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STABILITY OF VARIOUS ITERATIVE TYPE FUNCTIONAL EQUATIONS IN MENGER $\varphi-$ NORMED SPACES

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Abstract. The objective of this study to examine some stability results concerning the iterative type functional equations like Gamma, Schröder functional equations and also generalize the stability results of quintic and sextic functional equations (QF Equations and SF Equations) in complete Menger $\varphi-$normed spaces.

1. Introduction

The first stability problem was established by Ulam [37] in 1940. He raised a question whether there exists an exact homomorphism close to approximate homomorphism. The solution of Ulam problem was given by Hyers [10] in Banach spaces. Last some decades, stability problems have been studied by several mathematicians. In order to have more knowledge on the stability of various functional equations and also stability problems in probabilistic and fuzzy normed spaces, see[3-10, 18-21, 24-29]. Radu [30] gave an answer of Ulam's doubt strongly by using the fixed point method. In [1, 2, 19-20], the authors studied the theory of fixed point for the probabilistic stability of functional equations.

In this paper, we employ this method to find the stability of the iterative, quintic and sextic functional equations in Menger probabilistic $\varphi-$ normed spaces originated by Golet in [9]. During this article, we denote complete Menger probabilistic $\varphi-$ normed space by CMP $\varphi-$ normed space.

Definition 1.1. [21] *"A function $F:\mathbb{R}\to[0,1]$ is called a distribution function if it is non-decreasing and left continuous with sup $F(t)=1$ and inf $F(t)=0$. The class of all distribution functions F with $F(0)=0$ is denoted by D^+ and $\varepsilon_\circ$ is the specific distribution function defined through*

$$\varepsilon_\circ \ = \begin{cases} 0, & t\leq 0 \\ 1, & t>0." \end{cases}$$

Let φ be a function defined on the real field $\mathbb{R}$ into itself, with the following properties:

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(a) $\varphi(-t) = \varphi(t)$ for every $t \in \mathbb{R}$;
(b) $\varphi(1) = 1$;
(c) φ is strictly increasing and continuous on $[0, \infty)$, $\varphi(0) = 0$ and

$$\lim_{\alpha \to \infty} \varphi(\alpha) = \infty$$

where $\alpha \in \mathbb{R}$.

Examples of such functions are:
$\varphi(\alpha) = |\alpha|$; $\varphi(\alpha) = |\alpha|^l,\ l \in (0, \infty)$; $\varphi(\alpha) = \frac{2\alpha^{2n}}{|\alpha|+1}$, $n \in \mathbb{N}$.

Definition 1.2. [9] *" A Menger probabilistic $\varphi-$normed space is a triple (Z, ν, T), where Z is a real vector space, T is a continuous $t-$ norm and ν is defined from Z into D^+ such that the following conditions hold:*

(PN1) $\nu_z(t) = \varepsilon_o(t)$ for all $t > 0$ if and only if $z = 0$;
(PN2) $\nu_{\alpha z}(t) = \nu_z(\frac{t}{\varphi(\alpha)})$ for all z in Z , $\alpha \neq 0$ and $t > 0$;
(PN3) $\nu_{z+y}(t + s) \geqslant T(\nu_z(t), \nu_y(s))$ for all $z, y \in Z$ and $t, s \geqslant 0$."

Definition 1.3. [9] *"Let (Z, ν, T) be a Menger probabilistic $\varphi-$normed space.*
(1) A sequence $\{z_n\}$ in Z is said to be convergent to z in Z in the topology τ if for every $t > 0$ and $\varepsilon > 0$, there exists positive integer $\mathbb{N}$ such that $\nu_{z_n - z}(t) > 1 - \varepsilon$ whenever $n \geqslant \mathbb{N}$.
(2) A sequence $\{z_n\}$ in Z is called Cauchy if for every $t > 0$ and $\varepsilon > 0$, there exists positive integer $\mathbb{N}$ such that $\nu_{z_n - z_m}(t) > 1 - \varepsilon$ whenever $n, m \geq \mathbb{N}$.
(3) A Menger probabilistic $\varphi-$ normed space (Z, ν, T) is said to be complete if every Cauchy sequence in Z is convergent to a point in Z."

Let (Y, ν, T_M) be a CMP $\varphi-$ normed space, Z be a vector space and G be a function from $Z \times \mathbb{R}$ into $[0, 1]$, in such a way that $G(z, .) \in D^+\ \forall\ z$. Taking the set $F = \{h : Z \to Y : h(0) = 0\}$ and the function d_G defined on $F \times F$ by

$$d_G(h, \psi) = inf\{u \in \mathbb{R}^+ : \nu_{h(z)-\psi(z)}(ut) \geq G(z, t)\ for\ all\ z \in Z\ and\ t > 0\}$$

where inf $\phi = +\infty$. The next lemma can be showed as in [20]:

Lemma 1.4. *([19,20]) " d_G is a complete generalized metric on F".*

2. Probabilistic stability of iterative functional equation

A general iterative functional equation can be presented as

$$F(z, f^1(z), f^2(z), ..., f^m(z)) = 0, \tag{2.1}$$

where $m \geq 2$, is one of the iterative functional equations [15] and was studied in many papers. We mention here some classical functional equations as

- Gamma Functional Equation

$$f(z + 1) = (z + 1)f(z)$$

- Schröder Functional Equation

$$f(g(z)) = sf(z)$$

Our main result is the following stability theorem for the iterative functional equation.

Theorem 2.1. *Let (Y, ν, T_m) be a CMP $\varphi-$normed space, Z be a real vector space and let $f : Z \to Y$ be a $\Phi-$ approximate solution of the equation*

$$F(z, f^1(z), f^2(z), ..., f^m(z)) = 0$$

in the sense that

$$\nu_{F(z,f^1(z),f^2(z),...,f^m(z))}(t) \geq \Phi(z, t) \tag{2.2}$$

for all $z \in Z, t > 0$, where Φ is function from Z to D^+. If there exists $\alpha \in (0, \varphi(m))$ for all $z \in Z, t > 0$ such that

$$\Phi(f^m(z))(\alpha t) \geq \Phi(z)(t) \tag{2.3}$$

and

$$lim_{n\to\infty}\Phi(m^n(z), m^n f^1(z), ..., m^n f^m(z))\Big(\frac{t}{\varphi(\frac{1}{m^n})}\Big) = 1$$

then there exists one and only one function $h : Z \to Y$ such that

$$\nu_{h(z)-f(z)}(t) \geq \Phi(z)((\varphi(m) - \alpha)t).$$

Moreover,

$$h(z) = lim_{n\to\infty} \tfrac{F(f^m(z))^n}{m^n}.$$

Proof. Let $G(z, t) = \Phi(z)(\varphi(m)t)$, $F = \{h : Z \to Y | h(0) = 0\}$ and the function d_G defined as $F \times F$ by

$$d_G(h, \psi) = inf\{u \in \mathbb{R}^+ : \nu_{h(z)-\psi(z)}(ut) \geq G(z, t)\}.$$

By using the lemma 1.4, we obtain (F, d_G) is a generalized metric space which is complete. Now, we assume the linear function $J : F \to F$ defined by

$$J(h(z)) = \frac{1}{m}h(f^m(z)).$$

It is convenient to verify that J is a self mapping on F which is strictly contractive together the Lipschitz constant $k = \frac{\alpha}{\varphi(m)}$. In fact, let h, ψ be functions lies in F gives that $d_G(h, \psi) < \epsilon$. Then

$$\nu_{Jh(z)-J\psi(z)}\big(\tfrac{\alpha}{\varphi(m)}\epsilon t\big) \geq G(z, t)$$

hence

$$\begin{aligned}\nu_{Jh(z)-J\psi(z)}\big(\frac{\alpha}{\varphi(m)}\epsilon t\big) &= \nu_{h(f^m(z))-\psi(f^m(z))}(\alpha\epsilon t)\\ &\geq G(f^m(z), \alpha t)\end{aligned}$$

Since, $G(f^m(z), \alpha t) \geq G(z, t)$ then $\nu_{Jh(z)-J\psi(z)}\big(\frac{\alpha}{\varphi(m)}\epsilon t\big) \geq G(z, t)$ i.e.,

$$d_G(h, \psi) < \epsilon \Rightarrow d_G(J_h, J_\psi) \leq \frac{\alpha}{\varphi(m)}\epsilon.$$

This implies that

$$d_G(J_h, J_\psi) \leq \frac{\alpha}{\varphi(m)} d_G(h, \psi),\ \forall\ h, \psi \in E.$$

It follows that $d_G(f, Jf) \leq 1$ from

$$\nu_{f(z)-m^{-1}f(f^m(z))}(t) \geq G(z, t).$$

By using Luxemburg theorem (see[17]), we derive the presence of a function $h : Z \to Y$ in such a way that

$$h(f^m(z)) = mh(z) \text{ for all } z \in Z.$$

Also, $d_G(f,h) \leq \frac{1}{1-k} d(f, Jf) \Rightarrow d_G(f,h) \leq \frac{1}{1-\frac{\alpha}{\varphi(m)}}$ from which it instantly follows

$$\nu_{h(z)-f(z)}\big(\tfrac{\varphi(m)}{\varphi(m)-\alpha} t\big) \geq G(z,t).$$

This yields

$$\nu_{h(z)-f(z)}(t) \geq G\big(z, \tfrac{\varphi(m)-\alpha}{\varphi(m)} t\big)$$

hence, we obtain the conclusion

$$\nu_{h(z)-f(z)}(t) \geq \Phi(z)((\varphi(m)-\alpha)t) \qquad \forall z \in Z \text{ and } t > 0.$$

Thus,

$$d_G(u,v) < \epsilon \Rightarrow \nu_{u(z)-v(z)}(t) \geq G\left(z, \frac{t}{\epsilon}\right)$$

and $d_G(J^n f, h) \to 0$, it follows $lim_{n\to\infty} \frac{F(f^m(z))^n}{m^n} = h(z) \quad \forall z \in Z$. For the confirmation of the additivity of h in the natural way, see [18, 22]. Actually, since T_m is a continuous t-norm then $x \to v_x$ is continuous and thus, see [34, Chapter 12],

$$\begin{aligned}
&\nu_{F(z,h^1(z),h^2(z),\ldots,h^m(z))}(t)\\
&= lim_{n\to\infty} \nu_F(m^n(z), m^n f^1(z), \ldots, m^n f^m(z))\left(\frac{t}{\varphi(\frac{1}{m^n})}\right)\\
&\geq lim_{n\to\infty} \Phi(m^n(z), m^n f^1(z), \ldots, m^n f^m(z))\left(\frac{t}{\varphi(\frac{1}{m^n})}\right) = 1
\end{aligned}$$

We conclude that $\nu_{F(z,h^1(z),h^2(z),\ldots,h^m(z))} = 1$ which implies

$$F(z, h^1(z), h^2(z), \ldots, h^m(z)) = 0.$$

The uniqueness of h is due to the verity that h is the specific fixed point of J which belongs the $\{\psi \in F : d_G(f,\psi) < \infty\}$ i.e., same with property $\nu_{h(z)-f(z)}(Ct) \geq G(z,t)$ where $C \in (0,\infty)$ and $t > 0$. □

2.1. **Probabilistic stability of Gamma functional equation.**

Throughout the subsection, we examine the stability of the following Gamma functional equation in CMP $\varphi-$ normed space

$$f(z+p) = g(z)f(z)$$

Theorem 2.2. *Let f be a function from Z into a CMP $\varphi-$ normed space (Y,ν,T_m), Z be a real vector space, $p \in R$, $g(z) \neq 0$ with $f(0) = 0$ and let $G : Z \to D^+$ be a function with the property $G(z+p, \alpha t) \geq G(z,t)$ where $\alpha \in (0, \varphi(g(z)))$, $z \in Z$ and $t > 0$. If*

$$\nu_{f(z+p)-g(z)f(z)}(t) \geq G\left(z, \frac{\varphi(g(z))-\alpha}{\varphi(g(z))} t\right)$$

then there exists one and only one function $h : Z \to Z$ such that $h(z+p) = g(z)f(z)$. Moreover,

$$h(z) = lim_{n\to\infty} \tfrac{f[(z+p)^n]}{(g[(z)])^n}$$

for all $z \in Z$ and $t > 0$.

Proof. Assume the set $F = \{h : Z \to Y | h(0) = 0\}$ and the function d_G defined as $F \times F$ by

$$d_G(h, \psi) = inf\{u \in R^+ : \nu_{h(z)-\psi(z)}(ut) \geq G(z, t)\}.$$

By lemma 1.4, we obtain (F, d_G) is a generalized metric space which is complete . Now, let us assume the linear function $J : F \to F$ defined as

$$J(h(z)) = \frac{1}{g(z)} h(z + p).$$

We prove that J is a self mapping on F which is strictly contractive together the Lipschitz constant $k = \frac{\alpha}{\varphi(g(z))}$. In fact, let h, ψ in F be such that $d_G(h, \psi) < \epsilon$. Then $\nu_{h(z)-\psi(z)}(\epsilon t) \geq G(z, t)$, hence

$$\begin{aligned} \nu_{Jh(z)-J\psi(z)}\left(\frac{\alpha}{\varphi(g(z))}\epsilon t\right) &= \frac{1}{g(z)}\nu_{h(z+p)-\psi(z+p)}\left(\frac{\alpha}{\varphi(g(z))}\epsilon t\right) \\ &= \nu_{h(z+p)-\psi(z+p)}(\alpha\epsilon t) \geq G(z + p, \alpha t). \end{aligned}$$

As $G(z + p, \alpha t) \geq G(z, t)$, after this $\nu_{Jh(z)-J\psi(z)}\big(\frac{\alpha}{\varphi(g(z))}\epsilon t\big) \geq G(z, t)$, i.e.,

$$d_G(h, \psi) < \epsilon \Rightarrow d_G(J_h, J_\psi) \leq \frac{\alpha}{\varphi(g(z))}\epsilon.$$

This yields

$$d_G(J_h, J_\psi) \leq \frac{\alpha}{\varphi(g(z))} d_G(h, \psi),\ \forall h, \psi \in E.$$

Now, it follows from

$$\nu_{g(z)f(z)-f(z+p)}(t) \geq G(z, t)$$

that $d_G(f, Jf) \leq 1$. By using Luxemburg theorem (see [17]), we derive the presence of a fixed point i.e., the existence of a function $h : Z \to Y$ in such a way that

$$h(z + p) = g(z)h(z) \text{ for all } z \in Z.$$

Also, $d_G(u, v) < \epsilon$, this indicates from $d_G(J^n f, h) \to 0$ that

$$\nu_{h(z)-v(z)}(t) \geq G\big(z, \tfrac{t}{\epsilon}\big),$$

it follows that $lim_{n\to\infty} \frac{f[(z+p)^n]}{(g[(z)])^n} = g(z)$. Also $d_G(f, h) \leq \frac{1}{1-k} d(f, Jf)$ indicates the inequality $d_G(f, h) \leq \frac{1}{1-\frac{\alpha}{\varphi(g(z))}}$ from which instantly follows

$$\nu_{h(z)-f(z)}(t) \geq G\big(z, \tfrac{\varphi(g(z))-\alpha}{\varphi(g(z))} t\big).$$

The uniqueness of h is due to the fact that h is the specific fixed point of J with the property

$$\nu_{h(z)-f(z)}(Ct) \geq G(z, t)$$

where $C \in (0, \infty)$. $\square$

Corollary 2.3. *Let f be a function from Z into a CMP $\varphi-$ normed space (Y, ν, T_M), Z be a real vector space with $f(0) = 0$ and let $G : Z \to D^+$ be a function with the property $G(z + p, \alpha t) \geq G(z, t)$ where $\alpha \in (0, \varphi(4)), z \in Z$ and $t > 0$. If*

$$\nu_{f(z+p)-4f(z)}(t) \geq G(z, t)$$

and

$$lim_{n\to\infty}\ \alpha^n \varphi\bigg(\frac{1}{2^{2n}}\bigg) = 0,$$

then the formula $h(z) = lim_{n\to\infty}\ \frac{f(z+p^n)}{2^{2n}}$ defines one and only one function h : $Z \to Z$ in such a way that $\nu_{h(z)-f(z)}(t) \geq G(z, Mt)$ where $M = \frac{\varphi(4)-\alpha}{\varphi(4)}$.

Proof. By setting $z = p$, we obtain $\nu_{f(2z)-4f(z)}(t) \geq G(z, t)$ hence $\nu_{\frac{1}{4}f(2z)-f(z)} \geq G(z, t)$ where

$$G(z, t) = G\bigg(z, \frac{t}{\varphi(\frac{1}{4})}\bigg).$$

□

2.2. **Probabilistic stability of the Schröder functional equation.**

Throughout the subsection, we examine the stability of the following Schröder functional equation in CMP $\varphi-$ normed space

$$f(g(z)) = sf(z)$$

Theorem 2.4. *Let f be a function from Z into a CMP φ-normed space (Y, ν, T_m), Z to be a real vector space, $s \neq 0$ with $f(0) = 0$ and let $G : Z \to D^+$ be a function with the property $\exists\ \alpha \in (0, \varphi(s))$ for all $z \in Z, t > 0$ in such a way that*

$$G(g(z), \alpha t) \geq G(z, t). \tag{2.4}$$

If

$$\nu_{f(g(z))-sf(z)}(t) \geqslant G(z, t)$$

then there exists one and only one function $h : Z \to Y$ in such a way that $h(g(z)) = sh(z)$ and

$$\nu_{h(z)-f(z)}(t) \geqslant G\bigg(z, \frac{\varphi(s) - \alpha}{\varphi(s)}\bigg).$$

Moreover, $h(z) = lim_{n\to\infty} \frac{f(g(z)^n)}{s^n}$.

Proof. Assume the set $F = \{h : Z \to Y : h(0) = 0\}$ and the function d_G defined as $F \times F$ by

$$d_G(h, \psi) = inf\{u \in \mathbb{R}^+ : \nu_{h(z)-\psi(z)}(ut) \geqslant G(z, t)\forall z \in Z, t > 0\}.$$

By Lemma 1.4, we obtain (F, d_G) is a generalized metric space which is complete. Now , let us assume the linear function J : $F \times F$ defined by

$$J_{h(z)} = \frac{1}{s}h(g(z)).$$

We prove that J is a self mapping on F which is strictly contractive together the Lipschitz constant $k = \frac{\alpha}{\varphi(s)}$. Let h, ψ be functions lies in F gives that $d_G(h,\psi) < \epsilon$. After this, for all $z \in Z, t > 0$, we obtain $\nu_{h(z)-\psi(z)}(\epsilon t) \geqslant G(z,t)$, hence

$$\begin{aligned}\nu_{Jh(z)-J\psi(z)}\left(\frac{\alpha}{\varphi(s)}\epsilon t\right) &= \nu_{\frac{1}{s}(h(g(z))-\psi(g(z)))}\left(\frac{\alpha}{\varphi(s)}\epsilon t\right)\\ &= \nu_{h(g(z))-\psi(g(z))}(\alpha\epsilon t)\\ &\geqslant G(g(z),\alpha t).\end{aligned}$$

Since $G(g(z),\alpha t) \geqslant G(z,t)$ then $\nu_{Jh(z)-J\psi(z)}\big(\frac{\alpha}{\varphi(s)}\epsilon t\big) \geqslant G(z,t)$ that is,

$$d_G(h,\psi) < \epsilon \Rightarrow d_G(J_h, J_\psi) \leqslant \frac{\alpha}{\varphi(s)}\epsilon.$$

This implies that

$$d_G(J_h, J_\psi) \leqslant \frac{\alpha}{\varphi(s)} d_G(h,\psi),$$

for all h, ψ in E. Now $\nu_{f(z)-\frac{1}{s}f(g(z))}(t) \geqslant G(z,t)$ it follows that $d_G(f, Jf) \leqslant 1$. Using Luxemburg theorem(see [17]), we derive the existence of a fixed point of J i.e., the existence of function $h : Z \to Y$ in such a way that $h(g(z)) = sh(z)$ for all $z \in Z$ and

$$d_G(u,v) < \epsilon \Rightarrow \nu_{u(z)-v(z)}(t) \geqslant G\left(z, \frac{t}{\epsilon}\right),$$

from $d_G(J^n f, h) \to 0$, it follows that $lim_{n\to\infty}\frac{f((g(z))^n)}{s^n}$, for any $z \in Z$. Also , $d_G(f,h) \leqslant \frac{1}{1-k} d(f, Jf)$ implies $d_G(f,h) \leqslant \frac{1}{1-\frac{\alpha}{\varphi(s)}}$ from which it instantly follows $\nu_{h(z)-f(z)}\big(\frac{\varphi(s)}{\varphi(s)-\alpha}\big) \geqslant G(z,t)$. By means of this

$$\nu_{h(z)-f(z)}(t) \geqslant G\left(z, \frac{\varphi(s)-\alpha}{\varphi(s)}t\right).$$

The uniqueness of h is due to the verity that h is the specific fixed point of J with the property : there is $C \in (0,\infty)$ in such a way that

$$\nu_{h(z)-f(z)}(Ct) \geqslant G(z,t).$$

□

3. Probabilistic stability for the QF and SF equations

Recall that the functional equation

$$\begin{aligned}&f(z+3y) - 5f(z+2y) + 10f(z+y) - 10f(z)\\ &+5f(z-y) - f(z-2y) = 120f(y)\end{aligned} \tag{3.1}$$

is called QF equation as $f(z) = cz^5$ is a solution. In [38], Xu et. al. firstly studied the stability problem for the QF equation. The functional equation

$$\begin{aligned}&f(z+3y) - 6f(z+2y) + 15f(z+y) - 20f(z) +\\ &15f(z-y) - 6f(z-2y) + f(z-3y) = 720f(y)\end{aligned} \tag{3.2}$$

is called SF equation since $f(z) = cz^6$ is a solution. In [38], Xu et. al. firstly studied the stability problem for the SF equation. Later, the stability of QF and SF equations have been established by several mathematicians [16, 33, 39, 40].

Theorem 3.1. *Let f be a function from Z into a CMP φ-normed space (Y, ν, T_M), Z be a real vector space with f(0)= 0 and Φ be a function from Z^2 to D^+ ($\Phi_{(z,y)}$ is denoted by $\Phi_{z,y}$) in such a way that, for some $0 < \alpha < \varphi(32)$,*

$$\Phi_{(2z,2y)}(\alpha t) \geqslant \Phi_{z,y}(t). \tag{3.3}$$

If

$$\nu_{f(z+3y)-5f(z+2y)+10f(z+y)-10f(z)+5f(z-y)-f(z-2y)-120f(y)}(t) \geqslant \Phi_{z,y}(t) \tag{3.4}$$

for all $z, y \in Z$ and

$$lim_{n\to\infty}\alpha^n\varphi\left(\frac{1}{2^{5n}}\right) = 0 \tag{3.5}$$

then the formula $h(z) = lim_{n\to\infty}\frac{f(2^nz)}{2^{5n}}$ defines one and only one quintic function $h : Z \to Y$ in such a way that

$$\nu_{h(z)-f(z)}(t) \geqslant \Phi_{z,z}(Mt) \tag{3.6}$$

where $M = \frac{\varphi(32)-\alpha}{\varphi(32)\varphi\left(\frac{1}{64}\right)}$.

Proof. By putting $z = y$ in (3.1) , we obtain

$$\nu_{f(4z)-5f(3z)+10f(2z)-10f(z)-f(-z)-120f(z)}(t) \geq \Phi_{z,z}(t).$$

It follows that

$$\nu_{f(4z)-5f(3z)+10f(2z)-10f(z)-f(-z)-120f(z)}(t) \geqslant G(z, t)$$

where $G(z, t) = \Phi_{z,z}\left(\frac{t}{\varphi(\frac{1}{64})}\right)$. From theorem (2.2), we infer that

$$h(z) = lim_{n\to\infty}\frac{f(2^nz)}{2^{5n}}$$

is the unique function $h : Z \to Y$ in such a way that $h(2z) = 2^5h(z)$ and

$$\nu_{h(z)-f(z)}(t) \geqslant \Phi_{z,z}\left(\frac{\varphi(32)-\alpha}{\varphi(32)\varphi(\frac{1}{64})}\right).$$

It is sufficient to show the mapping h is quintic, when h is a solution of quintic equation. We have

$$\begin{aligned}
\nu_{h(z)+h(y)-h(z+y)}(t) \quad & \geqslant Min\{\nu_{h(z)-\frac{f(2^nz)}{2^{5n}}}(\frac{t}{4}), \\
& \nu_{h(y)-\frac{f(2^ny)}{2^{5n}}}(\frac{t}{4}), \\
& \nu_{h(z+y)-\frac{f(2^n(z+y))}{2^{5n}}}(\frac{t}{4}), \\
& \nu_{\frac{f(2^n(z+y))}{2^{5n}}-\frac{f(2^nz)}{2^{5n}}-\frac{f(2^ny)}{2^{5n}}}(\frac{t}{4})\}.
\end{aligned}$$

The first three terms on R.H.S. of the above inequality approaches to 1 as $n \to \infty$. Furthermore, let us observe from (3.3) it instantly follows by mathematical induction on n that $\Phi_{2^nz,2^ny}(\alpha^nt) \geqslant \Phi_{z,y}(t)$, hence

$$\Phi_{2^nz,2^ny}(t) \geqslant \Phi_{z,y}\left(\frac{t}{\alpha^nt}\right). \tag{3.7}$$

Then by using (3.4), we obtain

$$\nu_{\frac{f(2^n(z+y))}{2^{5n}}-\frac{f(2^nz)}{2^{5n}}-\frac{f(2^ny)}{2^{5n}}}\left(\frac{t}{4}\right)=\nu_{\frac{f(2^n(z+y))}{2^{5n}}-\frac{f(2^nz)}{2^{5n}}-\frac{f(2^ny)}{2^{5n}}}\left(\frac{t}{4\varphi(\frac{1}{2^{5n}})}\right)$$
$$\geqslant \Phi_{2^nz,2^ny}\left(\frac{t}{4\varphi(\frac{1}{2^{5n}})}\right)$$
$$\geqslant \Phi_{z,y}\left(\frac{t}{4\alpha^n\varphi(\frac{1}{2^{5n}})}\right).$$

From (3.7) we derive that the fourh term also approaches to 1 when n approaches to ∞, achieving h is quintic. □

Theorem 3.2. *Let f be a function from Z into a CMP $\varphi-$normed space (Y,ν,T_M), Z be a real vector space with $f(0)=0$ and let $\Phi : Z^2 \to D^+$ be a function with the property $\exists\ \alpha \in (0,\varphi(2^6))\ \forall z,y \in Z, t>0$ such that*

$$\Phi_{2z,2y} \geqslant \Phi_{z,y}(t). \tag{3.8}$$

If

$$\nu_{f(z+3y)-6f(z+2y)+15f(z+y)-20f(z)+15f(z-y)-6f(z-2y)+f(z-3y)-720f(y)}(t)$$
$$\geq \Phi_{z,y}(t) \tag{3.9}$$

and

$$lim_{n\to\infty}\ \alpha^n\varphi\left(\frac{1}{2^{6n}}\right)=0 \tag{3.10}$$

then the formula $h(z)=lim_{n\to\infty}\ \frac{f(2^nz)}{2^{6n}}$ defines one and only one sextic function $h: Z\to Y$ such that $\nu_{h(z)-f(z)}(t)\geq \Phi_{z,z}(Mt)$ where

$$M=\frac{\varphi(64)-\alpha}{\varphi(64)\varphi(\frac{1}{128})}.$$

Proof. By putting $z=y$ in (3.2), we obtain

$$\nu_{f(z+3y)-6f(z+2y)+15f(z+y)-20f(z)+15f(z-y)-6f(z-2y)+f(z-3y)-720f(y)}(t)$$
$$\geq \Phi_{z,z(t)},$$

hence

$$\nu_{f(z+3y)-6f(z+2y)+15f(z+y)-20f(z)+15f(z-y)-6f(z-2y)+f(z-3y)-720f(y)}(t)$$
$$\geq \Phi_{z,z}\left(\frac{t}{\varphi(\frac{1}{128})}\right).$$

Let $G(z,t)=\Phi_{z,z}\left(\frac{t}{\varphi(\frac{1}{128})}\right)$. From theorem 2.2 it follows the presence of a unique function $h: Z\to Y$ in such a way that $h(2z)=2^6h(z)$, for all $z\in Z$ and

$$\nu_{h(z)-f(z)}(t)\geq \Phi_{z,z}\left(\frac{\varphi(2^6)-\alpha}{\varphi(2^6)\varphi(\frac{1}{128})}\right).$$

Moreover, $lim_{n\to\infty}\ \frac{f(2^nz)}{2^{6n}}$. The proof of the fact that h has a sextic function is similar to the proof of the linearity in the preceeding theorem. □

4. PARTICULAR CASES

For specific choices of φ, Φ and ν, one can acquire stability theorems for different functional equations in RN-spaces or in linear normed spaces.

Theorem 4.1. *Let (Y, ν, T_M) be a CMP $\varphi-$ normed space, Z be a real vector space and Φ be a function from Z^2 to D^+ in such a way that, for some $(0 < \alpha < 32), \Phi_{2z,2y}(\alpha t) \geq \Phi_{z,y}(t)$ for all $z \in Z, t > 0$. If $f : Z \to Y$ is a function with $f(0) = 0$ and*

$$\nu_{f(z+3y)-5f(z+2y)+10f(z+y)-10f(z)+5f(z-y)-f(z-2y)-120f(y)}(t) \geq \Phi_{x,y}(t).$$

then there exists one and only one quintic function $h : Z \to Y$ in such way that

$$\nu_{f(z)-h(z)}(t) \geq \Phi_{z,0}(2(32-\alpha)t).$$

Proof. The completion follows by assuming $\varphi(\alpha) = |\alpha|$ in theorem 3.1 (we observe that $\frac{\varphi(32)-\alpha}{\varphi(32)\varphi(\frac{1}{64})} = \varphi(2(32-\alpha)))$. The condition $lim_{n\to\infty}\ \alpha^n \varphi\left(\frac{1}{2^{5n}}\right) = 0$ is fulfilled, as it diminishes to

$$lim_{n\to\infty} \left(\frac{\alpha}{32}\right)^n = 0.$$

□

Theorem 4.2. *Let (Y, ν, T_M) be a complete RN-space, $(Z, \|.\|)$ be a real normed linear space and q be non negative real number. If $f : Z \to Y$ is a function in such a way that*

$$\begin{aligned} &\nu_{f(z+3y)-5f(z+2y)+10f(z+y)-10f(z)+5f(z-y)-f(z-2y)-120f(y)}(t) \\ &\geq \frac{t}{t + \|z\|^q + \|y\|^q} \end{aligned} \tag{4.1}$$

and $1 < q < 5$, then there exists one and only one quintic function $h : Z \to Y$ in such a way that

$$\nu_{f(z)-h(z)}(t) \geq \frac{(32^q-2)t}{((32^q-2)t + 2^{-q}\|z\|^q)} \qquad \forall z \in Z, t > 0. \tag{4.2}$$

Proof. Cosider the function $\Phi : Z^2 \to D^+$ defined by

$$\Phi_{z,y}(t) = \frac{t}{t + \|\|\|^q + \|y\|^q}$$

and let $\varphi(t) = |t|^q \quad (t \in \mathbb{R})$, where $1 < q < 5$, $\alpha = 32$. It is instant that $0 < 32 < \varphi(32)$, $\Phi_{2z,2y}(\alpha t) \geq \Phi_{z,y}(t)$ and

$$lim_{n\to\infty}\ \alpha^n \varphi\left(\frac{1}{32^n}\right) = lim_{n\to\infty}\ 32^{(1-q)n} = 0.$$

Now the completion follows from theorem 3.1. □

Theorem 4.3. *Let $(Z, \|.\|)$ be a real normed vector space, (Y, ν, T_M) be a complete RN-space and q be non negative real number. If $f : Z \to Y$ is a function such that*

$$\begin{aligned} &\nu_{f(z+3y)-5f(z+2y)+10f(z+y)-10f(z)+5f(z-y)-f(z-2y)-120f(y)}(t) \\ &\geq \frac{t}{t + \|z\|^q + \|y\|^q} \end{aligned} \tag{4.3}$$

and $\frac{1}{5} < q < 1$, then there exists one and only one quintic function $h : Z \to Y$ such that

$$\nu_{f(z)-h(z)}(t) \geq \frac{(32^q - 2)t}{((32^q - 2)t + 2^{-q}\|z\|^q)} \qquad \forall z \in Z, t > 0. \tag{4.4}$$

Proof. Cosider the function $\Phi : Z^2 \to D^+$ defined through

$$\Phi_{z,y}(t) = \frac{t}{t + \|z\|^q + \|y\|^q}$$

and let $\varphi(t) = |t|^q$, $(t \in R)$, where $\frac{1}{5} < q < 1, \alpha = 2$. It is instant that $0 < 2 < \varphi(32)$, $\Phi_{2z,2y}(\alpha t) \leq \Phi_{z,y}(t) \qquad \forall z \in Z, t > 0$ and

$$lim_{n\to\infty}\ \alpha^n \varphi\left(\frac{1}{32^n}\right) = lim_{n\to\infty}\ 2^{(1-5q)n} = 0.$$

Now the completion follows from the theorem 3.1. □

Corollary 4.4. *([16:Theorem 2], with $\delta = 0, \theta = 2$) Let $\frac{1}{5} < q < 1$ be fixed and $f : Z \to Y$ be a function between real Banach spaces that satisfies the inequality*

$$\begin{aligned} &\|f(z+3y) - 5f(z+2y) + 10f(z+y) \\ &-10f(z) + 5f(z-y) - f(z-2y) - 120f(y)\| \\ &\leq \|z\|^q + \|y\|^q \end{aligned}$$

for all $z, y \in X$, then there exists one and only one quintic function $h : Z \to Y$ in such a way that

$$\|f(z) - h(z)\| \leq \frac{2^{-q}}{32^q - 2}\|z\| \qquad \forall z \in Z.$$

Proof. Consider the induced RN-space (Z, ν, T_M), where $\nu_z(t) = \frac{t}{t+\|z\|^q}$. Then (4.3) is equivalent to

$$\begin{aligned} &\|f(z+3y) - 5f(z+2y) + 10f(z+y) - 10f(z) \\ &+5f(z-y) - f(z-2y) - 120f(y)\| \\ \leq\ &\|z\|^q + \|y\|^q, \end{aligned}$$

while (4.4) is identical to

$$\|f(z) - h(z)\| \leq \frac{2^{-q}}{32^q - 2}\|z\|.$$

□

Theorem 4.5. *Let (Y, ν, T_M) be a CMP $\varphi-$ normed space, Z be a real vector space and Φ be a function from Z^2 to D^+ in such a way that, for some $(0 < \alpha < 64)$, $\Phi_{2z,2y}(\alpha t) \geq \Phi_{z,y}(t)$. If $f : Z \to Y$ is a function with $f(0) = 0$ and*

$$\nu_{f(z+3y)-6f(z+2y)+15f(z+y)-20f(z)+15f(z-y)-6f(z-2y)+f(z-3y)-720f(y)}(t) \geq \Phi_{z,y}(t).$$

then there exists one and only one sextic function $h : Z \to Y$ in such a way that

$$\nu_{f(z)-h(z)}(t) \geq \Phi_{z,0}(2(64 - \alpha)t) \qquad \forall z \in Z, t > 0.$$

Proof. The completion follows by assuming $\varphi(\alpha) = |\alpha|$ in theorem 3.1 (we observe that $\frac{\varphi(64)-\alpha}{\varphi(64)\varphi(\frac{1}{128})} = \varphi(2(64-\alpha))$). The condition

$$lim_{n\to\infty}\alpha^n\varphi\left(\frac{1}{2^{6n}}\right) = 0$$

is fulfilled, as it diminishes to $lim_{n\to\infty}\left(\frac{\alpha}{64}\right)^n = 0$. □

Theorem 4.6. *Let (Y, ν, T_M) be a complete RN-space, $(Z, \|.\|)$ be a real normed vector space and q be non negative real number. If $f : Z \to Y$ is a function in such a way that*

$$\nu_{f(z+3y)-6f(z+2y)+15f(z+y)-20f(z)+15f(z-y)-6f(z-2y)+f(z-3y)-720f(y)}(t) \geq \frac{t}{t+\|z\|^q+\|y\|^q} \tag{4.5}$$

and $1 < q < 6$, then there exists one and only one sextic function $h : Z \to Y$ in such a way that

$$\nu_{f(z)-h(z)}(t) \geq \frac{(64^q-2)t}{((64^q-2)t+2^{-q}\|z\|^q)} \qquad \forall z \in Z, t > 0. \tag{4.6}$$

Proof. Cosider the function $\Phi : Z^2 \to D^+$ defined by

$$\Phi_{z,y}(t) = \frac{t}{t+\|z\|^q+\|y\|^q}$$

and let $\varphi(t) = |t|^q \quad (t \in \mathbb{R})$, where $1 < q < 6$, $\alpha = 64$. It is instant that $0 < 64 < \varphi(64)$, $\Phi_{2z,2y}(\alpha t) \geq \Phi_{z,y}(t)$ and

$$lim_{n\to\infty}\ \alpha^n\varphi\left(\frac{1}{64^n}\right) = lim_{n\to\infty}\ 64^{(1-q)n} = 0.$$

Now the completion follows from Theorem 3.2. □

Theorem 4.7. *Let (Y, ν, T_M) be a complete RN-space, $(Z, \|.\|)$ be a real normed vector space and q be non negative real number. If $f : Z \to Y$ is a function in such a way that*

$$\nu_{f(z+3y)-6f(z+2y)+15f(z+y)-20f(z)+15f(z-y)-6f(z-2y)+f(z-3y)-720f(y)}(t) \geq \frac{t}{t+\|z\|^q+\|y\|^q} \tag{4.7}$$

and $\frac{1}{6} < q < 1$, then there exists one and only one sextic function $h : Z \to Y$ in such a way that

$$\nu_{f(z)-h(z)}(t) \geq \frac{(64^q-2)t}{((64^q-2)t+2^{-q}\|z\|^q)} \qquad \forall z \in Z, t > 0. \tag{4.8}$$

Proof. Cosider the function $\Phi : Z^2 \to D^+$ defined by

$$\Phi_{z,y}(t) = \frac{t}{t+\|z\|^q+\|y\|^q}$$

and let $\varphi(t) = |t|^q$, $(t \in \mathbb{R})$, where $\frac{1}{6} < q < 1, \alpha = 2$. It is instant that $0 < 2 < \varphi(64)$, $\Phi_{2z,2y}(\alpha t) \leq \Phi_{z,y}(t)$ and

$$lim_{n\to\infty}\alpha^n\varphi\left(\frac{1}{32^n}\right) = lim_{n\to\infty}\ 2^{(1-6q)n} = 0.$$

Now the completion follows from the theorem 3.2. ☐

Corollary 4.8. *([16:Theorem 3], with $\delta = 0, \theta = 2$) Let $\frac{1}{6} < q < 1$ be fixed and $f : Z \to Y$ be a function between real Banach spaces that satisfies the inequality*

$$\begin{aligned}&\|f(z+3y) - 6f(z+2y) + 15f(z+y) - 20f(z)\\&+15f(z-y) - 6f(z-2y) + f(z-3y) - 720f(y)\|\\&\leq \|z\|^q + \|y\|^q\end{aligned}$$

for all $z, y \in Z$, then there exists one and only one sextic function $h : Z \to Y$ in such a way that

$$\|f(z) - h(z)\| \leq \frac{2^{-q}}{64^q - 2}\|z\| \qquad \forall z \in Z.$$

Proof. Consider the induced RN-space (Z, ν, T_M), where $\nu_z(t) = \frac{t}{t+\|z\|^q}$. Then (4.7) is equivalent to

$$\begin{aligned}&\|f(z+3y) - 6f(z+2y) + 15f(z+y) - 20f(z)\\&+15f(z-y) - 6f(z-2y) + f(z-3y) - 720f(y)\|\\&\leq \|z\|^q + \|y\|^q\end{aligned}$$

while (4.8) is identical to $\|f(z) - h(z)\| \leq \frac{2^{-q}}{64^q-2}\|z\|$. ☐

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