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UNCERTAINTY PRINCIPLE FOR THE FOURIER-LIKE MULTIPLIERS OPERATORS IN q-RUBIN SETTING

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Abstract. The aim of this project is establish the Heisenberg-Pauli-Weyl uncertainty principle and Donoho-Stark's uncertainty principle for the Fourier-like multipliers operators in q-Rubin setting.

1. Introduction

The q^2-analogue differential-difference operator ∂_q, also called q-Rubin's operator defined on $\mathbb{R}_q$ in [12, 13] by

$$\partial_q f(z) = \begin{cases} \dfrac{f(q^{-1}z) + f(-q^{-1}z) - f(qz) + f(-qz) - 2f(-z)}{2(1-q)z} & \text{if} \quad z \neq 0 \\ \lim\limits_{z\to 0} \partial_q f(z) \quad \text{in} \quad \mathbb{R}_q & \text{if} \quad z = 0. \end{cases}$$

This operator has correct eigenvalue relationships for analogue exponential Fourier analysis using the functions and orthogonalities of [11].

The q^2-analogue Fourier transform we employ to make our constructions and results in this paper is based on analogue trigonometric functions and orthogonality results from [11] which have important applications to q-deformed quantum mechanics. This transform generalizing the usual Fourier transform, is given by

$$\mathcal{F}_q(f)(x) := K \int_{-\infty}^{+\infty} f(t) e(-itx; q^2) d_q t, \quad x \in \widetilde{\mathbb{R}}_q.$$

In this paper we study the Fourier multiplier operators $\mathcal{T}_m$ defined for $f \in L_q^2$ by

$$\mathcal{T}_m f(x) := \mathcal{F}_q^{-1}\left(m_a \mathcal{F}_q(f)\right)(x), \quad x \in \mathbb{R}_q,$$

where the function m_a is given by

$$m_a(x) = m(ax).$$

These operators are a generalization of the multiplier operators $\mathcal{T}_m$ associated with a bounded function m and given by $\mathcal{T}_m(f) = \mathcal{F}^{-1}(m\mathcal{F}(f))$, where $\mathcal{F}(f)$ denotes

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the ordinary Fourier transform on $\mathbb{R}^n$. These operators made the interest of several Mathematicians and they were generalized in many settings, (see for instance [1, 2, 15, 19]).

In this work we are interested the L^2 uncertainty principles for the q^2-analogue Fourier transform. The uncertainty principles play an important role in harmonic analysis. These principles state that a function f and its Fourier transform $\mathcal{F}(f)$ cannot be simultaneously sharply localized. Many aspects of such principles are studied for several Fourier transforms.

Recently, we have investigated the behaviour of the q^2-analogue Fourier transform in many setting [16, 17, 18]. Many uncertainty principles have already been proved for the q^2-analogue Fourier transform [3, 5]. The authors have established in [3] the Heisenberg-Pauli-Weyl inequality for the q^2-analogue Fourier transform, by showing that

$$\|f\|_{q,2} \leq C_q \||x|f\|_{q,2} \||y|\mathcal{F}_q(f)\|_{q,2}. \tag{1.1}$$

for every f in L^2_q such that xf and $y\mathcal{F}_q(f)$ are in L^2_q, where

$$C_q = 1 + q + q^{\frac{-1}{2}} + q^{\frac{3}{2}}. \tag{1.2}$$

In the present paper we are interested in proving an analogue of Heisenberg-Pauli-Weyl uncertainty principle For the operators $\mathcal{T}_m$. More precisely, we will show, for $f \in L^2_q$

$$\|f\|^2_{q,2} \leq K^{-1} C_q \||y|\mathcal{F}_q(f)\|_{q,2} \left(\int_{-\infty}^{\infty} \int_0^{\infty} |x|^2 |\mathcal{T}_m f(x)|^2 \frac{d_q a}{a} d_q x \right)^{\frac{1}{2}},$$

provided m be a function in L^2_q satisfying the admissibility condition

$$\int_0^{\infty} |m_a(x)| \frac{d_q a}{a} = 1, \quad \text{a.e. } x \in \mathbb{R}^+_q. \tag{1.3}$$

Moreover, for $\beta, \delta \in [1, \infty)$ and $\varepsilon \in \mathbb{R}$, such that $\beta\varepsilon = (1-\varepsilon)\delta$, we will show

$$\|f\|_{q,2} \leq C_q^{\beta\varepsilon} K^{-\beta\varepsilon(\frac{1}{2\beta'} + \frac{1}{2\delta'})} \left\| |x|^{\beta} \mathcal{T}_m f \right\|^{\varepsilon}_{q,2} \left\| |y|^{\delta} \mathcal{F}_q(f) \right\|^{1-\varepsilon}_{q,2}.$$

Using the techniques of Donoho and Stark [6], we show uncertainty principle of concentration type for the L^2 theory. Let f be a function in L^2_q and $m \in L^1_q \cap L^2_q$ satisfying the admissibility condition (1.3). If f is ϵ-concentrated on Ω and $\mathcal{T}_m f$ is ν-concentrated on Σ, then

$$\|m\|_{q,1} |\Omega|^{\frac{1}{2}} \left(\int \int_{\Sigma} \frac{1}{a^2} d\mu_q(a,x) \right)^{\frac{1}{2}} \geq K^{\frac{1}{2}} \left(1 - (\epsilon + \nu)\right),$$

where μ_q is the measure on $\mathbb{R}^+_q \times \mathbb{R}_q$ given by $d\mu_q(x)(a,x) := (d_q a/a) d_q x$.

This paper is organized as follows. In section 2, we recall some basic harmonic analysis results related with q^2-analogue Fourier transform and we introduce preliminary facts that will be used later.

In section 3, we establish Heisenberg-Pauli-Weyl uncertainty principle For the operators $\mathcal{T}_m$.

The last section of this paper is devoted to Donoho-Stark's uncertainty principle for q^2-Fourier multiplier operators.

2. Notations and preliminaries

Throughout this paper, we assume $0 < q < 1$ and we refer the reader to [8, 10] for the definitions and properties of hypergeometric functions. In this section we will fix some notations and recall some preliminary results. We put $\mathbb{R}_q = \{\pm q^n : n \in \mathbb{Z}\}$ and $\widetilde{\mathbb{R}}_q = \mathbb{R}_q \cup \{0\}$. For $a \in \mathbb{C}$, the q-shifted factorials are defined by

$$(a;q)_0 = 1; \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), n = 1,2,...; \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

We denote also

$$[a]_q = \frac{1-q^a}{1-q}, \quad a \in \mathbb{C} \quad \text{and} \quad [n]_q! = \frac{(q;q)_n}{(1-q)^n}, \quad n \in \mathbb{N}.$$

A q-analogue of the classical exponential function is given by (see [12, 13])

$$e(z;q^2) = \cos(-iz;q^2) + i\sin(-iz;q^2), \tag{2.1}$$

where

$$\cos(z;q^2) = \sum_{n=0}^{+\infty} q^{n(n+1)}\frac{(-1)^n z^{2n}}{[2n]q!}, \quad \sin(z;q^2) = \sum_{n=0}^{+\infty} q^{n(n+1)}\frac{(-1)^n z^{2n+1}}{[2n+1]q!}, \tag{2.2}$$

satisfying the following inequality for all $x \in \mathbb{R}_q$

$$|\cos(x;q^2)| \leq \frac{1}{(q;q)_\infty}, \quad \sin(x;q^2)| \leq \frac{1}{(q;q)_\infty} \quad \text{and} \quad |e(ix;q^2)| \leq \frac{2}{(q;q)_\infty}. \tag{2.3}$$

The q-differential-difference operators is defined as (see [12, 13])

$$\partial_q f(z) = \begin{cases} \dfrac{f(q^{-1}z) + f(-q^{-1}z) - f(qz) + f(-qz) - 2f(-z)}{2(1-q)z} & \text{if} \quad z \neq 0 \\ \lim\limits_{z\to 0} \partial_q f(z) \quad \text{in} \quad \mathbb{R}_q & \text{if} \quad z = 0 \end{cases}$$

and we denote a repeated application by

$$\partial_q^0 f = f, \quad \partial_q^{n+1} f = \partial_q(\partial_q^n f).$$

The q-Jackson integrals are defined by (see [9])

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{+\infty} q^n f(aq^n),$$

$$\int_a^b f(x)d_qx = (1-q)\sum_{n=0}^{+\infty} q^n(bf(bq^n) - af(aq^n))$$

and

$$\int_{-\infty}^{+\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{+\infty} q^n\left\{f(q^n) + f(-q^n)\right\},$$

provided the sums converge absolutely.

In the following we denote by

- $\mathcal{C}_{q,0}$ the space of bounded functions on $\mathbb{R}_q$, continued at 0 and vanishing a ∞.
- $\mathcal{C}_q^p$ the space of functions p-times q-differentiable on $\mathbb{R}_q$ such that for all $0 \leq n \leq p$. $\partial_q^p f$ is continuous on $\mathbb{R}_q$.

- $\mathcal{D}_q$ the space of functions infinitely q-differentiable on $\mathbb{R}_q$ with compact supports.
- $\mathcal{S}_q$ stands for the q-analogue Schwartz space of smooth functions over $\mathbb{R}_q$ whose q-derivatives of all order decay at infinity. $\mathcal{S}_q$ is endowed with the topology generated by the following family of semi-norms:

$$\|u\|_{M,\mathcal{S}_q}(f) := \sup_{x\in\mathbb{R};k\leq M}(1+|x|)^M|\partial_q^k u(x)| \quad \text{for all} \quad u\in\mathcal{S}_q \quad \text{and} \quad M\in\mathbb{N}.$$

- $\mathcal{S}'_q$ the space of tempered distributions on $\mathbb{R}_q$, it is the topological dual of $\mathcal{S}_q$.
- $L_q^p = \left\{ f : \|f\|_{q,p} = \left(\int_{-\infty}^{+\infty}|f(x)|^p d_qx\right)^{\frac{1}{p}} < \infty \right\}$.
- $L_q^\infty = \left\{ f : \|f\|_{q,\infty} = \sup_{x\in\mathbb{R}_q}|f(x)| < \infty \right\}$.

The q^2-Fourier transform was defined by R. L. Rubin defined in [12], as follow

$$\mathcal{F}_q(f)(x) = K\int_{-\infty}^{+\infty} f(t)e(-itx;q^2)d_qt, \quad x\in\widetilde{\mathbb{R}}_q$$

where

$$K = \frac{(q;q^2)_\infty}{2(q^2;q^2)_\infty(1-q)^2}.$$

To get convergence of our analogue functions to their classical counterparts as $q\uparrow 1$ as in [11, 13], we impose the condition that $1-q = q^{2m}$ for some integer m. Therefore, in the remainder of this paper, letting $q\uparrow 1$ subject to the condition

$$\frac{\log(1-q)}{\log(q)} \in 2\mathbb{Z}.$$

It was shown in ([7, 12]) that the q^2-Fourier transform $\mathcal{F}_q$ verifies the following properties:

(a) If $f,\ uf(u)\in L_q^1$, then

$$\partial_q(\mathcal{F}_q)(f)(x) = \mathcal{F}_q(-iuf(u))(x).$$

(b) If $f,\ \partial_q f\in L_q^1$, then

$$\mathcal{F}_q(\partial_q(f))(x) = ix\mathcal{F}_q(f)(x). \tag{2.4}$$

(c) If $f\in L_q^1$, then $\mathcal{F}_q(f)\in\mathcal{C}_{q,0}$ and we have

$$\|\mathcal{F}_q(f)\|_{q,\infty} \leq \frac{2K}{(q;q)_\infty}\|f\|_{q,1}. \tag{2.5}$$

(d) If $f\in L_q^1$, then, we have the reciprocity formula

$$\forall t\in\mathbb{R}_q, \quad f(t) = K\int_{-\infty}^{+\infty}\mathcal{F}_q(f)(x)e(itx;q^2)d_qx. \tag{2.6}$$

(e) The q^2-Fourier transform $\mathcal{F}_q$ is an isomorphism from $\mathcal{S}_q$ onto itself and we have, for all $f\in\mathcal{S}_q$

$$\mathcal{F}_q^{-1}(f)(x) = \mathcal{F}_q(f)(-x) = \overline{\mathcal{F}_q(\overline{f})}(x). \tag{2.7}$$

(f) $\mathcal{F}_q$ is an isomorphism from L_q^2 onto itself, and we have

$$\|\mathcal{F}_q(f)\|_{2,q} = \|f\|_{q,2}, \quad \forall f \in L_q^2 \tag{2.8}$$

and

$$\forall t \in \mathbb{R}_q, \quad f(t) = K \int_{-\infty}^{+\infty} \mathcal{F}_q(f)(x) e(itx; q^2) d_q x.$$

The q-translation operator $\tau_{q;x}, x \in \mathbb{R}_q$ is defined on L_q^1 by (see [12])

$$\tau_{q,y}(f)(x) = K \int_{-\infty}^{+\infty} \mathcal{F}_q(f)(t) e(itx; q^2) e(ity; q^2) d_q t, \quad y \in \mathbb{R}_q,$$
$$\tau_{q,0}(f)(x) = (f)(x).$$

It was shown in [12] that the q-translation operator can be also defined on L_q^2. Furthermore, it verifies the following properties

(a) For $f, g \in L_q^1$, we have

$$\tau_{q,y} f(x) = \tau_{q,x} f(y), \quad \forall x, y \in \mathbb{R}_q$$

and

$$\int_{-\infty}^{+\infty} \tau_{q,y}(f)(-x) g(x) d_q x = \int_{-\infty}^{+\infty} f(x) \tau_{q,y}(g)(-x) d_q x, \quad \forall y \in \widetilde{\mathbb{R}}_q.$$

(b) For all $f \in L_q^1$ and all $y \in \mathbb{R}_q$, we have(see [4])

$$\int_{-\infty}^{+\infty} \tau_{q,y}(f)(x) d_q x = \int_{-\infty}^{+\infty} f(x) d_q x. \tag{2.9}$$

(c) For all $y \in \mathbb{R}_q$ and for all $f \in L_q^p$,$1 \leq p \leq \infty$, we have $\tau_{q,y}(f) \in L_q^p$ (see[4]) and

$$\|\tau_{q,y} f\|_{q,p} \leq M \|f\|_{q,p}, \tag{2.10}$$

where

$$M = \frac{4(-q,q)_\infty}{(1-q)^2 q (q,q)_\infty} + 2C, \quad \text{with} \quad C = K^2 \|e(\cdot, q^2)\|_{\infty,q} \|e(\cdot, q^2)\|_{1,q}. \tag{2.11}$$

(d) $\tau_{q;y} f$ is an isomorphism for $f \in L_q^2$ onto itself and we have

$$\|\tau_{q,y} f\|_{q,2} \leq \frac{2}{(q,q)_\infty} \|f\|_{q,2}, \quad \forall y \in \widetilde{\mathbb{R}}_q. \tag{2.12}$$

(e) Let $f \in L_q^2$, then

$$\mathcal{F}_q(\tau_{q,y} f)(\lambda) = e(i\lambda y; q^2) \mathcal{F}_q(f)(\lambda), \quad \forall y \in \widetilde{\mathbb{R}}_q. \tag{2.13}$$

The q-convolution product is defined by using the q-translation operator, as follow For $f \in L_q^2$ and $g \in L_q^1$, the q-convolution product is given by

$$f * g(y) = K \int_{-\infty}^{+\infty} \tau_{q,y} f(x) g(x) d_q x.$$

The q-convolution product satisfying the following properties:

(a) $f * g = g * f$.
(b) $\forall f, g \in L_q^1 \cap L_q^2, \quad \mathcal{F}_q(f *_q g) = \mathcal{F}_q(f) \mathcal{F}_q(g)$.
(c) $\forall f, g \in \mathcal{S}_q, \quad f *_q g \in \mathcal{S}_q$.
(d) $f * g \in L_q^2$ if and only if $\mathcal{F}_q(f) \mathcal{F}_q(g) \in L_q^2$ and we have

$$\mathcal{F}_q(f * g) = \mathcal{F}_q(f) \mathcal{F}_q(g).$$

(e) Let $f, g \in L_q^2$. Then we have

$$\|f * g\|_{q,2}^2 = K\|\mathcal{F}_q(f)\mathcal{F}_q(g)\|_{q,2}^2, \tag{2.14}$$

and

$$f * g = \mathcal{F}_q^{-1}\left(\mathcal{F}_q(f)\mathcal{F}_q(g)\right). \tag{2.15}$$

(f) If $f, g \in L_q^1$ then $f * g \in L_q^1$ and

$$\|f * g\|_{q,1} = KM\|f\|_{q,1}\|g\|_{q,1}. \tag{2.16}$$

Definition 2.1. *Let $a \in \mathbb{R}_q^+$, $m \in L_q^2$ and f a smooth function on $\mathbb{R}_q$. We define the q^2-Fourier L^2-multiplier operators $\mathcal{T}_m$ for a regular function f on $\mathbb{R}_q$ as follow*

$$\mathcal{T}_m f(x) = \mathcal{F}_q^{-1}\left(m_a \mathcal{F}_q(f)\right)(x), \quad x \in \mathbb{R}_q, \tag{2.17}$$

where the function m_a is given by

$$m_a(x) = m(ax).$$

Remark. *Let $a \in \mathbb{R}_q^+$, $m \in L_q^2$ and f, we can write the operator $\mathcal{T}_m$ as*

$$\mathcal{T}_m f(x) = \mathcal{F}_q^{-1}(m_a) * f(x), \quad x \in \mathbb{R}_q, \tag{2.18}$$

where

$$\mathcal{F}_q^{-1}(m_a)(x) = \frac{1}{a}\mathcal{F}_q^{-1}(m)(\frac{x}{a}).$$

3. Heisenberg-Pauli-Weyl uncertainty principle

This section is devoted to establish Heisenberg-Pauli-Weyl uncertainty principle for the q^2-Fourier multiplier operator $\mathcal{T}_m$.

Theorem 3.1. *Let m be a function in L_q^2 satisfying the admissibility condition (1.3). Then, for $f \in L_q^2$, we have*

$$\|f\|_{q,2}^2 \leq K^{-1}C_q \||y|\mathcal{F}_q(f)\|_{q,2} \left(\int_{-\infty}^{\infty}\int_0^{\infty} |x|^2 |\mathcal{T}_m f(x)|^2 \frac{d_q a}{a} d_q x\right)^{\frac{1}{2}}. \tag{3.1}$$

Proof. Let $f \in L_q^2$. The inequality (3.1) holds if

$$\||y|\mathcal{F}_q(f)\|_{q,2} = +\infty$$

or

$$\int_{-\infty}^{\infty}\int_0^{\infty} |x|^2 |\mathcal{T}_m f(x)|^2 \frac{d_q a}{a} d_q x = +\infty.$$

Let us now assume that

$$\||y|\mathcal{F}_q(f)\|_{q,2} + \int_{-\infty}^{\infty}\int_0^{\infty} |x|^2 |\mathcal{T}_m f(x)|^2 \frac{d_q a}{a} d_q x < +\infty.$$

Inequality (1.1) leads to

$$\int_{-\infty}^{\infty} |\mathcal{T}_m f(x)|^2 d_q x \leq C_q \left(\int_{-\infty}^{\infty} |x|^2 |\mathcal{T}_m f(x)|^2 d_q x\right)^{\frac{1}{2}} \times \left(\int_{-\infty}^{\infty} |y|^2 |\mathcal{F}_q(\mathcal{T}_m f(.))(y)|^2 d_q y\right)^{\frac{1}{2}}.$$

Integrating with respect to $d_q a/a$, we get

$$\begin{aligned}\|\mathcal{T}_m f)\|_{q,2}^2 &< C_q\int_0^\infty\left(\int_{-\infty}^{\infty}|x|^2|\mathcal{T}_m f(x)|^2 d_qx\right)^{\frac{1}{2}}\\&\times\left(\int_{-\infty}^{\infty}|y|^2|\mathcal{F}_q(\mathcal{T}_m f(.))(y)|^2 d_qy\right)^{\frac{1}{2}}\frac{d_qa}{a}.\end{aligned}$$

From Plancherel formula for the q^2-Fourier multiplier operators [14, Theorem 3.1] and Schwartz's inequality, we obtain

$$\begin{aligned}\|f\|_{q,2}^2 &< K^{-1}C_q\left(\int_0^\infty\int_{-\infty}^{\infty}|x|^2|\mathcal{T}_m f(x)|^2 d_qx\frac{d_qa}{a}\right)^{\frac{1}{2}}\\&\times\left(\int_0^\infty\int_{-\infty}^{\infty}|y|^2|\mathcal{F}_q(\mathcal{T}_m f(.))(y)|^2 d_qy\frac{d_qa}{a}\right)^{\frac{1}{2}}.\end{aligned}$$

According to relation (2.17), Fubini-Tonnelli's theorem and the admissibility condition (1.3), we obtain

$$\begin{aligned}\int_0^\infty\int_{-\infty}^{\infty}|y|^2|\mathcal{F}_q(\mathcal{T}_m f(.))(y)|^2 d_qy\frac{d_qa}{a} &= \int_0^\infty\int_{-\infty}^{\infty}|y|^2|m_a(y)|^2|\mathcal{F}_q(f)(y)|^2 d_qy\frac{d_qa}{a}\\&= \int_{-\infty}^{\infty}|y|^2|\mathcal{F}_q(f)(y)|^2 d_qy.\end{aligned}$$

This gives the result and completes the proof of the theorem. □

Theorem 3.2. *Let m be a function in L_q^2 satisfying the admissibility condition (1.3) and $\beta,\delta\in[1,\infty)$. Let $\varepsilon\in\mathbb{R}$, such that $\beta\varepsilon=(1-\varepsilon)\delta$ then, for all $f\in L_q^2$, we have*

$$\|f\|_{q,2}\le C_q^{\beta\varepsilon}K^{-\beta\varepsilon\frac{1}{2\beta'}+\frac{1}{2\delta'}}\left\||x|^\beta\mathcal{T}_m f\right\|_{q,2}^{\varepsilon}\left\||y|^\delta\mathcal{F}_q(f)\right\|_{q,2}^{1-\varepsilon}. \tag{3.2}$$

Proof. Let $f\in L_q^2$. Then the inequality (3.2) holds if

$$\left\||x|^\beta\mathcal{T}_m f\right\|_{q,2}^{\varepsilon}=+\infty \quad\text{or}\quad \left\||y|^\delta\mathcal{F}_q(f)\right\|_{q,2}^{1-\varepsilon}=+\infty.$$

Let us now assume that $f\in L_q^2$ with $f\neq 0$ such that

$$\left\||x|^\beta\mathcal{T}_m f\right\|_{q,2}^{\varepsilon}+\left\||y|^\delta\mathcal{F}_q(f)\right\|_{q,2}^{1-\varepsilon}<+\infty,$$

therefore, for all $\beta>1$, we have

$$\left\||x|^\beta\mathcal{T}_m f\right\|_{q,2}^{\frac{1}{\beta}}\|\mathcal{T}_m f\|_{q,2}^{\frac{1}{\beta'}}=\left\||x|^2|\mathcal{T}_m f|^{\frac{2}{\beta}}\right\|_{q,\beta}^{\frac{1}{2}}\left\||\mathcal{T}_m f|^{\frac{2}{\beta'}}\right\|_{q,\beta'}^{\frac{1}{2}},$$

with $\beta'=\frac{\beta}{\beta-1}$.
Applying the Hölder's inequality, we get

$$\||x|\mathcal{T}_m f\|_{q,2}\le\|\mathcal{T}_m f\|_{q,2}^{\frac{1}{\beta'}}\left\||x|^\beta\mathcal{T}_m f\right\|_{q,2}^{\frac{1}{\beta}}.$$

According to Plancherel formula for the q^2-Fourier multiplier operators [14, Theorem 3.1], we have for all $\beta\ge 1$

$$\||x|\mathcal{T}_m f\|_{q,2}\le K^{\frac{1}{2\beta'}}\|f\|_{q,2}^{\frac{1}{\beta'}}\left\||x|^\beta\mathcal{T}_m f\right\|_{q,2}^{\frac{1}{\beta}}, \tag{3.3}$$

with equality if $\beta = 1$. In the same manner, for all $\delta \geq$ and using Plancherel formula (2.8), we get

$$\left\||y|\mathcal{F}_q(f)\right\|_{q,2} \leq K^{\frac{1}{2\delta'}} \|f\|_{q,2}^{\frac{1}{\delta'}} \left\||y|^{\delta}\mathcal{F}_q(f)\right\|_{q,2}^{\frac{1}{\delta}}, \tag{3.4}$$

with equality if $\delta = 1$. By using the fact that $\beta\varepsilon = (1-\varepsilon)\delta$ and according to inequalities (3.3) and (3.4), we have

$$\left(\frac{\left\||x|\mathcal{T}_m f\right\|_{q,2}\left\||y|\mathcal{F}_q(f)\right\|_{q,2}}{\|f\|_{q,2}^{\frac{1}{\beta'}+\frac{1}{\delta'}}}\right)^{\beta\varepsilon} \leq K^{\frac{1}{2\beta'}+\frac{1}{2\delta'}} \left\||x|^{\beta}\mathcal{T}_m f\right\|_{q,2}^{\varepsilon} \left\||y|^{\delta}\mathcal{F}_q(f)\right\|_{q,2}^{1-\varepsilon},$$

with equality if $\beta = \delta = 1$. Next by Theorem 3.1, we obtain

$$\|f\|_{q,2} \leq C_q^{\beta\varepsilon} K^{-\beta\varepsilon(\frac{1}{2\beta'}+\frac{1}{2\delta'})} \left\||x|^{\beta}\mathcal{T}_m f\right\|_{q,2}^{\varepsilon} \left\||y|^{\delta}\mathcal{F}_q(f)\right\|_{q,2}^{1-\varepsilon},$$

which completes the proof of the theorem. □

4. Donoho-Stark's uncertainty principle

Definition 4.1. *(1) A subset $E \subset \mathbb{R}_q$ is said to be measurable subset of $\mathbb{R}_q$ if*

$$|E| = \int_{-\infty}^{\infty} \chi_E(x) d_q x < \infty,$$

where χ_E is the characteristic function of the set E.

(2) Let Ω be a measurable subset of $\mathbb{R}_q$, we say that the function $f \in L_q^2$ is ϵ-concentrated on Ω, if

$$\|f - \chi_\Omega f\|_{q,2} \leq \epsilon \|f\|_{q,2}. \tag{4.1}$$

(3) Let Σ be a measurable subset of $\mathbb{R}_q^+ \times \mathbb{R}_q$ and let $f \in L_q^2$. We say that $\mathcal{T}_m f$ is ν-concentrated on Σ, if

$$\|\mathcal{T}_m f - \chi_A \mathcal{T}_m f\|_{q,2} \leq \nu \|\mathcal{T}_m\|_{q,2}. \tag{4.2}$$

We need the following Lemma for the proof of Donoho-Stark's uncertainty principle for the q^2-Fourier multiplier operator.

Lemma 4.2. *Let $m, f \in L_q^1 \cap L_q^2$. Then the operators $\mathcal{T}_m$ satisfy the following integral representation*

$$\mathcal{T}_m(f)(x) = \frac{1}{a}\int_{-\infty}^{\infty} \Psi_q(\frac{x}{a}, \frac{y}{a}) f(y) d_q y, \quad (a, x) \in \mathbb{R}_q^+ \times \mathbb{R}_q,$$

where

$$\Psi_q(x, y) = \int_{-\infty}^{\infty} m(z) e(itx, q^2) e(-itx, q^2) d_q(z).$$

Proof. The result follows from the definition of the q^2-Fourier multiplier operator (2.17) and the inversion formula of the q^2-Fourier transform (2.6) using Fubini-Tonnelli's theorem. □

Theorem 4.3. *Let f be a function in L_q^2 and $m \in L_q^1 \cap L_q^2$ satisfying the admissibility condition (1.3). If f is ϵ-concentrated on Ω and $\mathcal{T}_m f$ is ν-concentrated on Σ, then*

$$\|m\|_{q,1} |\Omega|^{\frac{1}{2}} \left(\int\int_{\Sigma} \frac{1}{a^2} d\mu_q(a, x)\right)^{\frac{1}{2}} \geq K^{\frac{1}{2}} \left(1 - (\epsilon + \nu)\right),$$

where μ_q is the measure on $\mathbb{R}_q^+ \times \mathbb{R}_q$ given by $d\mu_q(x)(a, x) := (d_q a/a) d_q x$.

Proof. Let f be a function in L^2_q. Assume that $0 < \mu_q(\Omega) < \infty$ and

$$\int\int_{\Sigma} \frac{1}{a^2} d\mu_q(a,x) < \infty.$$

According to [14, Theorem 2.3] and inequalities (4.1)-(4.2), we get

$$\begin{aligned} \|\mathcal{T}_m f - \chi_\Sigma \mathcal{T}_m(\chi_\Omega f)\|_{q,2} &\leq \|\mathcal{T}_m f - \chi_\Sigma \mathcal{T}_m f\|_{q,2} + \|\chi_\Sigma \mathcal{T}_m(f - \chi_\Omega f)\|_{q,2} \\ &\leq \nu \|\mathcal{T}_m f\|_{q,2} + \|\chi_\Sigma \mathcal{T}_m(f - \chi_\Omega f)\|_{q,2} \\ &\leq K^{\frac{1}{2}}(\epsilon + \nu)\|f\|_{q,2}. \end{aligned}$$

By triangle inequality it follows that

$$\begin{aligned} \|\mathcal{T}_m f\|_{q,2} &\leq \|\mathcal{T}_m f - \chi_\Sigma \mathcal{T}_m(\chi_\Omega f)\|_{q,2} + \|\chi_\Sigma \mathcal{T}_m(\chi_\Omega f)\|_{q,2} \\ &\leq K^{\frac{1}{2}}(\epsilon + \nu)\|f\|_{q,2} + \|\chi_\Sigma \mathcal{T}_m(\chi_\Omega f)\|_{q,2}. \end{aligned} \tag{4.3}$$

On the other hand, we have

$$\|\chi_\Sigma \mathcal{T}_m(\chi_\Omega f)\|_{q,2} = \left(\int\int_{\Sigma} |\mathcal{T}_m(\chi_\Omega f)(x)|^2 d\mu_q(a,x)\right)^{\frac{1}{2}}$$

and moreover $m, \chi_\Omega f \in L^1_q \cap L^2_q$, then by Lemma 4.2, we obtain

$$|\mathcal{T}_m(\chi_\Omega f)(x)| \leq \frac{1}{a}\|m\|_{1,\alpha}\|f\|_{q,2}|\Omega|^{\frac{1}{2}}.$$

Therefore, thus

$$\|\chi_\Sigma \mathcal{T}_m(\chi_\Omega f)\|_{q,2} \quad \leq \quad \|m\|_{1,\alpha}\|f\|_{q,2}|\Omega|^{\frac{1}{2}} \left(\int\int_{\Sigma} \frac{1}{a^2} d\mu_q(a,x)\right)^{\frac{1}{2}}.$$

Hence, according to last inequality and (4.3)

$$\|\mathcal{T}_m(f)\|_{q,2} \quad \leq \quad \|m\|_{1,\alpha}\|f\|_{q,2}|\Omega|^{\frac{1}{2}} \left(\int\int_{\Sigma} \frac{1}{a^2} d\mu_q(a,x)\right)^{\frac{1}{2}} + K^{\frac{1}{2}}(\epsilon + \nu)\|f\|_{q,2}.$$

According to Plancherel formula [14, Theorem 2.3], we obtain

$$\|m\|_{\alpha,1}|\Omega|^{\frac{1}{2}} \left(\int\int_{\Sigma} \frac{1}{a^2} d\mu_q(a,x)\right)^{\frac{1}{2}} \geq K^{\frac{1}{2}}\left(1 - (\epsilon + \nu)\right),$$

which completes the proof of the theorem. □

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