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ON A CLASS OF ANALYTIC MULTIVALENT FUNCTIONS IN q-ANALOGUE ASSOCIATED WITH LEMNISCATE OF BERNOULLI

BAKHTIAR AHMAD[1], MASLINA DARUS[2], NASIR KHAN[3,*], RAEES KHAN[4], MUHAMMAD GHAFFAR KHAN[5]

Abstract. The object of the paper is to examine some various interseting properties of analytic multivalent functions in q-analogue associated with the lemniscate of Bernoulli.

1. Introduction

First of all we recall some basic definitions and concepts of Geometric Function Theory which are useful to understand the notions used in our main work, so we present first the class $\mathcal{A}_p$ of analytic multivalent functions $f(z)$ in the region $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$, with the representation

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \ \ (z \in \mathfrak{D}, \, p \in N). \tag{1.1}$$

For $p = 1$, it becomes the well-known class of analytic functions $\mathcal{A}$.
Moreover, for two functions f and g analytic in $\mathfrak{D}$, we say that the function f is subordinate to the function g and write as

$$f \prec g \qquad \text{or} \qquad f(z) \prec g(z),$$

if there exists a Schwarz function w which is analytic in $\mathfrak{D}$ with

$$w(0) = 0 \qquad \text{and} \qquad |w(z)| < 1,$$

such that

$$f(z) = g(w(z)).$$

Furthermore, if the function g is univalent in $\mathfrak{D}$ then we have the following equivalence (*cf., eg.,* [17], see also [18]) ::

$$f(z) \prec g(z) \qquad (z \in \mathfrak{D}) \Rightarrow f(0) = g(0) \quad \text{and} \quad f(\mathfrak{D}) \subset g(\mathfrak{D}).$$

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Let $\mathcal{SL}^*$ be the class of functions defined by

$$S_L = \left\{ f \in \mathcal{A} : \left| \left(\frac{zf'(z)}{f(z)} \right)^2 - 1 \right| < 1 \right\}.$$

The subclass $\mathcal{SL}^*$ which motivates the researchers was investigated by Sokół et al. [22], containing functions $f \in \mathcal{A}$ such that $\frac{zf'(z)}{f(z)}$ lies in the region bounded by the right-half of the Bernoulli lemniscate given by $\left| w(z)^2 - 1 \right| < 1$. In terms of subordination, the class $\mathcal{SL}^*$ consists of normalized analytic functions f satisfying

$$\frac{zf'(z)}{f(z)} \prec \sqrt{1+z}.$$

This class was further investigated by [5, 7, 23, 24].

A function $h(z)$ is said to be in the class $P[A,B]$, if it is analytic in E with $p(0) = 1$ and

$$h(z) \prec \frac{1+Az}{1+Bz}, 1 \leq B < A \leq 1,$$

equivalently we can write

$$\left| \frac{h(z) - 1}{A - Bh(z)} \right| < 1$$

This class was introduced by Janowski [10] and explored by a few creators like [21, 6, 2, 16, 25, 26, 19, 20, 27, 13].

The Calculus without the concept of limits which is called q-calculus has evolved as key component in different fields of sciences and mathematics. Due to its numerous physical and mathematical applications it attracted a lot of researchers. The q-analogue of derivative and integral were introduced and studied by Jackson [8, 9]. Srivastava and Bansal [28, pp. 62] used the q-analogue of derivative in Geometric function theory by introducing the q-generalization of starlike functions for the first time, see also [29, pp. 347 *et seq.*]. More details of the topic can be seen in [11, 4, 15, 3, 14].

The q-derivative (or q-difference)[9] D_q of a function f defined is in a given subset of $\mathbb{C}$ by

$$(D_q f)(z) = \begin{cases} \frac{f(z) - f(qz)}{(1-q)z} & (z \neq 0) \\ f'(0) & (z = 0). \end{cases} \tag{1.2}$$

provided that $f'(0)$ exists.

From Definition (1), we can observe that

$$\lim_{q \longrightarrow 1-} (D_q f)(z) = \lim_{q \longrightarrow 1-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z),$$

for a differentiable function f in a given subset of $\mathbb{C}$. It is readily known from (1.1) and (1.2) that

$$(D_q f)(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}. \tag{1.3}$$

where

$$[n]_q = \frac{1-q^n}{1-q} = 1 + \sum_{l=1}^{n} q^l, \quad [0]_q = 0.$$

Now we define $\mathcal{SL}_{p,q}^{*}$, the class of analytic multivalent functions in q-analogue associated with the lemniscate of Bernoulli as

$$\mathcal{SL}_{p,q}^{*}=\left\{f(z)\in\mathcal{A}_{p}:\frac{zD_{q}f\left(z\right)}{\left[p\right]_{q}f(z)}\prec\sqrt{1+z},\,z\in\mathfrak{D}\right\},$$

we note that if $q\to1^{-}$ then $\mathcal{SL}_{p,q}^{*}$ becomes $\mathcal{SL}_{p}^{*}$,the class of analytic multivalent functions in domain of lemniscate of bernoulie, investigated by Qaiser et. al [12].

In recent past Ali et al. [5] have invrstigated and studied differential subordinations $1+\alpha\frac{zh'(z)}{h^{n}(z)}\prec\sqrt{1+z}$ and found that $h(z)\prec\sqrt{1+z}$ where $n=0,1,2$ for some particular range of α. Similar kind of differential subordinations are also discussed by various authors. In this article we are investigating some properties of analytic multivalent functions in q-analogue associated with lemniscate of Bernoulli. We determine some conditions on α so that $1+\alpha\frac{z^{1-p}D_{q}f(z)}{[p]_{q}}$, $1+\alpha\frac{zD_{q}f(z)}{[p]_{q}f(z)}$, $1+\alpha\frac{z^{1-p}D_{q}f(z)}{[p]_{q}(f(z))^{2}}$ and $1+\alpha\frac{z^{1-2p}D_{q}f(z)}{[p]_{q}(f(z))^{3}}$ are in Janowski domain and $\frac{f(z)}{z^{p}}\prec\sqrt{1+z}$. Then using this we discuss the conditions so that a function will belong to $\mathcal{SL}_{p,q}^{*}$. To avoide repetations it is admitted once that $-1\leq B<A\leq1$, $q\in(0,1)\,,z\in\mathfrak{D}$, $p\in N$.
For proving main results we need the following Lemma.

Lemma 1.1. [1] *(q-jack's lemma) Let $w\left(z\right)$ be analytic in $\mathfrak{D}=\left\{z\in\mathbb{C}:\left|z\right|<1\right\}$ with $w\left(0\right)=0$. If $\left|w\left(z\right)\right|$ attains its maximum value on the circle $\left|z\right|=r$ at a point $z_{0}=re^{i\theta}$, for $\theta\in\left[-\pi,\pi\right]$, we can write that for $0<q<1$*

$$z_{0}D_{q}w\left(z_{0}\right)=mw\left(z_{0}\right),$$

where m is real and $m\geq1$.

2. Main Results

Theorem 2.1. *If $(z)\in\mathcal{A}_{p}$, such that*

$$1+\frac{\alpha z^{1-p}D_{q}f\left(z\right)}{\left[p\right]_{q}}\prec\frac{1+Az}{1+Bz},$$

with

$$\left|\alpha\right|\geq\frac{2^{\frac{3}{2}}\left[p\right]_{q}\left(A-B\right)}{1-\left|B\right|-4p\left(1+\left|B\right|\right)},\tag{2.1}$$

then

$$\frac{f\left(z\right)}{z^{p}}\prec\sqrt{1+z}.$$

Proof. Define the function h by

$$1+\frac{\alpha z^{1-p}D_{q}f\left(z\right)}{\left[p\right]_{q}}=h\left(z\right),\tag{2.2}$$

where $h\left(z\right)$ is analytic and $h(0)=1$. Also consider

$$\frac{f\left(z\right)}{z^{p}}=\sqrt{1+w\left(z\right)}.\tag{2.3}$$

For proving the result it is enough to show that $\left|w\left(z\right)\right|<1$.
By carrying out logarithmic differentiation in (2.3), and using (2.2) we get

$$h\left(z\right)=1+\frac{\alpha zD_{q}w\left(z\right)}{2\left[p\right]_{q}\sqrt{1+w\left(z\right)}}+\frac{\alpha p\sqrt{1+w\left(z\right)}}{\left[p\right]_{q}}.$$

Also

$$\begin{aligned} &\left|\frac{h(z)-1}{A-Bh(z)}\right| \\ = &\left|\frac{\frac{\alpha z D_q w(z)}{2[p]_q\sqrt{1+w(z)}}+\frac{\alpha p\sqrt{1+w(z)}}{[p]_q}}{A-B\left(1+\frac{\alpha z D_q w(z)}{2[p]_q\sqrt{1+w(z)}}+\frac{\alpha p\sqrt{1+w(z)}}{[p]_q}\right)}\right| \\ = &\left|\frac{\alpha z D_q w(z)+2p\alpha(1+w(z))}{2[p]_q(A-B)\sqrt{1+w(z)}-B(\alpha z D_q w(z)+2p\alpha(1+w(z)))}\right|. \end{aligned}$$

Now if $w(z)$ attains its maximum value at some $z=z_0$ and $|w(z_0)|=1$. Then by Lemma (1.1), there exists a number $m\geq 1$ such that, $z_0 D_q w(z_0)=mw(z_0)$. And suppose that $w(z_0)=e^{i\theta}$,for $\theta\in[-\pi,\pi]$. Then for $z_0\in\mathfrak{D}$, we have

$$\begin{aligned} &\left|\frac{h(z_0)-1}{A-Bh(z_0)}\right| \\ = &\left|\frac{\alpha m w(z_0)-2p\alpha(1+w(z_0))}{2[p]_q(A-B)\sqrt{1+w(z_0)}-\alpha B(mw(z_0)+2p(1+w(z_0)))}\right| \\ \geq &\frac{|\alpha|\left(m-2p\left(\left|1+e^{i\theta}\right|\right)\right)}{2[p]_q(A-B)\sqrt{|1+e^{i\theta}|}+|\alpha||B|(m+2p(|1+e^{i\theta}|))} \\ \geq &\frac{|\alpha|(m-4p)}{2^{\frac{3}{2}}[p]_q(A-B)+|B||\alpha|(m+4p)}=\phi(m) \end{aligned}$$

Now by elementary calculus we have

$$\phi'(m)=\frac{2^{\frac{3}{2}}[p]_q(A-B)|\alpha|+8|\alpha|^2 p|B|}{\left(2^{\frac{3}{2}}[p]_q(A-B)+|B||\alpha|(m+4p)\right)^2}>0,$$

which shows that $\phi(m)$ is an increasing function and hence it will have its minimum value at $m=1$ and so

$$\left|\frac{h(z_0)-1}{A-Bh(z_0)}\right|\geq\frac{|\alpha|(1-4p)}{2^{\frac{3}{2}}[p]_q(A-B)+|B||\alpha|(1+4p)}.$$

Now by (2.1) we have

$$\left|\frac{h(z_0)-1}{A-Bh(z_0)}\right|\geq 1,$$

which contradicts the fact that $h(z)\prec\frac{1+Az}{1+Bz}$, and so $|w(z)|<1$ and so we get the desired result. □

Corollary 2.2. *If* $(z)\in\mathcal{A}_p$, *such that*

$$1+\frac{\alpha z D_q f(z)}{[p]_q^2 f(z)}\left(p+1+\frac{zD_q^2 f(z)}{\partial_q f(z)}-\frac{zD_q f(z)}{f(z)}\right)\prec\frac{1+Az}{1+Bz}, \tag{2.4}$$

with

$$|\alpha|\geq\frac{2^{\frac{3}{2}}[p]_q(A-B)}{1-|B|-4p(1+|B|)},$$

then $f(z)\in\mathcal{SL}^*_{p,q}$.

Proof. Let us consider a function

$$l(z) = \frac{z^{p+1} D_q f(z)}{[p]_q f(z)}, \tag{2.5}$$

where $l(z)$ is analytic and $l(0) = 1$. With some calculations we obtain

$$z^{1-p} D_q l(z) = \frac{z D_q f(z)}{[p]_q f(z)} \left(p + 1 + \frac{z D_q^2 f(z)}{\partial_q f(z)} - \frac{z D_q f(z)}{f(z)} \right). \tag{2.6}$$

Using (2.5) and (2.6) we obtain

$$1 + \frac{\alpha z^{1-p} D_q l(z)}{[p]_q} \prec \frac{1 + Az}{1 + Bz}.$$

Now by the application of Theorem (2.1) we get

$$\frac{l(z)}{z^p} = \frac{z D_q f(z)}{p f(z)} \prec \sqrt{1 + z}.$$

and so $f(z) \in \mathcal{SL}^*_{p,q}$. □

Theorem 2.3. *If* $(z) \in \mathcal{A}_p$, *such that*

$$1 + \alpha \frac{z D_q f(z)}{[p]_q f(z)} \prec \frac{1 + Az}{1 + Bz}, \tag{2.7}$$

with

$$|\alpha| \geq \frac{4 [p]_q (A - B)}{1 - |B| - 4p (1 + |B|)} \tag{2.8}$$

then

$$\frac{f(z)}{z^p} \prec \sqrt{1 + z}.$$

Proof. Setting a function $h(z)$ as

$$h(z) = 1 + \alpha \frac{z D_q f(z)}{[p]_q f(z)}$$

Then for

$$\frac{f(z)}{z^p} = \sqrt{1 + w(z)}$$

with some calculations we obtain that

$$h(z) = 1 + \frac{\alpha z D_q w(z)}{2 [p]_q (1 + w(z))} + \frac{\alpha p}{[p]_q}$$

and so

$$\begin{aligned} & \left| \frac{h(z) - 1}{A - B h(z)} \right| \\ = \quad & \left| \frac{\frac{\alpha z D_q w(z)}{2[p]_q (1 + w(z))} + \frac{\alpha p}{[p]_q}}{A - B \left(1 + \frac{\alpha z D_q w(z)}{2[p]_q (1 + w(z))} + \frac{\alpha p}{[p]_q} \right)} \right| \\ = \quad & \left| \frac{\alpha z D_q w(z) + 2p\alpha (1 + w(z))}{2 [p]_q (A - B) (1 + w(z)) - B (\alpha z D_q w(z) + 2p\alpha (1 + w(z)))} \right| \end{aligned}$$

Now if $w(z)$ attains its maximum value at some $z = z_0$ and $|w(z_0)| = 1$. Then by Lemma (1.1), there exists a number $m \geq 1$ such that, $z_0 D_q w(z_0) = mw(z_0)$. And suppose that $w(z_0) = e^{i\theta}$,for $\theta \in [-\pi, \pi]$. Then for $z_0 \in \mathfrak{D}$, we have

$$\begin{aligned}
&\left|\frac{h(z_0)-1}{A-Bh(z_0)}\right| \\
= &\left|\frac{\alpha m w(z_0) + 2p\alpha(1+w(z_0))}{2[p]_q(A-B)(1+w(z_0)) - B(\alpha m w(z_0) + 2p\alpha(1+w(z_0)))}\right| \\
\geq &\frac{|\alpha| m - 2p|\alpha| \left|1+e^{i\theta}\right|}{2[p]_q(A-B)|1+e^{i\theta}| + |B||\alpha| m + 2p|B||\alpha||1+e^{i\theta}|} \\
= &\frac{|\alpha| m - 2p|\alpha|\sqrt{2+2\cos\theta}}{2\left((A-B)[p]_q + |B||\alpha| p\right)\sqrt{2+2\cos\theta} + |B||\alpha| m} \\
\geq &\frac{|\alpha|(m-4p)}{4\left((A-B)[p]_q + |B||\alpha| p\right) + |B||\alpha| m} = \phi(m)
\end{aligned}$$

Now let

$$\phi'(m) = \frac{4(A-B)|\alpha|[p]_q + 8p|\alpha|^2|B|}{\left(4\left((A-B)[p]_q + |B||\alpha| p\right) + |B||\alpha| m\right)^2} > 0$$

which shows that $\phi(m)$ is an increasing function and hence it will have its minimum value at $m = 1$ and so

$$\left|\frac{h(z_0)-1}{A-Bh(z_0)}\right| \geq \frac{|\alpha|(1-4p)}{4\left((A-B)[p]_q + |B||\alpha| p\right) + |B||\alpha|}.$$

Now by(2.8) we have

$$\left|\frac{h(z_0)-1}{A-Bh(z_0)}\right| \geq 1$$

which contradicts (2.7), and so $|w(z)| < 1$ and so we get the desired proof. □

Corollary 2.4. *If $f(z) \in \mathcal{A}_p$ such that*

$$1 + \frac{\alpha}{[p]_q}\left(p + 1 + \frac{zD_q^2 f(z)}{\partial_q f(z)} - \frac{zD_q f(z)}{f(z)}\right) \prec \frac{1+Az}{1+Bz},$$

with

$$|\alpha| \geq \frac{4[p]_q(A-B)}{1-|B|-4p(1+|B|)},$$

holds then $f(z) \in \mathcal{SL}^*_{p,q}$.

Theorem 2.5. *If $f(z) \in \mathcal{A}_p$ such that*

$$1 + \alpha\frac{z^{1-p}D_q f(z)}{[p]_q (f(z))^2} \prec \frac{1+Az}{1+Bz}, \tag{2.9}$$

with

$$|\alpha| \geq \frac{2^{\frac{5}{2}}[p]_q(A-B)}{1-|B|-4p(1+|B|)} \tag{2.10}$$

then

$$\frac{f(z)}{z^p} \prec \sqrt{1+z}.$$

Proof. Here we define a function

$$h(z) = 1 + \alpha \frac{z^{1-p} D_q f(z)}{[p]_q (f(z))^2}$$

Then for

$$\frac{f(z)}{z^p} = \sqrt{1 + w(z)}.$$

Using some simplification we obtain that

$$h(z) = 1 + \frac{\alpha z D_q w(z)}{2[p]_q (1 + w(z))^{\frac{3}{2}}} + \frac{\alpha p}{[p]_q \sqrt{1 + w(z)}}.$$

Therefore,

$$\begin{aligned} & \left| \frac{h(z) - 1}{A - Bh(z)} \right| \\ = & \left| \frac{\frac{\alpha z D_q w(z)}{2[p]_q (1+w(z))^{\frac{3}{2}}} + \frac{\alpha p}{[p]_q \sqrt{1+w(z)}}}{A - B\left(1 + \frac{\alpha z D_q w(z)}{2[p]_q (1+w(z))^{\frac{3}{2}}} + \frac{\alpha p}{[p]_q \sqrt{1+w(z)}}\right)} \right| \\ = & \left| \frac{\alpha z D_q w(z) + 2p\alpha (1 + w(z))}{2[p]_q (A - B)(1 + w(z))^{\frac{3}{2}} - B\alpha z D_q w(z) - 2p\alpha B (1 + w(z))} \right| \end{aligned}$$

Now if $w(z)$ attains its maximum value at some $z = z_0$ and $|w(z_0)| = 1$. Then by Lemma (1.1), there exists a number $m \geq 1$ such that, $z_0 D_q w(z_0) = m w(z_0)$. And suppose that $w(z_0) = e^{i\theta}$,for $\theta \in [-\pi, \pi]$. Then for $z_0 \in \mathfrak{D}$, we have

$$\begin{aligned} & \left| \frac{h(z_0) - 1}{A - Bh(z_0)} \right| \\ = & \left| \frac{\alpha m w(z_0) + 2p\alpha (1 + w(z_0))}{2[p]_q (A - B)(1 + w(z_0))^{\frac{3}{2}} - B\alpha m w(z_0) - 2p\alpha B (1 + w(z_0))} \right| \\ \geq & \frac{|\alpha| m - 2p |\alpha| \left|1 + e^{i\theta}\right|}{2[p]_q (A - B) |1 + e^{i\theta}|^{\frac{3}{2}} + |B| |\alpha| m + 2p |\alpha| |B| |1 + e^{i\theta}|} \\ = & \frac{|\alpha| (m - 4p)}{2^{\frac{5}{2}} [p]_q (A - B) + |B| |\alpha| m + 4p |\alpha| |B|} \\ \geq & \frac{|\alpha| (m - 4p)}{2^{\frac{5}{2}} [p]_q (A - B) + |B| |\alpha| m + 4p |\alpha| |B|} = \phi(m). \end{aligned}$$

Now let

$$\phi'(m) = \frac{2^{\frac{5}{2}} [p]_q |\alpha| (A - B) + 8 |\alpha|^2 |B| p}{\left(2^{\frac{5}{2}} [p]_q (A - B) + B |\alpha| m + 4p |\alpha| B\right)^2} > 0$$

which shows that $\phi(m)$ is an increasing function and hence it will have its minimum value at $m=1$ and so

$$\left|\frac{h(z_0)-1}{A-Bh(z_0)}\right| \geq \frac{|\alpha|(1-4p)}{2^{\frac{5}{2}}[p]_q(A-B)+|B||\alpha|+4p|\alpha||B|}.$$

Now by (2.10) we have

$$\left|\frac{h(z_0)-1}{A-Bh(z_0)}\right| \geq 1$$

Hence a contradiction to (2.9), and so $|w(z)|<1$ and so we get the required proof. □

Corollary 2.6. *If $f(z)\in\mathcal{A}_p$ such that*

$$1+\frac{\alpha f(z)}{z^{2p+1}D_q f(z)}\left(p+1+\frac{zD_q^2 f(z)}{\partial_q f(z)}-\frac{zD_q f(z)}{f(z)}\right)\prec\frac{1+Az}{1+Bz},$$

with

$$|\alpha|\geq\frac{2^{\frac{5}{2}}[p]_q(A-B)}{1-|B|-4p(1+|B|)},$$

then $f(z)\in\mathcal{SL}^*_{p,q}$.

Theorem 2.7. *If $f(z)\in\mathcal{A}_p$ such that*

$$1+\alpha\frac{z^{1-2p}D_q f(z)}{[p]_q(f(z))^3}\prec\frac{1+Az}{1+Bz},$$

with

$$|\alpha|\geq\frac{8[p]_q(A-B)}{1-|B|-4p(1+|B|)}, \tag{2.11}$$

then

$$\frac{f(z)}{z^p}\prec\sqrt{1+z}.$$

Proof. Let us define a function

$$h(z)=1+\alpha\frac{z^{1-2p}D_q f(z)}{[p]_q(f(z))^3}.$$

Then if

$$\frac{f(z)}{z^p}=\sqrt{1+w(z)}.$$

Using some calculations we obtain that

$$h(z)=1+\frac{\alpha zD_q w(z)}{2[p]_q(1+w(z))^2}+\frac{\alpha p}{[p]_q(1+w(z))}$$

and so

$$\begin{aligned}
&\left|\frac{h(z)-1}{A-Bh(z)}\right| \\
= &\left|\frac{\frac{\alpha z D_q w(z)}{2[p]_q(1+w(z))^2}+\frac{\alpha p}{[p]_q(1+w(z))}}{A-B\left(1+\frac{\alpha z D_q w(z)}{2[p]_q(1+w(z))^2}+\frac{\alpha p}{[p]_q(1+w(z))}\right)}\right| \\
= &\left|\frac{\alpha z D_q w(z)+2p\alpha(1+w(z))}{2[p]_q(A-B)(1+w(z))^2-B\alpha z D_q w(z)-2p\alpha B(1+w(z))}\right|
\end{aligned}$$

Now if $w(z)$ attains its maximum value at some $z=z_0$ and $|w(z_0)|=1$. Then by Lemma (1.1), there exists a number $m\geq 1$ such that, $z_0 D_q w(z_0)=mw(z_0)$. And suppose that $w(z_0)=e^{i\theta}$,for $\theta\in[-\pi,\pi]$. Then for $z_0\in\mathfrak{D}$, we have

$$\begin{aligned}
&\left|\frac{h(z_0)-1}{A-Bh(z_0)}\right| \\
= &\left|\frac{\alpha m w(z_0)+2p\alpha(1+w(z_0))}{2[p]_q(A-B)(1+w(z_0))^2-B\alpha m w(z_0)-2p\alpha B(1+w(z_0))}\right| \\
\geq &\frac{|\alpha|m-2p|\alpha|\left|1+e^{i\theta}\right|}{2[p]_q(A-B)\left|1+e^{i\theta}\right|^2+|B||\alpha|m+2p|\alpha||B|\left|1+e^{i\theta}\right|} \\
= &\frac{|\alpha|m-2p|\alpha|\sqrt{2+2\cos\theta}}{2[p]_q(A-B)\left(\sqrt{2+2\cos\theta}\right)^2+|B||\alpha|m+2p|\alpha||B|\sqrt{2+2\cos\theta}} \\
\geq &\frac{|\alpha|(m-4p)}{8[p]_q(A-B)+|B||\alpha|m+4p|\alpha||B|}=\phi(m)
\end{aligned}$$

Now let

$$\phi'(m)=\frac{8[p]_q|\alpha|(A-B)+8|\alpha|^2|B|p}{\left(8[p]_q(A-B)+|B||\alpha|m+4p|\alpha||B|\right)^2}>0$$

which shows that $\phi(m)$ is an increasing function and hence it will have its minimum value at $m=1$ and so

$$\left|\frac{h(z_0)-1}{A-Bh(z_0)}\right|\geq\frac{|\alpha|(1-4p)}{8[p]_q(A-B)+|B||\alpha|+4p|\alpha||B|}.$$

Now as

$$\left|\frac{h(z_0)-1}{A-Bh(z_0)}\right|\geq 1,$$

which is a contradiction to the fact that $h(z)\prec\frac{1+Az}{1+Bz}$, and so $|w(z)|<1$ and so we get the desired result. □

Corollary 2.8. *If $f(z)\in\mathcal{A}_p$ such that*

$$1+\frac{\alpha p(f(z))^2}{z^{3p+2}(D_q f(z))^2}\left(p+1+\frac{zD_q^2 f(z)}{\partial_q f(z)}-\frac{zD_q f(z)}{f(z)}\right)\prec\frac{1+Az}{1+Bz},$$

with

$$|\alpha|\geq\frac{8[p]_q(A-B)}{1-|B|-4p(1+|B|)},$$

then $f(z) \in \mathcal{SL}_{p,q}^{*}$.

3. Conclusion

The generalized form of analytic functions in lemniscate of Bernoulli were introduced with the help of subordinations. Using the well known Janowski functions, various interesting characterizations were formulated for this newly defined class. The idea of q-calculus were utilized in this article as it is an interesting revelation in this field. Basic (or q-) series and basic (or q-) polynomials, especially the basic (or q-) gamma and (or q-)-hypergeometric functions and basic (or q-) hypergeometric polynomials, are applicable particularly in several diverse areas (see, for example, [[30], pp. 350-351] and [[31], p. 328)]. Moreover, in this recently-published survey-cum-expository review article by Srivastava [30], the so-called (or q-)-calculus was exposed to be a rather trivial and inconsequential variation of the classical q-calculus, the additional parameter p being redundant (see, for details, [[30], p. 340]). This observation by Srivastava [31] will indeed apply also to any attempt to produce the rather straightforward (p, q)-variants of the results which we have presented in this paper.

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Bakhtiar Ahmad
Department of Mathematics, Government Degree College Mardan, Pakistan
E-mail address: pirbakhtiarbacha@gmail.com

Maslina Darus
Department of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600, Bangi, Selangor, Malaysia
E-mail address: maslina@ukm.edu.my

Nasir Khan
Department of Mathematics, FATA University, TSD Darra, NMD Kohat, Pakistan
E-mail address: dr.nasirkhan@fu.edu.pk

Raees Khan
Department of Mathematics, FATA University, TSD Darra, NMD Kohat, Pakistan
E-mail address: raeeskhan@fu.edu.pk

Muhammad Ghaffar Khan
Institute of Numerical Sciences, Kohat University of Science and Technology, Kohat, Pakistan
E-mail address: ghaffarkhan020@gmail.com