $u_0 \in C$ and $\bar{x} \in \tilde{A}_1(u_0) = (A_1(u_0))_{a_1(u_0)}$, $\bar{y} \in \tilde{A}_2(u_0) = (A_2(u_0))_{a_2(u_0)}$ such that*

$$\langle N(\bar{x}, \bar{y}), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))} - 1)\rangle + F(g(u_0), v) \nleq_{\text{int}K(u_0)} 0, \ \forall v \in C,$$

(ii) *there exists $u_0 \in C$ such that*

$$\langle N(\bar{r}, \bar{s}), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))} - 1)\rangle + F(g(u_0), v) \nleq_{\text{int}K(u_0)} \alpha_g(v - u_0),$$
$$\forall v \in C, \ \bar{r} \in \tilde{A}_1(v) = (A_1(v))_{a_1(v)}, \ \bar{s} \in \tilde{A}_2(v) = (A_2(v))_{a_2(v)}.$$

If $N(x, y, z) = N(x)$ and A_2, $A_3 \equiv 0$, a zero mapping, and $g \equiv I$, an identity mapping then Theorem 3.1 reduces to the following corollary

Corollary 3.2. *Let C be a nonempty closed convex bounded subset of a real Euclidean space E_1 and (E_2, K) an ordered Euclidean space induces by a pointed closed convex cone K. Let $K : C \to 2^{E_2}$ be a closed convex pointed cone valued mapping with $\text{int}K(u) \neq \emptyset$ and $E_2 \setminus (\text{int}K(u))$ be an upper semicontinuous mapping. Let $\eta : C \times C \to E_1$ be an affine in the first argument with $\eta(u, u) = 0$ for all $u \in C$. Let $F : C \times C \to E_2$ be a $K(u)$-convex in the second argument with the condition $F(u, u) = 0$ for all $u \in C$. Let $N : \mathfrak{F}(L(E_1, E_2)) \to \mathfrak{F}(L(E_1, E_2))$ be a Lipschitz continuous mapping with all arguments, $\tilde{A}_1 : C \to 2^{L(E_1,E_2)}$ be a nonempty upper semi continuous compact valued mapping induced by fuzzy mappings $A_1 : C \to \mathfrak{F}(L(E_1, E_2))$, that is $\tilde{A}_1(u) = (A_1(u))_{a_1(u)}$ with $a_1 : E_1 \to [0, 1]$. If $\tilde{A}_1$ is H-hemicontinuous and α-relaxed exponentially (γ, η)-monotone with respect to N. Then the following two statements (i) and (ii) are equivalent:*

(i) *there exist* $u_0 \in C$ *and* $\bar{x} \in \tilde{A}_1(u_0) = (A_1(u_0))_{a_1(u_0)}$ *such that*

$$\langle N(\bar{x}), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))} - 1)\rangle + F(g(u_0), v) \nleq_{\mathrm{int}K(u_0)} 0, \ \forall v \in C,$$

(ii) *there exists* $u_0 \in C$ *such that*

$$\langle N(\bar{r}), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))} - 1)\rangle + F(g(u_0), v) \nleq_{\mathrm{int}K(u_0)} \alpha_g(v - u_0),$$

$$\forall v \in C, \ \bar{r} \in \tilde{A}_1(v) = (A_1(v))_{a_1(v)}.$$

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us to improve this article.

References

[1] T. Antczak, *(p, r)-invexity in multiobjective programming*, Eur. J. Oper. Res., **152** (2004), 72-87.

[2] J. P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, John Willy and Sons, Inc. New York (1984).

[3] L. Brouwer, *Zur invarianz des n-dimensional gebietes*, Math. Ann., **71**(3)(1912), 305-313.

[4] G. Y. Chen and G. M. Cheng, *Vector variational inequality and vector optimizations,* SpringerVerlag, 285, Lecture Notes in Economics and Mathematical Systems, (1967).

[5] L. C. Ceng and J. C. Yao, *On generalized variational-like inequalities with generalized monotone multivalued mappings*, Appl. Math. Lett., **22**(3) (2009), 428-434.

[6] S. S. Chang and Y. G. Zhu, *On variational inequalities for fuzzy mappings*, Fuzzy sets and Systems, **32(3)**, (1989), 359-367

[7] S. S. Chang, Salahuddin, M. K. Ahmad and X. R. Wang, *Generalized vector variational like inequalities in fuzzy environment*, Fuzzy sets and systems, **265** (2015), 110-120.

[8] X. P. Ding, M. K. Ahmad and Salahuddin, *Fuzzy generalized vector variational inequalities and complementarity problems*, Nonlinear Functional Analysis and Applications, **13(2)** (2008), 253-263.

[9] X. P. Ding and E. Tarafdar, *Generalized variational like inequalities with pseudomonotone set valued mappings*, Archiv der Mathematik, **74** (2000), 3471-3476.

[10] K. Fan, *A generalization of tychonoffs fixed point theorem*, Math. Ann., **142** (1961), 305-310.

[11] A. Farajzadeh, M. A. Noor and K. I. Noor, *Vector nonsmooth variational-like inequalities and optimization problems*, Nonlinear Anal., **71** (2009), 3471-3476.

[12] F. Giannessi, *Theorems of alternative, quadratic programs, and complementarity problems, In Variational Inequalities and Complementarity Problems*, Edited by R. W. Cottle, F. Giannessi and J. L. Lions), John Wiley Sons, Chichester, England, (1980).

[13] S. C. Ho and H. C. Lai, *Optimality and duality for nonsmooth minimax fractional programming problem with exponential (p, r)-invexity*, J. Nonlinear Convex Anal. Oper. Res., **13** (2012), 433-447.

[14] S. S. Irfan, M. Rahaman, I Ahmad, R. Ahmad and S. Husain, *Generalized nonsmooth exponential-type vector variational-like inequalities and nonsmooth vector optimization problems in Asplund spaces*, Mathematics, **7(4), 345**, (2019), 1-11.

[15] A. Jayswal, S. Choudhury and R. U. Verma, *Expenential type vector variational-like inequalities and vector optimization problems with exponential type invexities*, J. Appl. Math. Comput., **45 (1-2)** (2014), 87-97.

[16] D. Kinderlehrer and G. Stampacchia, *An introduction to variational inequalities and their applications in Pure Applied Maths*, Academic Press, New York, (1980).

[17] G. M. Lee, D. S. Kim and B. S. Lee, *Vector variational inequalities for fuzzy mappings*, Nonlinear Analysis Forum **4** (1999), 119-129.

[18] S. B. Nadler,, *Multi-valued contraction mappings*, Pac. J. Math., **30(2)** (1969), 475-488.

[19] M. Oveisiha and J. Zafarani, *Generalized Minty vector variational-like inequalities and vector optimization problems in Asplund spaces,* . Optim. Lett., **7**, (2013), 709-721.

[20] S. Park, B. S. Lee and G. M. Lee, *A general vector valued variational inequalities and its fuzzy extension*, Internal J. Math. Math. Sci., **21**, (1998), 637-642.

[21] T. W. Reiland, *Nonsmooth invexity*, Bull. Aust. Math. Soc., **42** (1990), 437-446.

[22] M. Rezaie and J. Zafarani, *Vector optimization and variational-like inequalities*, J. Glob. Optim., **43** (2009), 47-66.
[23] C. H. Su and V. M. Sehgal, *Some fixed point theorems for condensing multifunctions in locally convex spaces*, Proceedings of the National Academy of Sciences of United States of America, **50** (1975), 150-154.
[24] F. Usman and S. A. Khan, *A generalized mixed vector variational-like inequality problem*, Nonlinear Analysis: Theory Methods Appl., **71(11)** (2009), 5354-5362.
[25] K. Q. Wu and N. J. Huang, *Vector variational-like inequalities with relaxed $\eta - \alpha$-Pseudomonotone mappings in Banach spaces*, J. Math. Inequal, **1** (2007), 281-290.
[26] G. Xiao, Z. Fan and X. Qi, *Existence results for generalized nonlinear vector variational like inequalities with set valued mapping*, Appl. Math. Lett., **23** (2010), 44-47.
[27] L. A. Zadeh, *Fuzzy sets*, Inf. Control, **8** (1965), 338-353.
[28] H. J. Zimmermann, *Fuzzy set theory and its applications*, Kluwer Academic Publishers, Dordrecht, (1988).

Mohammad Farid
Deanship of Educational Services, Qassim University, Buraidah 51452, Al-Qassim, Saudi Arabia
E-mail address: mohdfrd55@gmail.com

Syed Shakaib Irfan (Corresponding Author)
College of Engineering, Qassim University, Buraidah 51452, Al-Qassim, Saudi Arabia
E-mail address: shakaib@qec.edu.sa

Iqbal Ahmad
College of Engineering, Qassim University, Buraidah 51452, Al-Qassim, Saudi Arabia
E-mail address: iqbal@qec.edu.sa

Preeti Shukla
Department of Mathematics, Faculty of Science, Integral University, Lucknow - 226026, U.P., India
E-mail address: shuklapreeti1991@gmail.com