Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 14 Issue 3(2022), Pages 20-35.
https://doi.org/10.54671/BMAA-2022-3-2

ABSOLUTE DERIVATIVE OF SET-VALUED MAPS

MOHAMAD MUSLIKH, ADEM KILICMAN

Abstract. The aim of this article is to develop a differential calculus for set-valued functions with values in metric spaces of the family of all compact and convex subsets. Using only the metric Hausdorff on metric spaces and without using the Hukuhara difference, a concept of the derivative is introduced for set-valued functions in such metric spaces. The comparison with the other derivative definitions of the set-valued functions and the relation to the Lipschitz conditions was also investigated in this paper.

1. Introduction

Differential calculus for set-valued functions with compact and convex values was introduced by M. Hukuhara in 1967 [6]. He is using the concept of difference of two sets in $\mathbb{R}^k$ that was introduced. Thenceforth many researchers develop it with various different points of view. In 1970, Banks and Jacobs are using the embedding technique to define the derivative of set-valued functions on Banach spaces [2]. Whereas, either Bridgland Jr [3] or De Blasi [5] utilizes the Hausdorff metric induced by the norm on the Banach spaces. In [9], Lasota and Strauss gave the definition of a set-valued derivative for single-valued map from $f : \mathbb{R}^k \to \mathbb{R}^k$. Furthermore, S. Markov introduces the generalization of the Hukuhara differential [10]. Generally, all of the results obtained by researchers are the differential calculus on the normed linear spaces.

The study on derivative for set-valued functions with values in abstract metric space is still undeveloped. Motivated by this considerations, the purpose of this article is to generalize a differential calculus for set-valued functions from abstract metric space to the other abstract metric spaces. The concept that we introduce has the advantage that is avoiding the use of the arithmetic operation that indeed have not in metric spaces. The notion of derivative in this article extends the concept of metric derivative of set-valued function introduced by author et.al in [11].

In 1971 E. Braude (see [13]), K. Skaland [14] in 1975, and Charatonic in 2012 introduce absolute derivatives for single-valued functions from abstract metric space to the other abstract metric spaces. Base on their results, we apply them for the

1991 *Mathematics Subject Classification.* 54C60; 28A15.
Key words and phrases. Set-valued; Defferentiation.

Submitted September 18, 2021. Published July 7, 2022.
Communicated by M. Mursaleen.

set-valued functions.

2. Preliminaries

In 1971, E. Braude has been introduced the derivative for functions with values in a metric spaces. (see [13]).

Definition 2.1. *Let (X, d) and (Y, ρ) be two metric spaces and let $p \in X$ be a limit point of X. Then $f : X \longrightarrow Y$ is said to be **metrically differentiable** at p if there is a real number $f'(p)$ and for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\left| \frac{\rho(f(x), f(y))}{d(x,y)} - f'(p) \right| < \epsilon, \tag{2.1}$$

for all $x \neq y \in X$ with $0 < d(x,p) < \delta$ and $0 < d(y,p) < \delta$.

In 1975, K. Skaland defined the same things, but it was weaker than Braude's definition [14].

Definition 2.2. *Let (X, d) and (Y, ρ) be two metric spaces and let $p \in X$ be a limit point. The function $f : X \longrightarrow Y$ is said to be **differentiable** at p if there is a real number $f'(p)$ and for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\left| \frac{\rho(f(x), f(p))}{d(x,p)} - f'(p) \right| < \epsilon, \tag{2.2}$$

for every $x \in X$ with $0 < d(x,p) < \delta$.

A non-negative real number $f'(p)$ is called the *metrically derivative* or the *quasiderivative* of the function f at the point $p \in X$ (see [13], [14]). Recently, differentiation in metric spaces, as discussed in [4], explain two kinds derivative, namely the *absolute derivative* (Definition 2.2) and the *strongly absolute derivative* (Definition 2.1).

Suppose (X, d) and (Y, ρ) are two metric spaces. Then we use the notation $\mathcal{P}_0(X)$ (resp. $\mathcal{CB}(X)$, $\mathcal{K}(X)$ and $\mathcal{KC}(X)$) as the family of all non-empty (resp. closed-bounded, compact and compact-convex) subsets of X.

The mapping $F : X \longrightarrow \mathcal{P}_0(Y)$ is called ***set-valued functions*** if the map $F(x) \in \mathcal{P}_0(Y)$ for each $x \in X$. The function $f : X \longrightarrow Y$ is said to be ***selection*** of F if $f(x) \in F(x)$ for all $x \in X$. The ***image of the set*** $A \subset X$ is of the form as follows,

$$F(A) = \bigcup_{x \in A} F(x).$$

Suppose A and B are two subsets of a metric space (X, d). The Hausdorff distance between A and B is the distance function $H : \mathcal{P}_0(X) \times \mathcal{P}_0(X) \longrightarrow \mathbb{R}^+$ defined as

$$H(A, B) = \max \{d(A, B), d(B, A)\}, \tag{2.3}$$

where $d(A, B) = \sup_{a \in A} d(a, B)$. It is clear that $d(A, B) \neq d(B, A)$.

The Hausdorff distance H is a metric on the family $\mathcal{CB}(X)$ called Hausdorff metric. If X is a complete, then metric space $(\mathcal{CB}(X), H)$ is also complete.

Let $A = \{x\}$, $B = \{y\} \subset X$ and let C be a nonempty subset of X. The Hausdorff metric of the subsets is defined as

(i) $H(A, B) = H(\{x\}, \{y\}) = d(x, y)$
(ii) $H(\{z\}, C) = d(z, C)$.

The concept of continuous set-valued maps on the metric space (X, d) is defined as follow. Let $A \subset X$ and let $F : A \longrightarrow \mathcal{CB}(Y)$ be a set-valued mapping. We say that F is continuous at $p \in A$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$H(F(x), F(p)) < \epsilon,$$

for all $x \in N_\delta(p)$.

Let $A \subset X$ and let $F : A \longrightarrow \mathcal{K}(Y)$ be a set-valued mapping. F is said to be *Lipschitz* with respect to Hausdorff metric H in $\mathcal{K}(Y)$ if there exists $L \geq 0$ such that

$$H(F(x), F(y)) \leq Ld(x, y),$$

for all $x, y \in A$. The infimum of all real numbers L satisfying the above condition is called the *Lipschitz constant* of F that denoted by $\mathrm{Lip}(F)$. A Lipschitz continuous mapping is obviously continuous. For a fixed set $A \subset X$, the distance function $d_A(x) = d(A, x)$ for all $x \in X$ is a Lipschitz continuous function with $\mathrm{Lip}(d_A) \leq 1$.

Suppose $\mathcal{I}(\mathbb{R}) = \{[a, b] \mid a, b \in \mathbb{R}, a < b\}$. In [12], R.E. Moore et al introduced an absolute value of the interval. The absolute value of an interval $[a, b]$ is the maximum of the absolute values of its endpoints.

$$|[a, b]| = \max\{|a|, |b|\}.$$

The Hausdorff distance function on $\mathcal{I}(\mathbb{R})$ is a metric defined as $H : \mathcal{I}(\mathbb{R}) \times \mathcal{I}(\mathbb{R}) \longrightarrow [0, \infty)$ by

$$H(I, J) = \max\{|a - c|, |b - d|\}, \tag{2.4}$$

where $I = [a, b]$ and $J = [c, d]$. The pair $(\mathcal{I}(\mathbb{R}), H)$ is a complete and separable metric spaces.

Suppose U, V is subsets of $\mathbb{R}^k$. The form $U + V = \{u + v \mid u \in U, v \in V\}$ and $\alpha U = \{\alpha \cdot u \mid u \in U, \alpha \in \mathbb{R}\}$ defines the *Minkowski sum* and the *scalar multiplication*. It is well known that addition is commutative, associative and with neutral element $\{\Theta\}$. If $\alpha = 1$, scalar multiplication gives the "inverse" $-U = (-1)U = \{-u \mid u \in U\}$ but, in general, $U - U \neq \{\Theta\}$, namely $-U$ does not give the inverse with respect to Minkowski sum (unless $U = \{u\}$ is singleton). The implication of this fact that Minkowski sum is not valid (The Minkowski difference written as $U - V = U + (-1)V$).

To solve of such problem has been introduced the Hukuhara difference (h-difference) by M. Hukuhara in 1967 and defined $U \overset{h}{-} V = W \Longleftrightarrow U = V + W$ for each $U, V, W \in \mathcal{KC}(\mathbb{R}^k)$. An important properties of the Hukuhara difference is that $U \overset{h}{-} U = \{\Theta\}$ and $(U + V) \overset{h}{-} V = U$. The Hukuhara difference is unique, but it does not always exists [6].

Proposition 2.3. *Let $U, V \in \mathcal{KC}(\mathbb{R}^k)$. The necessary and sufficient conditions of the difference $U \overset{h}{-} V$ exists if $u \in \delta(U)$ there exists at least a point w such that $u \in V + \{w\} \subset U$.*

The h-difference $U \overset{h}{-} V$ exists if $Diam(\mathrm{U}) \geq Diam(\mathrm{V})$, where $Diam(U) = \sup\{\|u - v\| \mid u, v \in U\}$.

In 1969, S Markov [10] introduced to the concept of the Hukuhara difference which more general is called generalization Hukuhara difference (gh-difference) and it is defined as follows.

Definition 2.4. *Let $U, V \in \mathcal{KC}(\mathbb{R}^k)$. The gh-difference of two sets U and V defined as $U \overset{gh}{-} V = W$ if it satisfies (a) $U = V + W$ or (b) $V = U + (-1)W$.*

It is also possible that $U = V + W$ and $V = U + (-1)W$ holds simultaneously. In the case part (a) the gh-difference is equivalent to the h-difference.

Proposition 2.5. *Let $U, V \in \mathcal{KC}(\mathbb{R}^k)$. If $U \overset{gh}{-} V$ exists, it has the following properties:*

(i) $U \overset{gh}{-} U = \{\Theta\}$;
(ii) *If $U \overset{gh}{-} V$ exists, then $V \overset{gh}{-} U$ exists and $V \overset{gh}{-} U = -(U \overset{gh}{-} V)$;*
(iii) *If $U \overset{gh}{-} V$ exists then also $(-U) \overset{gh}{-} (-V)$ does and $-(U \overset{gh}{-} V) = (-U) \overset{gh}{-} (-V)$;*
(iv) $(U \overset{gh}{-} V) = (V \overset{gh}{-} U) = W$ *if and only if $W = \{\Theta\}$ and $U = V$.*

The gh-difference always exists for any two intervals in $\mathcal{I}(\mathbb{R})$.

Proposition 2.6. *Suppose $I = [x^-, x^+]$ and $J = [y^-, y^+]$ are intervals in $\mathcal{I}$. The gh-difference of two intervals I and J always exists and*

$$I \overset{gh}{-} J = [x^-, x^+] \overset{gh}{-} [y^-, y^+] = [z^-, z^+] \tag{2.5}$$

where $z^- = \min\{(x^- - y^-), (x^+ - y^+)\}$ and $z^+ = \max\{(x^- - y^-), (x^+ - y^+)\}$.

By making use of generalization of the Hukuhara difference, we can define the Hausdorff distance function as: $H : \mathcal{KC}(\mathbb{R}^k) \times \mathcal{KC}(\mathbb{R}^k) \longrightarrow [0, \infty)$ by following formula

$$H(U, V) = \|U \overset{gh}{-} V\| \tag{2.6}$$

for all $U, V \in \mathcal{KC}(\mathbb{R}^k)$ as long as $U \overset{gh}{-} V$ exists.

Proposition 2.7. *The Hausdorff distance on the equation 2.6 is a metric on $\mathcal{K}(\mathbb{R}^k)$.*

Proof. It is clear that $H(U,V) \geq 0$ and $H(U,V) = 0$ if and only if $U = V$. Whereas,the symmetry property is following

$$\begin{aligned} H(U,V) &= \|U \overset{gh}{-} V\| = |-1|\|U \overset{gh}{-} V\| = \|-(U \overset{gh}{-} V)\| \\ &= \|V \overset{gh}{-} U\| \quad (Proposition \quad 2.5(ii)) \\ &= H(V,U). \end{aligned}$$

By the definition 2.4, we have $A = U \overset{gh}{-} V \Leftrightarrow (a).V = U + (-A)$, $B = U \overset{gh}{-} W \Leftrightarrow (b).W = U + (-B)$ and $C = V \overset{gh}{-} W \Leftrightarrow (c).W = V + (-C)$. From (b) and (c), we obtain the equation (I). $U + (-B) = V + (-C)$. If both side of the equation (a) $V = U + (-A)$ added by the set $(-C)$, then we obtain the equation (II). $V + (-C) = U + (-A) + (-C)$. Since $V + (-C) = U + (-B)$, the equation (II) to be $U + (-B) = U + (-A) + (-C)$ or (-B)=(-A)+(-C). It means $-(U \overset{gh}{-} W) = -(U \overset{gh}{-} V) + (-(V \overset{gh}{-} W))$ so that we obtain

$$\begin{aligned} H(U,W) &= \|U \overset{gh}{-} W\| = \|(-1)(U \overset{gh}{-} W)\| = \|(-1)(U \overset{gh}{-} V) + (-1)(V \overset{gh}{-} W)\| \\ &= |-1|\|(U \overset{gh}{-} V) + (V \overset{gh}{-} W)\| = \|(U \overset{gh}{-} V) + (V \overset{gh}{-} W)\| \\ &\leq \|U \overset{gh}{-} V\| + \|V \overset{gh}{-} W\| \\ &= H(U,V) + H(V,W). \end{aligned}$$

□

Proposition 2.8. *If $U, V, U', V' \in \mathcal{KC}(\mathbb{R}^k)$ then*

$$H(tU, tV) = tH(U,V) \qquad \forall t \geq 0, \tag{2.7}$$

$$H(U + U', V + V') \leq H(U,V) + H(U',V'), \tag{2.8}$$

further,

$$H(U - U', V - V') \leq H(U,V) + H(U',V'), \tag{2.9}$$

provided the difference $U - U'$ and $V - V'$ exist. Moreover $\beta = \max\{\lambda, \mu\}$, we have

$$H(\lambda U, \mu V) \leq \beta H(U,V) + |\lambda - \mu| \left[H(U,\Theta) + H(V,\Theta)\right] \tag{2.10}$$

and

$$H(\lambda U, \lambda V) = \lambda H(U - V, \Theta) \tag{2.11}$$

if $U - V$ exists.

Next, we define the magnitude of a nonempty subset of U of $\mathbb{R}^k$ by

$$\|U\| = \sup\{\|u\| \mid u \in U\}, \tag{2.12}$$

or equivalent,

$$\|U\| = H(U,\Theta). \tag{2.13}$$

The norm $\|U\|$ is finite and the supremum in the equation 2.12 can be achieved because of $U \in \mathcal{K}(\mathbb{R}^k)$.

3. Derivative of Set-Valued

We begin by the notion of the derivative in the sense of the Hukuhara difference. M. Hukuhara in [6] introduced the definition as follows.

Definition 3.1. *Let $I \in \mathcal{I}(\mathbb{R})$ and let $F : I \longrightarrow \mathcal{KC}(\mathbb{R}^k)$ be a set-valued function. F is **Hukuhara differentiable** at $t_0 \in I$ if there exists $F'_h(t_0) \in \mathcal{KC}(\mathbb{R}^k)$ such that the limit*

$$\lim_{\Delta t \to 0^+} \frac{F(t_0 + \Delta t) \overset{h}{-} F(t_0)}{\Delta t} \tag{3.1}$$

and

$$\lim_{\Delta t \to 0^+} \frac{F(t_0) \overset{h}{-} F(t_0 - \Delta t)}{\Delta t}, \tag{3.2}$$

both exists and equals to $F'_h(t_0)$.

We note that using the difference quotient in (3.2) is not equivalent to using the difference quotient in

$$\lim_{\Delta t \to 0^-} \frac{F(t_0 + \Delta t) \overset{h}{-} F(t_0)}{\Delta t}. \tag{3.3}$$

Example 3.2. *Suppose $F : [0, 1] \to \mathcal{I}(\mathbb{R})$ is an interval-valued function with $F(t) = [t, 2t]$ for all $t \in [0, 1]$. F is Hukuhara differentiable for each $t \in (0, 1)$ with Hukuhara derivative $F'_h(t) = [1, 2]$ since*

$$\begin{aligned}
\lim_{\Delta t \to 0^+} \frac{F(t + \Delta t) \overset{h}{-} F(t)}{\Delta t} &= \lim_{\Delta t \to 0^+} \frac{[t + \Delta t, 2(t + \Delta t)] \overset{h}{-} [t, 2t]}{\Delta t} \\
&= \lim_{\Delta t \to 0^+} \frac{[\Delta t, 2\Delta t]}{\Delta t} \\
&= [1, 2]
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\Delta t \to 0^+} \frac{F(t) \overset{h}{-} F(t - \Delta t)}{\Delta t} &= \lim_{\Delta t \to 0^+} \frac{[t, 2t] \overset{h}{-} [(t - \Delta t), 2(t - \Delta t)]}{\Delta t} \\
&= \lim_{\Delta t \to 0^+} \frac{[\Delta t, 2\Delta t]}{\Delta t} \\
&= [1, 2].
\end{aligned}$$

In [6] obtained the results as follows.

Proposition 3.3. *If the set valued $F : [a, b] \longrightarrow \mathcal{KC}(\mathbb{R}^k)$ is Hukuhara differentiable on $[a, b]$, then the real valued function $t \longrightarrow diam(F(t))$, $t \in I$ is non decreasing on I.*

Proposition 3.4. *The set valued function $F : [a, b] \longrightarrow \mathcal{KC}(\mathbb{R}^k)$ is constant if, and only ,if we have*

$$F'_h(t) = 0$$

for all $t \in I$.

By the gh-difference, the derivative of a set-valued function has been introduced by Markov [10] as follows.

Definition 3.5. *Let $F : [a,b] \longrightarrow \mathcal{KC}(\mathbb{R}^k)$ be a set-valued function and suppose $t_0, t_0 + h \in (a,b)$. Then the gh-derivative of a set-valued function is defined as follows:*

$$F'_{gh}(t_0) = \lim_{h \to 0} \frac{F(t_0 + h) \overset{gh}{-} F(t_0)}{h}. \tag{3.4}$$

If $F'_{gh}(t_0) \in \mathcal{KC}(\mathbb{R}^k)$ exists and satisfies the equation (17)*, then F is said **generalized Hukuhara differentiable** (gh-differentiable) at the point $t_0 \in (a,b)$.*

The gh-difference $F(t_0+h) \overset{gh}{-} F(t_0)$ always exists if $\mathcal{KC}(\mathbb{R}^k) = \mathcal{I}(\mathbb{R})$ (Proposition 2.6). For the interval-valued function has been resulted as follows.

Theorem 3.6. *Let $F : [a,b] \longrightarrow \mathcal{I}$ be an interval-valued functions and $F(x) = [f(x), g(x)]$, where $f, g : [a,b] \longrightarrow \mathbb{R}$. F is gh-differentiable on (a,b) if and only if f and g are differentiable on (a,b) and*

$$F'_{gh}(x) = [\min\{f'(x), g'(x)\}, \max\{f'(x), g'(x)\}],$$

for all $x \in (a,b)$

This means that

$$F'_{gh}(x) = \begin{cases} [f'(x), g'(x)] & \text{if } f'(x) < g'(x), \\ [g'(x), f'(x)] & \text{if } f'(x) > g'(x) \ . \end{cases}$$

4. Absolute Derivative of Set-valued

In this section, we introduce the main result differential calculus for a set-valued function defined on abstract metric spaces with values in hypermetric spaces (metric spaces with subset elements). Definition of derivative, such as in the classical. In this section, we introduce the absolute derivative,namely the derivative for a set-valued function defined on abstract metric spaces with values on hypermetric spaces (metric spaces with subset elements). The classical definition of the derivative for single-valued function using the arithmetic structure. Likewise, a definition of derivatives for set-valued functions, such as that mentioned in the introduction, also involved arithmetic structure. Is it possible to define a derivative without arithmetic structure? Indeed the metric spaces have no structure arithmetic. of the derivative for single-valued function using the arithmetic structure. Likewise, a definition of derivatives for set-valued functions was mentioned in the introduction also involved arithmetic structure. Is it possible to define a derivative without arithmetic structure? Because indeed metric space has not arithmetic structure.

This is possible since there is a concept of the derivative for the function defined in differential manifolds and the manifolds involved do not generally have an arithmetic structure on their definitions. Therefore we make sure can define the "derivative" in the sense of the metric spaces. In this case, the role of arithmetic is significantly diminished.

The following is the definition of the derivative in question.

Definition 4.1. *Let (X,d) and (Y,ρ) be two metric spaces and let $p \in X$ be a limit point. The set-valued $F : X \longrightarrow \mathcal{KC}(Y)$ is called **absolutely differentiable***

at $p \in X$ if there exists a non-negative real number $F'_{abs}(p)$ with the property that for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left| \frac{H(F(x), F(p))}{d(x,p)} - F'_{abs}(p) \right| < \epsilon, \tag{4.1}$$

for every $x \in N_\delta(p) \subset X$. This means that the limit

$$\lim_{x \to p} \frac{H(F(x), F(p))}{d(p,x)} \tag{4.2}$$

axists and equals $F'_{abs}(p)$. Where, H is Hausdorff metric on $\mathcal{KC}(Y)$ induced by metric ρ

A non-negative real number $F'_{abs}(p)$ is called the "absolute derivative" of the set-valued functions F at the point $p \in X$. Furthermore, the absolute derivative of set-valued functions F is denoted by F'_{abs}.

The following theorems state the absolute derivative $F'_{abs}(p)$ is well-defined .

Theorem 4.2. *If the absolute derivative $F'_{abs}(p)$ is exist, then it is unique.*

Proof. Suppose $\epsilon > 0$. There exists $\delta_1, \delta_2 > 0$ such that for all $x \in N_{\delta_1}(p)$ we have

$$\left| \frac{H(F(x), F(p))}{d(x,p)} - F'_{abs}(p) \right| < \frac{\epsilon}{2}. \tag{4.3}$$

and for all $x \in N_{\delta_2}(p)$ we have

$$\left| \frac{H(F(x), F(p))}{d(x,p)} - G'_{abs}(p) \right| < \frac{\epsilon}{2} \tag{4.4}$$

respectively.
Let $\delta = \max\{\delta_1, \delta_2\}$. For all $x \in N_\delta(p)$ and from 4.3 and 4.4 we obtain

$$\begin{aligned} |F'_{abs}(p) - G'_{abs}(p)| &\leq \left| F'_{abs}(p) - \frac{H(F(x), F(p))}{d(x,p)} \right| + \left| \frac{H(F(x), F(p))}{d(x,p)} - G'_{abs}(p) \right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□

Some of the examples were given as follows : Suppose $X = [a,b]$ is a closed interval of real number and $\mathcal{KC}(Y) = \mathcal{KC}(\mathbb{R})$.

Example 4.3. *Let $F : [a,b] \longrightarrow \mathcal{KC}(\mathbb{R})$ be a set-valued with $F(t) = A$ is a constant for all $t \in [a,b]$. F is absolutely differentiable on (a,b) with derivative*

$$F'_{abs}(t) = \lim_{h \to 0} \frac{H(F(t+h), F(t))}{h} = \lim_{h \to 0} \frac{H(A,A)}{h} = 0$$

for all $t \in (a,b)$.

Example 4.4. *Let $F : [a,b] \longrightarrow \mathcal{KC}(\mathbb{R})$ be a set-valued with $F(t) = \{t\}$ for all $t \in [a,b]$. F is absolutely differentiable at $p \in (a,b)$ with derivative*

$$F'_{abs}(p) = \lim_{t \to p} \frac{H(F(t), F(p))}{|t-p|} = \lim_{t \to p} \frac{H(\{t\}, \{p\})}{|t-p|} = \lim_{t \to p} \frac{|t-p|}{|t-p|} = 1.$$

The following example is for the interval-valued function.

Example 4.5. *Let $F : [0,1] \longrightarrow \mathcal{I}(\mathbb{R})$ be a interval-valued with $F(t) = [t, 2t]$ for all $t \in [0,1]$. F is Hausdorff metrically differentiable at $p \in (0,1)$ with derivative*

$$\begin{aligned}
F'_{abs}(p) &= \lim_{x\to p} \frac{H(F(t), F(p))}{|t-p|} = \lim_{t\to p} \frac{H([t,2t],[p,2p])}{|t-p|} \\
&= \lim_{t\to p} \frac{\max\{|t-p|, |2t-2p|\}}{|t-p|} \\
&= \lim_{t\to p} \frac{|2t-2p|}{|t-p|} = 2.
\end{aligned}$$

The following example is a vector-valued function in the set form.

Example 4.6. *Let $F : [a,b] \longrightarrow \mathcal{KC}(\mathbb{R}^k)$ be a set-valued with $F(t) = f(t)\bar{B}$ for all $t \in (a,b)$, where f is real valued function differentiable on (a,b) , $f(t) \geq 0$ for all $t \in (a,b)$ and $\bar{B}$ is the closed unit ball in $\mathbb{R}^k$. Then F is absolutely differentiable for each $p \in (a,b)$ with derivative*

$$F'_{abs}(p) = |f'(p)|\|\bar{B}\|.$$

Because the limit

$$\begin{aligned}
F'_{abs}(p) &= \lim_{t\to p} \frac{H(F(t), F(p))}{|x-p|} = \lim_{t\to p} \frac{\|F(t) \overset{gh}{-} F(p)\|}{|x-p|} \\
&= \lim_{t\to p} \frac{\|f(t)\bar{B} \overset{gh}{-} f(p)\bar{B}\|}{|x-p|} = \lim_{t\to p} \frac{\|(f(t) \overset{gh}{-} f(p))\bar{B}\|}{|x-p|} \\
&= \lim_{t\to p} \frac{|f(t) \overset{gh}{-} f(p)|\|\bar{B}\|}{|x-p|} \\
&= \lim_{t\to p} \frac{|f(t) - f(p)|\|\bar{B}\|}{|x-p|} \\
&= |f'(p)|\|\bar{B}\|.
\end{aligned}$$

Let me several fundamental properties of absolutely differentiable set-valued functions are reviewed.

Theorem 4.7. *Let (X,d) and (Y,ρ) be two metric spaces. If set-valued $F : X \longrightarrow \mathcal{KC}(Y)$ is absolutely differentiable at a point $p \in X$ then F is continuous at $p \in X$.*

Proof.

$$\begin{aligned}
\lim_{x\to p} H(F(x), F(p)) &= \lim_{x\to p} \left[\frac{H(F(x), F(p))}{d(x,p)} d(x,p) \right] \\
&= \left[\lim_{x\to p} \frac{H(F(x), F(p))}{d(x,p)} \right] \left[\lim_{x\to p} d(x,p) \right] \\
&= F'_{abs}(p) \cdot 0 = 0.
\end{aligned}$$

□

Theorem 4.8. *If $F : X \longrightarrow \mathcal{KC}(Y)$ is absolutely differentiable at point $p \in X$ and $G : Y \longrightarrow \mathcal{KC}(Z)$ is absolutely differentiable on the set $F(p) \in \mathcal{KC}(Y)$, then the composition set-valued map $G \circ F : X \longrightarrow \mathcal{KC}(Z)$ is also differentiable at the point $p \in X$, and*

$$(G \circ F)'_{abs}(p) = G'_{abs}(F(p))F'_{abs}(p)$$

Proof. We observe the limit

$$\begin{aligned}
\lim_{d(x,p)\to 0} \frac{H((G\circ F)(x),(G\circ F)(p))}{d(x,p)} &= \lim_{d(x,p)\to 0} \frac{H(G(F(x)),G(F(p))}{d(x,p)} \\
&= \lim_{d(x,p)\to 0} \left[\frac{H(G(F(x)),G(F(p))}{H(F(x),F(p))}\frac{H(F(x),(F(p))}{d(x,p)}\right] \\
&= \left[\lim_{d(x,p)\to 0} \frac{H(G(F(x)),G(F(p)))}{H(F(x),F(p))}\right] \\
&\qquad \left[\lim_{d(x,p)\to 0} \frac{H(F(x),F(p))}{d(x,p)}\right] \\
&= \left[\lim_{H(F(x),F(p))\to 0} \frac{H(G(F(x)),G(F(p)))}{H(F(x),F(p))}\right] \\
&\qquad \left[\lim_{d(x,p)\to 0} \frac{H(F(x),F(p))}{d(x,p)}\right].
\end{aligned} \tag{4.5}$$

The last equality holds since $d(x,p)\to 0$ implies $H(F(x),F(p))\to 0$ (Theorem 4.7). By the hypothesis, we have the equality

$$\lim_{H(F(x),F(p))\to 0} \frac{H(G(F(x)),G(F(p)))}{H(F(x),F(p))} = G'_{abs}(F(p))$$

and

$$\lim_{d(x,p)\to 0} \frac{H(F(x),F(p))}{d(x,p)} = F'_{abs}(p).$$

This means the limit 4.5 exists and equals $(G\circ F)'_{abs}(p)$ so that $(G\circ F)'_{abs}(p) = G'_{abs}(F(p))F'_{abs}(p)$, and the proof is complete. □

It is well known fact that for a function $f:\mathbb{R}\longrightarrow\mathbb{R}$ this statement holds: if $f'=0$ then f is constant. In general, this is not true if $\mathbb{R}$ space is replaced topological space (see [4]). However, our result show that the derivative of set-valued function in the sense of the metric space is zero if and only if set-valued function is constant provided its metric space has the geometric property of being *rectifiably connected*.

In [1] be given the notion as follow. The metric space X is said to be *path connected* if for any two points x and y in X there exists a homeomophism $\gamma:[0,1]\longrightarrow X$ with $\gamma(0)=x$ and $\gamma(1)=y$. This function is called a path from x to y. A path is said *rectifiable* if its length is a finite number.

We now proceed to prove the following as standard result on the real line for the absolute derivative.

Theorem 4.9. *Let X be rectifiably connected metric space, and let Y be any metric space. The set-valued $F:X\longrightarrow\mathcal{KC}(Y)$ is absolutely differentiable on X. Then F is constant if and only if, $F'_{abs}(x)=0$ for every $x\in X$.*

Proof. If F is constant, it is trivial by definition direct. Conversely, suppose that F is not a constant, namely for every point $p\neq q\in X$, $F(p)\neq F(q)$.

Let E be a rectifiable arc from p to q in X, with length $l > 0$. We take the real number $\epsilon > 0$ with

$$\epsilon = \frac{H(F(p), F(q))}{l}. \tag{4.6}$$

Since $F'_{abs}(x) = 0$, for all $x \in X$, of course for each $x \in E$ there exists $\delta_x > 0$ such that for all $y \in N(x, \delta_x)$ we have

$$\frac{H(F(x), F(y))}{d(x, y)} < \epsilon. \tag{4.7}$$

Let $\mathcal{N} = \{N(x, \delta_x) \mid x \in E\}$ and the collection $\mathcal{C} = \{C \mid C$ is path component of some members of $\mathcal{N}\}$ is an open cover of E. Since E is compact there exists the points $c_1, c_2, \dots c_m \in E$ such that $\{C_j \mid c_j \in C_j, 1 \leq j \leq m\} \subset \mathcal{C}$ be a finite cover of E. Since for each members of $\mathcal{C}$ is connected, it follow that for each $j = 1, 2, \dots m, C_j \cap C_{j+1} \neq \emptyset$.
Let $p_0, p_1, \dots p_m \in E$ such that $p_0 = p, p_m = q$ and $p_j \in C_j \cap C_{j+1}$ for each $1 \leq j \leq m$. Then for each j,

$$H(F(p_j), F(c_{j+1})) + H(F(F(c_{j+1}), p_j)) < \epsilon\,[d(p_j, c_{j+1}) + d(c_{j+1}, p_j)] \tag{4.8}$$

by inequalities 4.7. From the equality 4.6 and the inequality 4.7 and 4.8, we obtain

$$\begin{aligned} H(F(p), F(q)) &= H(F(p_0), F(p_m)) \\ &\leq \sum_{j=0}^{m-1} H(F(p_j), F(c_{j+1})) + H(F(c_{j+1}), F(p_j)) \\ &< \sum_{j=0}^{m-1} \epsilon\,[d(p_j, c_{j+1}) + d(c_{j+1}, p_j)] \\ &= \epsilon \sum_{j=0}^{m-1} [d(p_j, c_{j+1}) + d(c_{j+1}, p_j)] = \epsilon l = H(F(p), F(q)), \end{aligned}$$

a contradiction and should be $F(p) = F(q)$ is a constant □

4.1. **Comparison with another definition of derivative.** In this subsection, we will comparisons our concept with the concept differentiability introduced by M. Hukuhara [6] and L. Stefanini [15].

Theorem 4.10. *If $F : [a, b] \longrightarrow \mathcal{KC}(\mathbb{R}^k)$ is Hukuhara differentiable at $t \in (a, b)$, then F is absolutely differentiable at $t \in (a, b)$, and in this case*

$$F'_{abs}(t) = \|F'_h(t)\|.$$

Proof. Suppose $x = t + \Delta t$. If $x \to t$ then $\Delta t \to 0$. One obtain for $\Delta t > 0$

$$\lim_{x \to t} \frac{H(F(x), F(t))}{|x - t|} = \lim_{\Delta t \to 0^+} \frac{H(F(t + \Delta t), F(t))}{\Delta t}.$$

Since F is Hukuhara differentiable at $t \in (a, b)$, this means the Hukuhara difference $F(t+\Delta t) \overset{h}{-} F(t)$ exists. By using Proposition 2.8 of the part 2.11 and 2.13 obtained as follows

$$\begin{aligned}\lim_{x\to t}\frac{H(F(x),F(t))}{|x-t|} &= \lim_{\Delta t\to 0^+}\frac{H(F(t+\Delta t),F(t))}{\Delta t}\\ &= \lim_{\Delta t\to 0^+}\frac{H(F(t+\Delta t)\overset{h}{-}F(t)),\Theta)}{\Delta t}\\ &= \lim_{\Delta t\to 0^+}\frac{\|F(t+\Delta t)\overset{h}{-}F(t))\|}{\Delta t}\\ &= \|F'_h(t)\|.\end{aligned}$$

While for the real number $\Delta t<0$ obtained (with $k=-\Delta t$)

$$\begin{aligned}\lim_{x\to t}\frac{H(F(x),F(t))}{|x-t|} &= \lim_{\Delta t\to 0^-}\frac{H(F(t),F(t+\Delta t))}{\Delta t}\\ &= \lim_{-\Delta t\to 0^+}\frac{H(F(t),F(t-(-\Delta t))}{-\Delta t}\\ &= \lim_{k\to 0^+}\frac{H(F(t),F(t-k))}{k}\end{aligned}$$

Since F is Hukuhara differentiable at $t\in I$ this means the Hukuhara difference $F(t)\overset{h}{-}F(t-k)$ exists. Therefore similarly, by using again the Proposition 2.8 of the part 2.11 and 2.13 obtained as follows.

$$\begin{aligned}\lim_{x\to t}\frac{H(F(x),F(t))}{|x-t|} &= \lim_{k\to 0^+}\frac{H(F(t),F(t-k))}{k}\\ &= \lim_{k\to 0^+}\frac{H(F(t)\overset{h}{-}F(t-k)),\Theta)}{k}\\ &= \lim_{k\to 0^+}\frac{\|F(t)\overset{h}{-}F(t-k))\|}{k}\\ &= \|F'_h(t)\|.\end{aligned}$$

Thus F is absolutely differentiable at $t\in(a,b)$ and $F'_{abs}(t)=\|F'_h(t)\|$. □

Example 4.11. *From Example 3.2, the Hukuhara derivative is $F'_h(p)=[1,2]$ and from Example 4.5, the absolute derivative is $F'_{abs}(p)=2$. This means*

$$F'_{abs}(p)=2=\sup\{|x|\mid x\in[1,2]=F'_h(p)\}=\|F'_h(p)\|.$$

The converse does not necessarily true. Suppose $F:[-1,1]\longrightarrow\mathcal{I}(\mathbb{R})$ and $F(t)=(1-t^2)[-2,1]$ for all $t\in[-1,1]$. Then F is absolutely differentiable at $t=0$ with the derivative

$$\begin{aligned}F'(0)=\lim_{t\to 0}\frac{H(F(t),F(0))}{t} &= \lim_{t\to 0}\frac{\max\{d(F(t),F(0)),d(F(0),F(t))\}}{t}\\ &= \lim_{t\to 0}\frac{\max\{d([-2(1-t^2],[-2,1]),d([-2,1],[-2(1-t^2])\}}{t}\\ &= \lim_{t\to 0}\frac{\max\{0,-2t^2\}}{t}=0.\end{aligned}$$

But F is not differentiable in the sense of Hukuhara at the point $t = 0$ because the Hukuhara difference $F(0+\Delta t) \overset{h}{-} F(0) = (1-(\Delta t)^2)[-2,1] \overset{h}{-} [-2,1]$ does not exist (as $\Delta t \to 0$). Hence there is no value for $F'_h(0)$.

The next main result shows that the Definition 3.5 is equivalent to our new definitions in the context of an interval-valued function.

Theorem 4.12. *The set-valued $F : [a,b] \longrightarrow \mathcal{I}(\mathbb{R})$ is gh-differentiable at $t_0 \in (a,b)$ if and only if F is Hausdorff metrically differentiable at $t_0 \in (a,b)$, and*

$$F'_{abs}(t_0) = |F'_{gh}(t_0)|.$$

Proof. Let $F(t) = [f(t), g(t)]$ be an interval-valued function for all $t \in [a,b]$, where f and g are two real valued functions on $[a,b]$. Since F is gh-differentiable at $t_0 \in (a,b)$, the real valued function f and g are differentiable at $t_0 \in (a,b)$ and

$$F'_{gh}(t_0) = [\min\{f'(t_0), g'(t_0)\}, \max\{f'(t_0), g'(t_0)\}]$$

by Theorem 3.6.
The absolute value of the interval $F'_{gh}(t_0)$ is

$$\begin{aligned} |F'_{gh}(t_0)\,| &= \max\{\min\{|f'(t_0)|, |g'(t_0)|\}, \max\{|f'(t_0)|, |g'(t_0)|\}\} \\ &= \max\{|f'(t_0)|, |g'(t_0)|\}. \end{aligned} \tag{4.9}$$

By the limit (as $h \longrightarrow 0^+$) and from the equality 2.4 and the equality 4.9, we obtained

$$\begin{aligned} \lim_{h\to 0^+} \frac{H(F(t_0+h), F(t_0))}{|h|} &= \lim_{h\to 0^+} \frac{1}{|h|} H\left([f(t_0+h), g(t_0+h)], [f(t_0), g(t_0)]\right) \\ &= \lim_{h\to 0^+} \frac{1}{|h|} \max\left\{|f(t_0+h) - f(t_0)|, |g(t_0+h) - g(t_0)|\right\} \\ &= \max\left\{\lim_{h\to 0^+} \frac{1}{|h|}\left|f(t_0+h) - f(t_0)\right|, \lim_{h\to 0^+} \frac{1}{|h|}|g(t_0+h) - g(t_0)|\right\} \\ &= \max\{|f'(t_0)|, |g'(t_0)|\} \\ &= |F'_{gh}(t_0)\,|. \end{aligned}$$

So that F is Hausdorff metrically differentiable at $t_0 \in (a,b)$ and $F'_{abs}(t_0) = |F'_{gh}(t_0)|$.

To the converse, the interval-valued function F is Hausdorff metrically differentiable at the point $t_0 \in (a,b)$. This means the limit

$$\begin{aligned} \lim_{h\to 0^+} \frac{H(F(t_0+h), F(t_0))}{|h|} &= \lim_{h\to 0^+} \frac{1}{|h|} H\left([f(t_0+h), g(t_0+h)], [f(t_0), g(t_0)]\right) \\ &= \lim_{h\to 0^+} \frac{1}{|h|} \max\left\{|f(t_0+h) - f(t_0)|, |g(t_0+h) - g(t_0)|\right\} \end{aligned}$$

exists and equals $F'_{abs}(t_0)$. Obviously that the limit both

$$\lim_{h\to 0^+} \frac{1}{|h|}|f(t_0+h) - f(t_0)|$$

and

$$\lim_{h\to 0^+} \frac{1}{|h|}|g(t_0+h)-g(t_0)|$$

exist. In the other words the real valued functions f and g are differentiable at the point $t_0 \in (a,b)$. By the Theorem 3.6, the set-valued functions F is gh-differentiable at $t_0 \in (a,b)$. Furthermore, we will show the equality $|F'_{gh}(t_0)| = F'_{abs}(t_0)$.
Since F is gh-differentiable at $t_0 \in (a,b)$, we obtain

$$\begin{aligned}
|F'_{gh}(t_0)| &= \lim_{h\to 0}\left|\frac{F(t_0+h) \overset{gh}{-} F(t_0))}{h}\right| \\
&= \lim_{h\to 0^+}\frac{1}{|h|}\left|[f(t_0+h), g(t_0+h)] \overset{gh}{-} [f(t_0), g(t_0)]\right| \\
&= \lim_{h\to 0^+}\frac{1}{|h|}|[\min\{(f(t_0+h)-f(t_0)),(g(t_0+h)-g(t_0))\}, \\
&\qquad \max\{(f(t_0+h)-f(t_0)),(g(t_0+h)-g(t_0))\}]| \qquad (4.10)
\end{aligned}$$

Without loss generality of the proof, we may assume $f(t_0+h)-f(t_0) \leq g(t_0+h)-g(t_0)$ so that the equality 4.10 to be

$$\begin{aligned}
|F'_{gh}(t_0)| &= \lim_{h\to 0^+}\frac{1}{|h|}\ |[f(t_0+h)-f(t_0), g(t_0+h)-g(t_0)]\ | \\
&= \lim_{h\to 0^+}\frac{1}{|h|}\max\left\{|(f(t_0+h)-f(t_0))|, |(g(t_0+h)-g(t_0))|\right\}. \\
&= \lim_{h\to 0^+}\frac{1}{|h|}H\left([f(t_0+h), g(t_0+h)], [f(t_0), g(t_0)]\right) \\
&= \lim_{h\to 0^+}\frac{H(F(t_0+h), F(t_0))}{|h|} = F'_{abs}(t_0).
\end{aligned}$$

Hence the proof is complete. □

Now, we can state as follows

Definition 4.13. *Suppose $F : [a,b] \longrightarrow \mathcal{I}(\mathbb{R})$ is an interval-valued function with $F(t) = [f(t), g(t)]$ where f, g is real valued function for each $t \in [a,b]$. F is called absolutely differentiable at $t_0 \in (a,b)$ if f and g are differentiable at $t_0 \in (a,b)$ and*

$$F'_{abs}(t_0) = \max[|f'(t_0)|, |g'(t_0)|].$$

The last of the main result we will be shown that the derivative F'_{abs} is Lipschitz with to respect the functional α (Lipshitz (α)). Let us introduce the following [5].

Definition 4.14. *Let X be a metric space and let $A \in \mathcal{B}(X)$. The measure $\alpha(A)$ of non-compactness of A is defined by*

$$\alpha(A) = \inf\{\epsilon > 0 \mid \exists K \in \mathcal{K}(X), A \subset \bar{S}_\epsilon(K)\}.$$

where $\bar{S}_\epsilon(K) = \{x \in X \mid d(x,K) \leq \epsilon\}$

The functional α is called Kuratowski's measure of non-compactness. The functional α has the properties

Lemma 4.15. *Let $A, B \in \mathcal{B}(X)$*

(a) $\alpha(A) = 0$ *if and only if $\bar{A}$ is compact*

(b) *if* $A \subset B$ *then* $\alpha(A) \leq \alpha(B)$
(c) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$

This following by D. Blasi [5].

Definition 4.16. *Let* U *a non-empty open subset of* X. *The set-valued* $F : U \longrightarrow \mathcal{K}(X)$ *is said to be Lipschitz* (α) *with constant* $L \geq 0$ *if for every* $A \in \mathcal{B}(X)$ *with* $A \subset U$, *we have*

$$\alpha(F(A)) \leq L\alpha(A). \tag{4.11}$$

Theorem 4.17. *If set-valued* $F : U \longrightarrow \mathcal{K}(X)$ *is Hausdorff metrically differentiable on* U, *then the derivative* $F'_{abs}(x)$ *is a Lipschitz* (α) *with constant* $L > 0$ *for every* $x \in U$.

Proof. Let $A \subset \mathcal{B}(X)$ and $A \subset U$. By hypothesis there exists $F'_{abs}(x) \in [0, \infty)$ for every $x \in A$. We have

$$F'_{abs}(A) = \bigcup_{x \in A} F'_{abs}(x)$$

Since $F'_{abs}(x)$ is singleton for each $x \in A$ (hence compact) and by Lemma 4.15 part (c), we obtain

$$\begin{aligned} \alpha(F'_{abs}(A)) &= \alpha\left(\bigcup_{x \in A} F'_{abs}(x)\right) \\ &= \sup\{\alpha(F'_{abs}(x)) \mid x \in A\} = \sup\{0\} = 0 \leq L\alpha(A). \end{aligned}$$

It is proved that F'_{abs} Lipschitz (α) with the same constant $L > 0$ □

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

References

[1] A. Adrei., *Quasiconformal Mappings on Certain Classes of Domain in Metric Spaces*, Bull. of the Transilvania University of Brasov, Series III, **5 54** (2012), 11–22.
[2] H. T. Banks and M. Q. Jacobs, *A differential calculus for multifunctions*, J. Math. Anal. Appl.**19** (1970), 246–272.
[3] T.F Bridgland jr., *Trajectory Integral of set-valued functions*, Pacific J. Math. **33 1** (1970), 43–67.
[4] W.J. Charathonik, M. Insall, *Absolute Differentiation in Metric Spaces*, Houston Journal of Mathematics **38 4** (2012), 1313–1328.
[5] F. S. De Blasi., *On the Differentiablity of multifunctions*, Pacific J. Math. **66 1** (1976), 67–80.
[6] M. Hukuhara, *Intgration des applications mesurables dont la valeur est un compact convex*, Funkcial. Ekvac. **10** (1967), 205–229.
[7] M. Kisielewicz, *Differential inclusion and optimal control*, Polish Scientific Publishers and Kluwer Academic Publishers (1991).
[8] V. Laksmikantham, T.G. Baskhar and Devi J. Vasundhara., *Theory of Set Differential Equation in Metric space*, Cambridge Scientific Publisher, (2006).
[9] A. Lasota and A Strauss., *Asymptotic Behavior for Differential Equations Which Cannot be Locally Linearized*, J. of Differential Equations **10** (1971), 152–172.
[10] S. Markov., *Calculus for Interval Function of a real variables* , Computing, **22** (1969), 325–337.
[11] M. Muslikh, A. Kilicman, S. Hasana, Norfifah., *The metric derivative of set-valued functions* , Adv. Pure Appl. Math. **10 3** (2019), 263–272.
[12] R.E. Moore, R.B. Kearfott and M.J. Cloud., *Introduction to Interval Analysis*, The society for Industrial and applied Mathematics, Philadelphia, USA, (2009).

[13] R. Moriello, N. *Partial Differentiation on a Metric Spaces* , Phi Mu Epsilon Journal, **5 10** (1974), No.10, 514–519.
[14] K. Skaland, *Differentiation on Metric Spaces*, Proc. S.D. Acad. Sci.**54** (1975), 75–77.
[15] L. Stefanini and B. Bede., *Generalized Hukuhara Differentiability of Interval Valued Functions and Interval Differential Equations*, Nonlinear Analysis **71** (2009), 1311–1328.
[16] L. Stefanini., *A generalization of Hukuhara difference and division for Interval and Fuzzy Arithmetic*, Fuzzy Sets and System **161 11** (2010), 1564–1584.

Mohamad Muslikh
Department of Mathematics, Faculty of Mathematics and Sciences, Universitas Brawijaya, Malang 65143, East Java, Indonesia.
E-mail address: mslk@ub.ac.id

Adem Kilicman
Department of Mathematics, Faculty of Sciences, Universiti Putra malaysia, Serdang, Selangor, Malaysia.
E-mail address: akilic@upm.edu.my