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LACUNARY STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES IN TOPOLOGICAL GROUPS

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Abstract. In this paper, we introduce a class of summability methods that can be applied to lacunary double statistical convergence in topological groups and we also prove some theorems .

1. Introduction

We recall that the concept of statistical convergence of sequences was first introduced by Fast [4] as an extension of the usual concept of sequential limits and also independently Schoenberg [16] for real and complex sequences. Maddox [7] extended statistical convergence to locally convex Hausdorff topological linear spaces in terms of strong summability and further in [6], statistical convergence to normed spaces was extended by Kolk [6]. Also in [1] and [2], Çakalli extended this notation to topological Hausdorff groups. Savas [15] introduced lacunary statistical convergence of double sequences in topological groups. Also double ideal lacunary statistical convergence in topological groups was studied by Savas (see, [14]). Note that, generalized double statistical convergence in topological groups is considered by Savas, (see, [13]). More results on double statistical convergence can be seen from [3, 11, 12].

The notion of the statistical convergence depends on the density of subsets of $\mathbb{N}$ A subset E of $\mathbb{N}$ is said to have density $\delta(E)$ if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k)$$

exist.

Note that if $K \subset \mathbb{N}$ is a finite set, then $\delta(K) = 0$, and for any set $K \subset \mathbb{N}$, $\delta(K^c) = 1 - \delta(K)$.

A sequence $x = (x_k)$ is statistically convergent to ξ if

$$\delta(\{k \in \mathbb{N} : |x_k - \xi| \geq \varepsilon\}) = 0$$

for every $\varepsilon > 0$, (see [5]). In this case ξ is called the statistical limit of x.

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By X, we will denote an abelian topological Hausdorff group, written additively, which satisfies the first axiom of countability. For a subset B of X, $s(B)$ will denote the set of all sequences $x = (x_k)$ such that x_k is in B for $k = 1, 2, ..., c(X)$ will denote the set of all convergent sequences.

A sequence $x = (x_k)$ in X is called to be statistically convergent to an element ξ of X if for each neighborhood U of 0, (see, [2])

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : x_k - \xi \notin U\}| = 0$$

and is called statistically Cauchy in X if for each neighborhood U of 0 there exists a positive integer $N = N(\varepsilon)$, depending on the neighborhood U such that

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : x_k - x_N \notin U\}| = 0$$

where the vertical bars indicate the number of elements in the enclosed set. The set of all statistically convergent sequences in X is denoted by $S(X)$ and the set of all statistically Cauchy sequences in X is denoted by $SC(X)$. It is known that $SC(X) = S(X)$ if X is complete.

By a lacunary sequence, we mean an increasing sequence $\theta = (k_r)$ of positive integers such that $k_0 = 0$ and $h_r : k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper, the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$.
Lacunary statistical convergence in topological groups was defined by Cakalli [1] as follows: A sequence $x = (x_k)$ is said to be $S_\theta-$convergent to ξ (or lacunary statistically convergent to ξ) if for each neighborhood U of 0,

$$lim_{r\to\infty} \frac{1}{h_r} |k \in I_r : x_k - \xi \notin U\}| = 0.$$

In a topological group X, by the convergence of a double sequence we mean the convergence in Pringsheims sense [10]. A double sequence $x = (x_{kl})$ in X is said to be convergent to a point ξ in X in the Pringsheims sense if for every neighborhood U of 0 there exists $N \in N$ such that $x_{kl} - \xi \in U$ whenever $k, l \ge N$. ξ is called the Pringsheim limit of x. A double sequence $x = (x_{kl})$ of points in X is said to be a Cauchy sequence if for every neighborhood U of 0 there exists two positive integers $N = N(\varepsilon)$ and $M = M(\varepsilon)$, depending on the neighborhood U such that $x_{kl} - x_{NM} \in U$.

The goal of this paper is to introduce the statistical convergence of double sequences in topological groups and to prove some useful theorems.

2. Definitions and Notation

The double sequence $\theta = \{(k_r, l_s)\}$ is called **double lacunary** if there exist two increasing of integers such that

$$k_0 = 0, h_r = k_r - k_{k-1} \to \infty \text{ as } r \to \infty$$

and

$$l_0 = 0, \bar{h}_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty.$$

Notations: $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \bar{h}_s$, θ is determined by $I_r = \{(k) : k_{r-1} < k \le k_r\}$, $I_s = \{(l) : l_{s-1} < l \le l_s\}$, $I_{r,s} = \{(k,l) : k_{r-1} < k \le k_r \;\&\; l_{s-1} < l \le l_s\}$, $q_r = \frac{k_r}{k_{r-1}}$, $\bar{q}_s = \frac{l_s}{l_{s-1}}$, and $q_{r,s} = q_r \bar{q}_s$. We will denote the set of all double lacunary sequences by $\mathbf{N}_{\theta_{r,s}}$.

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ has double lacunary density $\delta_2^{\theta}(K)$ if

$$P - lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : (k,l) \in K\}|$$

exists.

In 2005, R. F. Patterson and E. Savas [9] studied double lacunary statistically convergence by giving the definition for complex sequences as follows:

Definition 1. *Let θ be a double lacunary sequence; the double number sequence x is double lacunary statistical convergent to ξ provided that for every $\varepsilon > 0$,*

$$P - \lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : |x_{kl} - \xi| \geq \varepsilon\}| = 0.$$

In this case write $st_{\theta}^2 - \lim x = \xi$ or $x_{kl} \to \xi(st_{\theta}^2)$..

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be a two dimensional set of positive integers and let $K_{m,n}$ be the numbers of (i,j) in K such that $i \leq n$ and $j \leq m$. Then the two-dimensional analogue of natural case density can be defined as follows: The lower asymptotic density of K is defined as

$$P - \liminf_{m,n} \frac{K_{m,n}}{mn} = \delta_2(K).$$

In the case when the sequence $\{\frac{K_{m,n}}{mn}\}_{m,n=1,1}^{\infty,\infty}$ has a limit then we say that K has a natural density and is defined as

$$P - \lim_{m,n} \frac{K_{m,n}}{mn} = \delta_2(K).$$

Recently the studies of double sequences has a rapid growth. The concept of double statistical convergence, for complex case, was introduced by Mursaleen and Edely [8] while the idea of statistical convergence of single sequences was first studied by Fast [4]. Mursaleen and Edely has presented the double statistical convergence as follows:

Definition 2. *A double sequences $x = (x_{kl})$ is said to be P-statistically convergent to ξ provided that for each $\varepsilon > 0$*

$$P - \lim_{m,n} \frac{1}{mn} \{ \text{ number of } (k,l) : k < m \text{ and } l < n, |x_{kl} - \xi| \geq \varepsilon\} = 0.$$

In this case we write $st^2 - \lim_{k,l} x_{kl} = \xi$ and we denote the set of all statistical convergent double sequences by st^2.
Recently, statistical convergence of double sequences $x = (x_{kl})$ in a topological group was presented by Cakalli and Savas [3] as follows:

A sequence $x = (x_{kl})$ is called double statistically convergent to a point ξ of X if for each neighborhood U of 0, the set

$$\{(k,l), k \leq n; and; l \leq m : x_{kl} - \xi \notin U\}$$

has double natural density zero. In this case we write $S_2(X) - \lim_{k,l} x_{kl} = \xi$ and we write the set of all statistically convergent double sequences by $S_2(X)$.

Definition 3. *(See, [15]). A sequence $x = (x_{kl})$ is said to be S_{θ}^2-convergent to ξ of X (or lacunary double statistically convergent to ξ of X) if for each neighborhood U of 0, the set*

$$\{(k,l) \in I_{rs} : x_{kl} - \xi \notin U\}$$

has double natural density zero. In this case, we write

$$S_\theta^2 - \lim_{k,l\to\infty} x_{kl} = \xi \quad or \quad x_{kl} \to \xi(S_\theta^2)$$

and we write the set of all double lacunary statistically convergent sequences by $S_\theta^2(X)$.

3. Main theorems

Theorem 3.1. *A double sequence $x = (x_{kl})$ in X is double lacunary statistically convergent to ξ if and only if there exists a subset $K \subset \mathbb{N}\times\mathbb{N}$ such that $\delta_\theta^2(K) = 1$ and $\lim_{k,l\to\infty} x_{kl} = \xi$ where limit is being taken over the set X, i.e., $(k,l) \in X$.*

Proof. Necessity. Suppose that $x = (x_{kl})$ be double lacunary statistically convergent to ξ, and (U_i) be a base of nested closed neighborhoods of 0. Write

$$K_i = \{(k,l) \in I_{rs} : x_{kl} - \xi \notin U_i\}$$

$$M_i = \{(k,l) \in I_{rs} : x_{kl} - \xi \in U_i\} \qquad (i = 1, 2, ...)$$

Then $\delta_\theta^2(K_i) = 0$ and

$$M_1 \supset M_2 \supset ... \supset M_i \supset M_{i+1} \supset ... \tag{3.1}$$

and

$$\delta_\theta^2(M_i) = 1,\ i = 1, 2, ... \tag{3.2}$$

Now we shall show that for $(k,l) \in M_i$, (x_{kl}) is double lacunary statistical convergent to ξ. Assume that $x = (x_{kl})$ is not double lacunary statistical convergent to ξ so that there is a neighborhood U of 0 such that

$$x_{kl} - \xi \notin U$$

for in finitely many terms. Let $U_i \subset U$ $(i = 1, 2, ...)$ and $M_U = \{(k,l) : x_{kl} - \xi \in U\}$. Then

$$\delta_\theta^2(M_U) = 0$$

and by (3.1), $M_i \subset M_U$. Hence $\delta_\theta^2(M_i) = 0$ which is a contradiction to (3.2). Thus $x = (x_{kl})$ is convergent to ξ.

Sufficiency: Suppose that there exists a subset $K = \{(k,l)\} \subseteq \mathbb{N}\times\mathbb{N}\}$ such that $\delta_\theta^2(K) = 1$ and $\lim_{kl} x_{kl} = \xi$, i.e., there exists an $(n_o, m_o) \in \mathbb{N}\times\mathbb{N}$ such that for each neighborhood U of 0,

$$x_{kl} - \xi \in U \text{ for every } k \geq n_o, l \geq m_o.$$

Now

$$K_U = \{(k,l) : x_{kl} - \xi \notin U\} \subseteq \mathbb{N}\times\mathbb{N}\backslash\{(k_{i_o+1}, l_{i_o+1}), (k_{i_o+2}, l_{i_o+2}), ...\}.$$

Therefore

$$\delta_\theta^2(K_U) \leq 1 - 1 = 0.$$

It follows that x is double lacunary statistically convergent to ξ. □

Corollary 3.2. *If a double sequence $x = (x_{kl})$ is double lacunary statistically convergent to ξ, then there exists a double sequence (y_{kl}) such that $\lim_{k,l} y_{kl} = \xi$ and $\delta_\theta^2\{(k,l) : x_{kl} = y_{kl}\} = 1$, i.e., $x_{kl} = y_{kl}$ for almost all (k,l).*

In a topological group, double sequence $x = (x_{kl})$ is called double lacunary statistically Cauchy if for each neighborhood U of 0 there exists $N = N(U)$ and $M = M(U)$ such that the set $\{(k,l) \in I_{rs} : x_{kl} - x_{NM} \notin U\}$ has double natural density zero. In this case, we denote the set of all double lacunary statistically Cauchy sequences by $S_\theta^2 C(X)$.

Theorem 3.3. *Let X be complete topological group. A double sequence $x = (x_{kl})$ in X is double lacunary statistically convergent if and only if $x = (x_{kl})$ is double lacunary statistically Cauchy.*

We need the following lemma to prove the theorem.

Lemma 3.4. *Let be X a topological vector space over the field F. So if W is a neighborhood of 0 in X then there is a neighborhood U of 0 which is symmetric (that is $U = -U$) and which satisfies $W + W \subset U$.*

Proof. Let $x = (x_{kl})$ be double lacunary statistically convergent to ξ. Let U be any neighborhood of 0. Then we may choose a symmetric neighborhood W of 0 such that

$$W + W \subset U.$$

Then for this neighborhood W of 0, the set

$$\{(k,l) \in I_{rs} : x_{kl} - \xi \in W\}$$

has double natural lacunary density 0. For each neighborhood U of 0, the set $\{(k,l) \in I_{rs} : x_{kl} - \xi \notin U\}$ has double natural lacunary density zero. Then we may choose natural numbers M and N such that $x_{MN} - \xi \notin U$. Now write

$$A_U = \{(k,l) \in I_{rs} : x_{kl} - x_{MN} \notin U\}$$
$$B_W = \{(k,l) \in I_{rs} : x_{kl} - \xi \notin W\}$$
$$C_W = \{(M,N) \in I_{rs} : x_{MN} - \xi \notin W\}$$

Then $A_U \subset B_W \cup C_W$ and hence $\delta_\theta^2(A_U) \le \delta_\theta^2(B_W) + \delta_\theta^2(C_W) = 0$. Therefore we get that x is lacunary statistically Cauchy.

To prove the converse, suppose that there is a double lacunary statistically Cauchy sequence x but it is not double lacunary statistically convergent. Then we may find natural numbers M and N such that the set A_U has double natural lacunary density zero. It follows from this that the set

$$Z_U = \{(k,l) \in I_{rs} : x_{kl} - x_{MN} \in U\}$$

has double lacunary natural density 1. Therefore we may choose a symmetric neighborhood W of 0 such that $W + W \subset U$. Now take any fixed non-zero element ξ of X. Let $x_{kl} - x_{MN} = x_{kl} - \xi + \xi - x_{MN}$. It follows from this equality that $x_{kl} - x_{MN} \in U$ if $x_{kl} - \xi \in W$. Since x is not double lacunary statistically convergent to ξ, the set C_W has double lacunary natural density 1., i.e., the set $\{(k,l) \in I_{rs} : x_{kl} - \xi \notin W\}$ has double lacunary natural density 0. Hence the set $\{(k,l) \in I_{rs} : x_{kl} - x_{MN} \in U\}$ has double lacunary natural density 0, i.e., the set A_U has double lacunary natural density 1 which is a contradiction. Hence this completes the proof. □

Finally we conclude this paper by stating the following theorem which is from theorems 1 and 2 and the proof is easy and omitted.

Theorem 3.5. *If X is complete topological group, then the following conditions are equivalent:*

(a): x is double lacunary statistically convergent to ξ;
(b): x is double lacunary statistically Cauchy;
(c): there exists a subsequence y of x such that $\lim_{k,l} y_{kl} = \xi$.

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