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ON QUASI STATISTICAL CONVERGENCE OF FUZZY VARIABLE SEQUENCES

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Abstract. In this paper, we introduce and investigate the notion of quasi statistical convergence of fuzzy variable sequences in credibility space. We also establish the relationship of the notion with statistical convergence of fuzzy variable sequences. In addition, we examine some interrelationships between quasi statistical convergence almost surely, quasi statistical convergence in measure, quasi statistical convergence in mean, and quasi statistical convergence in distribution.

1. Introduction

In 1965, the notion of fuzzy sets was put forward to by Zadeh [32]. These days, it has extensive applications in various branches of engineering and science. A fuzzy variable is a function from a credibility space (expressed with the credibility measure) to the set of real numbers. The convergence of fuzzy variables is considerable component of credibility theory, which can be applied into real problems in mathematical finance and engineering. Fuzzy variable, possibility distribution and membership function was presented by Kaufmann [7]. Possibility measure, which is generally identified as supremum preserving set function on the power set of a nonempty set, is a fundamental notion in possibility theory but it is not self-dual. Since a self-dual measure is absolutely required in both theory and practice, Liu and Liu [14] investigated a self-duality credibility measure. The credibility measure plays the role of possibility measure in fuzzy world because it shares some fundamental features with possibility measure. Particularly, since Liu has begun the survey of credibility theory, and then many specific contents have been worked (see [8, 10, 12, 13, 15]). Contemplating sequence convergence plays a key role in credibility theory, Liu [11] put forward four kinds of convergence concept for fuzzy variables: convergence in credibility, convergence almost surely, convergence in mean, convergence in distribution. Additionally, based upon credibility theory, several convergence features of credibility distribution for fuzzy variables were examined by Jiang [5] and Ma [17].

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Wang and Liu [28] put forward the relationships among convergence in mean, convergence in credibility, convergence almost uniformly, convergence in distribution, and convergence almost surely. Moreover, several researchers highlighted convergence notions in classical measure theory, credibility theory, probability theory, and established the connections between them. For more details on credibility theory, one may refer to Chen et al. [2], Lin [9], Liu and Wang [16], Xia [29], and You [30, 31].

Statistical convergence was put forward by Fast [3] as a generalization of ordinary convergence for real sequences. Statistical convergence has also been studied in more general abstract spaces such as the fuzzy number space [21]. Further investigations in this direction and further applications of statistical convergence can be seen in [1, 4, 6, 18, 19, 23, 25, 26, 27]. The reader can refer to [20] for summability theory and related background.

In an attempt to generalize the notion of statistical convergence, in 2012 Özgüç and Yurdakadim [22] investigated the concept of quasi-statistical convergence in terms of quasi-density.

In [22], the notion of natural density was extended to quasi density by involving a sequence $c = (c_t)$ satisfying the following features:

$$c_t > 0 : \forall t \in \mathbb{N}, \lim_{t\to\infty} c_t = \infty \text{ and } \limsup_t \frac{c_t}{t} < \infty. \tag{1.1}$$

The quasi-density of a set $K \subseteq \mathbb{N}$ is defined by $\delta_c(K) = \lim_{t\to\infty} \frac{|K_t|}{c_t}$, provided the limit exists. It should be noted that if $c_t = t$, then the above definition turns to the definition of natural density. Throughout the paper, we write $c = (c_t)$ for the sequences satisfying the relation (1.1).

In [22], Özgüç and Yurdakadim introduced the notion of quasi statistical convergence of real-valued sequences as follows:

A sequence (w_i) is said to be quasi statistical convergent to w provided that for all $\eta > 0$,

$$\delta_c(\{i \in \mathbb{N} :| w_i - w |\geq \eta\}) = 0.$$

In this case, w is named the quasi statistical limit of the sequence (w_i) and symbolically it is expressed as $w_i \xrightarrow{st_q} w$. They mainly worked the relationship of quasi statistical convergence and statistical convergence and demonstrate that the condition $\inf_t \frac{c_t}{t} > 0$ along with (1.1), plays a significant role for the equivalence of the concepts.

It should be noted that, if we choose $c_t = t, : \forall t \in \mathbb{N}$, then the definition of quasi statistical convergence turns to the definition of statistical convergence.

The aim of the present paper is to investigate the new kind of convergence for fuzzy variables sequences. The following is how the paper is structured. The literature review is covered in Section 1 of the introduction. The key findings are then demonstrated in Section 2. That is, we intend to investigate the concept of quasi statistical convergence of fuzzy variables and to develop essential features of quasi statistical convergence in credibility. Section 3 concludes with the findings of the acquired results.

2. Preliminaries

A set function Cr is credibility measure if it supplies the subsequent axioms: Let Θ be a nonempty set, and the power set $\mathcal{P}(\Theta)$ of Θ (i.e., the largest algebra over Θ). Each element in $\mathcal{P}$ is called an event. For any $Y \in \mathcal{P}(\Theta)$, Liu and Liu [14] presented a crebility measure $\mathrm{Cr}\{Y\}$ to express the chance that fuzzy event Y occurs. Li and Liu [8] proved that a set function $\mathrm{Cr}\{.\}$ a crebility measure if and only if

Axiom i. $\mathrm{Cr}\{\Theta\} = 1$;

Axiom ii. $\mathrm{Cr}\{Y\} \leq \mathrm{Cr}\{Z\}$ whenever $Y \subset Z$;

Axiom iii. Cr is self-dual, i.e., $\mathrm{Cr}\{Y\} + \mathrm{Cr}\{Y^c\} = 1$, for any $Y \in \mathcal{P}(\Theta)$;

Axiom iv. $\mathrm{Cr}\{\cup_i Y_i\} = \sup_i \mathrm{Cr}\{Y_i\}$ for any collection $\{Y_i\}$ in $\mathcal{P}(\Theta)$ with $\sup_i \mathrm{Cr}\{Y_i\} < 0.5$.

The triplet $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ is named a crebility space. A fuzzy variable is investigated by Liu and Liu [14] as function from the crebility space to the set of real numbers.

Example 2.1. *Let $\Theta = \{\theta_1, \theta_2\}$. For this case, there are only four events: $\emptyset$, $\{\theta_1\}$, $\{\theta_2\}$, Θ. Determine $\mathrm{Cr}\{\Theta\} = 0$, $\mathrm{Cr}\{\theta_1\} = 0.7$, $\mathrm{Cr}\{\theta_2\} = 0.3$, and $\mathrm{Cr}\{\Theta\} = 1$. Then, the set function Cr is a credibility measure because it supplies the four axioms.*

Definition 2.1. *([14]) The expected value of fuzzy variable μ is given by*

$$E[\varpi] = \int_0^{+\infty} \mathrm{Cr}\{\varpi \geq r\}\, dr - \int_{-\infty}^{0} \mathrm{Cr}\{\varpi \leq r\}\, dr$$

provided that at least one of the two integrals is finite.

If there is a $H > 0$ such that

$$\mathrm{Cr}\{\varpi \leq -H\} = 0$$

and

$$\mathrm{Cr}\{\varpi \leq H\} = 1,$$

then fuzzy variable ϖ is named as essentially bounded.

Theorem 2.1. *(Wang and Liu [28]) When the sequence $\{\varpi_i\}$ convergence in credibility to μ, then $\{\varpi_i\}$ converges a.s. to μ.*

Theorem 2.2. *(Liu, [11]) When the sequence $\{\varpi_i\}$ convergence in mean to μ, then $\{\mu_i\}$ converges credibility to μ.*

A sequence $\{\varpi_i\}$ of fuzzy variables is named as uniformly essentially bounded (UEB, shortly) provided that there is a $M > 0$ such that for all k, we get

$$\mathrm{Cr}\{\varpi_i \leq -M\} = 0$$

and

$$\mathrm{Cr}\{\varpi_i \leq M\} = 1.$$

Theorem 2.3. *([16]) Presume that $\{\varpi_i\}$ is a sequence of UEB fuzzy variables. If $\{\varpi_i\}$ is convergent in credibility to ϖ, then*

$$\lim_{i \to \infty} E[\varpi_i] = E[\varpi].$$

Theorem 2.4. *([9]) Assume $f : \mathbb{R} \to \mathbb{R}$ be a convex function. Then, there is $c > 0$ so that*

$$|f(z_1) - f(z_2)| \leq c\,|z_1 - z_2|,$$

for any $z_1, z_2 \in \mathbb{R}$.

Theorem 2.5. *Let ϖ be a fuzzy variable. Then, for any given numbers $t > 0$ and $p > 0$, we have*

$$Cr\left\{|\varpi| \geq t\right\} \leq \frac{E\left[|\varpi|^p\right]}{t^p}. \tag{2.1}$$

Theorem 2.6. *([28]) If the sequence $\{\varpi_i\}$ convergence in credibility to μ, then $\{\varpi_i\}$ converges a.s. to μ.*

3. Main Results

In this section, based on existing quasi statistical convergence, we study the quasi statistical convergence in credibility and the quasi statistical Cauchy sequence in credibility. In order to better explain our results, we first put forward some significant definitions.

Definition 3.1. *Let $\varpi, \varpi_1, \varpi_2, ...$ be fuzzy variables identified on credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$. The sequence $\{\varpi_i\}$ is said to be quasi statistically convergent almost surely (q.st.a.s.) in credibility space to the fuzzy variable ϖ iff there exists $Y \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{Y\} = 1$ so that*

$$\lim_{t\to\infty} \frac{1}{c_t} \left|\left\{i \leq t : \left|\varpi_i(\theta) - \varpi(\theta)\right| \geq \eta\right\}\right| = 0$$

for all $\theta \in Y$, each $\eta > 0$. Symbolically, we denote $\varpi_i \overset{q.st.a.s.}{\to} \varpi$.

Definition 3.2. *The sequence $\{\varpi_i\}$ is named to be quasi statistically convergent in credibility to ϖ iff there is $Y \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{Y\} = 1$ so that*

$$\lim_{t\to\infty} \frac{1}{c_t} \left|\left\{i \leq t : \mathrm{Cr}\left\{|\varpi_i - \varpi| \geq \eta\right\} \geq \gamma\right\}\right| = 0$$

for all η, $\gamma > 0$. Symbolically, we indicate $qst(\mathrm{Cr}) - \lim \varpi_i = \varpi$ or $\varpi_i \overset{q.st.\mathrm{Cr}}{\to} \varpi$.

Remark. *When $c_t = t$, then Definition 3 is reduced to the statistical convergence of fuzzy variable sequences, (see [24]).*

Example 3.1. *Take $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ to be $\{\theta_1, \theta_2, ...\}$ with $\mathrm{Cr}\{\theta_s\} = \frac{2s}{4s+1}$ for $s = 1, 2, ...$ The fuzzy variables are determined as*

$$\varpi_i(\theta_s) = \begin{cases} 2s, & \text{if } s = i \\ 0, & \text{otherwise} \end{cases}$$

for $i = 1, 2, ...$, $\varpi = 0$. But, for $\gamma \in \left[\frac{1}{2}, 1\right)$ and for any small number $\eta > 0$,

$$\mathrm{Cr}\left\{|\varpi_i - 0| \geq \eta\right\} = \frac{2i}{4i+1} \to \frac{1}{2}.$$

Therefore, $\varpi_i \overset{q.st.\mathrm{Cr}}{\to} \varpi$, but it is not quasi convergent in credibility.

Example 3.2. *Quasi statistical convergence a.s. does not imply quasi statistical convergence in credibility. To demonstrate this, presume that $\Theta = \{\theta_1, \theta_2, ...\}$, $\mathrm{Cr}\{\theta_1\} = 1$ and $\mathrm{Cr}\{\theta_s\} = (s-1)/s$ for $s = 2, 3, ...$ and the fuzzy variables are indicated by*

$$\varpi_i(\theta_s) = \begin{cases} i, & \text{if } s = i \\ 0, & \text{otherwise} \end{cases}$$

for $i=1,2,...$ and $\varpi=0$. Then, we get $\varpi_i \overset{q.st.a.s.}{\to} \varpi$. But, for $\gamma\in\left(0,\frac{1}{2}\right)$ and any small number $\eta>0$, the sequence $\{\varpi_i\}$ is not quasi statistically convergent in credibility. Also,

$$\mathrm{Cr}\left\{|\varpi_i-\varpi|\geq\eta\right\}=\frac{i-1}{2i}\nrightarrow 0.$$

That is to say, the sequence $\{\varpi_i\}$ does not quasi converge in credibility to ϖ.

Example 3.3. *Conversely, quasi statistical convergence in credibility does not imply quasi statistical convergence a.s., too. For instance, $\Theta=\{\theta_1,\theta_2,...\}$, $\mathrm{Cr}\left\{\theta_s\right\}=1/s$ for $s=1,2,...$ and the fuzzy variables are established by*

$$\varpi_i\left(\theta_s\right)=\begin{cases}(s+1)/s, & \text{if } s=i,i+1,i+2,...\\ 0, & \text{otherwise}\end{cases}\tag{3.1}$$

for $i=1,2,...$ and $\mu=0$. At that time, for $\gamma\in\left[\frac{1}{2},1\right)$ and any small number $\eta>0$,

$$\lim_{t\to\infty}\frac{1}{c_t}\left|\left\{i\leq t:\mathrm{Cr}\left\{|\varpi_i-\varpi|\geq\eta\right\}\geq\gamma\right\}\right|=0$$

which claims that $\varpi_i \overset{q.st.\mathrm{Cr}}{\to} \varpi$. But, it is clear that $\varpi_i \overset{q.st.a.s.}{\nrightarrow} \varpi$.

Definition 3.3. *Suppose that $\{\varpi_i\}$ is a sequence of fuzzy variables with finite expected values identified on $\left(\Theta,\mathcal{P}\left(\Theta\right),Cr\right)$. The sequence $\{\varpi_i\}$ is called to be quasi statistically convergent in mean to the fuzzy variable ϖ if*

$$\lim_{t\to\infty}\frac{1}{c_t}\left|\left\{i\leq t:E\left[|\varpi_i-\varpi|\right]\geq\eta\right\}\right|=0$$

for all $\eta>0$. Symbolically, we indicate $\varpi_i \overset{q.st.E.}{\to} \varpi$.

Example 3.4. *Quasi statistical convergence in mean does not imply quasi statistical convergence a.s. Observe the fuzzy variables given by (3.1) which does not quasi statistically converge a.s. to ϖ. But*

$$E\left[|\varpi_i-\varpi|\right]=\frac{i+1}{2i^2}\to 0.$$

Also, for all $\eta>0$, we obtain

$$\lim_{t\to\infty}\frac{1}{c_t}\left|\left\{i\leq t:E\left[|\varpi_i-\varpi|\right]\geq\eta\right\}\right|=0$$

which means that $\varpi_i \overset{q.st.E.}{\to} \varpi$.

Example 3.5. *Quasi statistical convergence a.s. does not imply quasi statistical convergence in mean, too. For instance, $\Theta=\{\theta_1,\theta_2,...\}$, $\mathrm{Cr}\left\{\theta_s\right\}=1/s$ for $s=1,2,...$ and the fuzzy variables are characterized as*

$$\varpi_i\left(\theta_s\right)=\begin{cases}i, & \text{if } s=i\\ 0, & \text{otherwise}\end{cases}\tag{3.2}$$

for $i=1,2,...$ and $\mu=0$. Afterwards, we get $\varpi_i \overset{q.st.a.s.}{\to} \varpi$. However, for any $\eta\in\left(0,\frac{1}{2}\right)$,

$$\lim_{t\to\infty}\frac{1}{c_t}\left|\left\{i\leq t:E\left[|\varpi_i-\varpi|\right]\geq\eta\right\}\right|\neq 0.$$

In other words, we get $\varpi_i \overset{q.st.E.}{\nrightarrow} \varpi$.

Theorem 3.1. *For any fuzzy variables sequence* $\{\varpi_i\}$, $\varpi_i \overset{q.st.E.}{\rightarrow} \varpi$ *implies* $\varpi_i \overset{q.st.\mathrm{Cr}}{\rightarrow} \varpi$.

Proof. Presume $\varpi_i \overset{q.st.E.}{\rightarrow} \varpi$. For any taken $\eta, \gamma > 0$, with the aid of Markov inequality, we acquire

$$\begin{aligned}&\lim_{t\to\infty} \tfrac{1}{c_t} \left|\{i \le t : Cr\{|\varpi_i - \varpi| \ge \eta\} \ge \gamma\}\right| \\ &\le \lim_{t\to\infty} \tfrac{1}{c_t} \left|\left\{i \le t : \left(\tfrac{E[|\varpi_i - \varpi|]}{\eta}\right) \ge \gamma\right\}\right|.\end{aligned}$$

As a result, we obtain $\varpi_i \overset{q.st.\mathrm{Cr}}{\rightarrow} \varpi$. □

Example 3.6. *Quasi statistical convergence in credibility does not imply quasi statistical convergence in mean. Examine the fuzzy variables defined by (3.2) which does not quasi statistically converge in mean to* μ. *But, for* $\gamma \in \left[\frac{1}{2}, 1\right)$ *and any small number* $\eta > 0$, *we get*

$$\lim_{t\to\infty} \frac{1}{c_t} \left|\{i \le t : \mathrm{Cr}\{|\varpi_i - \varpi| \ge \eta\} \ge \gamma\}\right| = 0$$

for all $\eta, \gamma > 0$. *That is to say,* $\varpi_i \overset{q.st.\mathrm{Cr}}{\rightarrow} \varpi$.

The credibility distribution $\Phi : \mathbb{R} \to [0,1]$ of a fuzzy variable μ is presented by Liu [10] as

$$\Phi(y) : \mathrm{Cr}\{\theta \in \Theta : \varpi(\theta) \le y\}.$$

for each $y \in \mathbb{R}$. Namely, $\Phi(y)$ is the credibility that the fuzzy variable ϖ takes a value less than or equal to y.

Definition 3.4. *Presume that* $\Phi, \Phi_1, \Phi_2, \ldots$ *are the credibility distributions of fuzzy variables* $\varpi, \varpi_1, \varpi_2, \ldots$ *respectively. When the sequence* $\{\Phi_i\}$ *quasi statistical converges weakly to* Φ, *then we say that* $\{\varpi_i\}$ *quasi statistically converges in distribution to fuzzy variable to* ϖ, *or equivalently* $\{\varpi_i\}$ *quasi statistically converges in distribution to fuzzy variable to* ϖ *provided that for all* $\eta > 0$,

$$\lim_{t\to\infty} \frac{1}{c_t} \left|\{i \le t : |\Phi_i(\rho) - \Phi(\rho)| \ge \eta\}\right| = 0,$$

for all ρ *at which* $\Phi(\rho)$ *is continuous.*

Definition 3.5. *The sequence* $\{\varpi_i\}$ *is named to be quasi statistically convergent uniformly almost surely (q.st.u.a.s.) in credibility space to the fuzzy variable* ϖ *provided that for all* $\eta > 0$, $\exists \gamma > 0$ *and a sequence of events* $\{Y_i'\} \in \mathcal{P}(\Theta)$ *with* $\mathrm{Cr}\{Y_i'\} = 1$ *so that*

$$\begin{aligned}&\lim_{t\to\infty} \tfrac{1}{c_t} \left|\{i \le t : |\mathrm{Cr}(Y_i') - 0| \ge \eta\}\right| = 0 \\ &\quad\Rightarrow \lim_{t\to\infty} \tfrac{1}{c_t} \left|\{i \le t : |\varpi_i(x) - \varpi(x)| \ge \gamma\}\right| = 0,\end{aligned}$$

for all $\theta \in Y_i'$. *Symbolically, we denote* $\varpi_i \overset{q.st.u.a.s.}{\rightarrow} \varpi$.

Theorem 3.2. *If a fuzzy variable sequence* $\{\varpi_i\}$ *is quasi statistically convergent to* ϖ, *then it is statistically convergent to* ϖ.

Proof. Let a fuzzy variable sequence $\{\varpi_i\}$ be quasi statistically convergent to ϖ and $H := \sup\limits_t \frac{c_t}{t}$. Since

$$\frac{1}{t} \mid \{i \le t : |\varpi_i - \varpi| \ge \eta\} \mid \le \frac{H}{c_t} \mid \{i \le t : |\varpi_i - \varpi| \ge \eta\} \mid,$$

the proof follows immediately. ∎

Definition 3.6. *Take $\varpi, \varpi_1, \varpi_2, ...$ as fuzzy variables. The sequence $\{\varpi_i\}$ is named a statistically quasi-Cauchy sequence in credibility provided that there exists Y with $\mathrm{Cr}\{Y\} = 1$ and $M = M(\gamma)$ such that*

$$\lim_{t\to\infty} \frac{1}{c_t} \left|\{i \le t : \mathrm{Cr}\{|\varpi_i - \varpi_M| \ge \eta\} \ge \gamma\}\right| = 0$$

for all $\eta, \gamma > 0$.

Example 3.7. *Contemplate the crebility space $(\Theta, \mathcal{P}, \mathrm{Cr})$ to be $\{\theta_1, \theta_2, ...\}$ with $\mathrm{Cr}(\theta_1) = \frac{1}{2}$ and $\mathrm{Cr}(\theta_s) = \frac{1}{s}$, for $s = 2, 3, ...$ The fuzzy variables are determined as*

$$\varpi_i(\theta_s) = \begin{cases} s, & \text{if } i = s \\ 0, & \text{otherwise.} \end{cases}$$

For all $\gamma > 0$, taking $\eta \in (0,1)$ and considering $M = \left[\frac{2}{\gamma}\right] + 1$, we acquire

$$\lim_{t\to\infty} \frac{1}{c_t} \left|\{i \le t : \mathrm{Cr}\{|\varpi_i - \varpi_M| \ge \eta\} \ge \gamma\}\right| = 0.$$

As a result, $\{\varpi_i\}$ is called a statistically quasi-Cauchy sequence in credibility.

Theorem 3.3. *If a fuzzy variable sequence $\{\varpi_i\}$ is quasi statistically convergent in credibility, then it is statistically quasi-Cauchy sequence in credibility.*

Proof. Let $\varpi_i \overset{q.st.\mathrm{Cr}}{\rightarrow} \varpi$. At that time, there is $Y \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{Y\} = 1$ so that

$$\lim_{t\to\infty} \frac{1}{c_t} \left|\left\{i \le t : \mathrm{Cr}\left\{|\varpi_i - \varpi| \ge \frac{\eta}{2}\right\} \ge \frac{\gamma}{2}\right\}\right| = 0$$

for all η, $\gamma > 0$. Identify the sets K_1, K_2 and K_1^c, as follows:

$$K_1 = \left\{i \le t : \mathrm{Cr}\left\{|\varpi_i - \varpi| \ge \tfrac{\eta}{2}\right\} \ge \tfrac{\gamma}{2}\right\},$$

$$K_2 = \{i \le t : \mathrm{Cr}\{|\varpi_i - \varpi_q| \ge \eta\} \ge \gamma\}$$

and

$$K_1^c = \left\{i \le t : \mathrm{Cr}\left\{|\varpi_i - \varpi| \ge \frac{\eta}{2}\right\} < \frac{\gamma}{2}\right\}.$$

Now, we demonstrate $K_2 \subseteq K_1$. Presume in contrast that $K_1 \subseteq K_2$, $i \in K_2 \diagdown K_1$. Then

$$\mathrm{Cr}\left\{|\varpi_i - \varpi| \ge \frac{\eta}{2}\right\} < \frac{\gamma}{2}, \ \mathrm{Cr}\{|\varpi_i - \varpi_q| \ge \eta\} \ge \gamma.$$

Let $q \in K_1^c$. Afterwards, we get

$$\mathrm{Cr}\left\{|\varpi_q - \varpi| \ge \frac{\eta}{2}\right\} < \frac{\gamma}{2}.$$

Therefore, there is a $q = q(\gamma)$ so that

$$\begin{aligned} \gamma &\le \mathrm{Cr}\{|\varpi_i - \varpi_q| \ge \eta\} \\ &\le \mathrm{Cr}\left\{|\varpi_q - \varpi| \ge \tfrac{\eta}{2}\right\} + \mathrm{Cr}\left\{|\varpi_i - \varpi| \ge \tfrac{\eta}{2}\right\} \\ &< \tfrac{\gamma}{2} + \tfrac{\gamma}{2} = \gamma, \end{aligned}$$

which is a contradiction. So, $K_2 \subseteq K_1$. Therefore, we obtain

$$\lim_{t\to\infty} \frac{1}{c_t} \left|\{i \le t : \mathrm{Cr}\{|\varpi_i - \varpi_q| \ge \eta\} \ge \gamma\}\right| = 0.$$

As a result, $\{\varpi_i\}$ statistically quasi-Cauchy sequence in credibility. ∎

Definition 3.7. *A credibility space is named as quasi statistically complete in credibility if all statistically quasi-Cauchy sequence in credibility quasi statistical converges in credibility.*

Theorem 3.4. *Credibility space* $(\Theta, \mathcal{P}(\Theta), Cr)$ *is quasi statistically complete in credibility.*

Proof. Take $\{\varpi_i\}$ as a statistically quasi-Cauchy sequence in credibility. Then, there is Y with $\mathrm{Cr}\{Y\} = 1$ and $M = M(\gamma)$ so that

$$\lim_{t\to\infty} \frac{1}{c_t} \left|\{i \le t : \mathrm{Cr}\{|\varpi_i - \varpi_M| \ge \eta\} \ge \gamma\}\right| = 0$$

for all η, $\gamma > 0$. Presume in contrast that it is not quasi statistical convergence in credibility. Afterwards, there is Y with $\mathrm{Cr}\{Y\} = 1$ so that

$$\lim_{t\to\infty} \frac{1}{c_t} \left|\left\{i \le t : \mathrm{Cr}\left\{|\varpi_i - \varpi| \ge \frac{\eta}{2}\right\} \ge \frac{\gamma}{2}\right\}\right| \ne 0$$

for each η, $\gamma > 0$. Contemplate

$$B = \left\{i \le t : \mathrm{Cr}\left\{|\varpi_i - \varpi| \ge \frac{\eta}{2}\right\} \ge \frac{\gamma}{2}\right\}$$

and

$$C = \{i \le t : \mathrm{Cr}\{|\varpi_i - \varpi_M| \ge \eta\} \ge \gamma\}.$$

Thus

$$B^c = \left\{i \le t : \mathrm{Cr}\left\{|\varpi_i - \varpi| \ge \frac{\eta}{2}\right\} < \frac{\gamma}{2}\right\}.$$

Next we demonstrate $B \subseteq C$. Assume $C \subseteq B$ and $i \in B^c \cap C$. Then

$$\mathrm{Cr}\left\{|\varpi_i - \varpi| \ge \frac{\eta}{2}\right\} < \frac{\gamma}{2}, \ \mathrm{Cr}\{|\varpi_i - \varpi_M| \ge \eta\} \ge \gamma.$$

Take $M \in B^c$, then we acquire

$$\mathrm{Cr}\left\{|\varpi_M - \varpi| \ge \frac{\eta}{2}\right\} < \frac{\gamma}{2}.$$

Hence, there is a $M = M(\gamma)$ so that

$$\begin{aligned}\gamma &\le \mathrm{Cr}\{|\varpi_i - \varpi_M| \ge \eta\} \\ &\le \mathrm{Cr}\left\{|\varpi_M - \varpi| \ge \tfrac{\eta}{2}\right\} + \mathrm{Cr}\left\{|\varpi_i - \varpi| \ge \tfrac{\eta}{2}\right\} \\ &\le \tfrac{\eta}{2} + \tfrac{\eta}{2} = \gamma,\end{aligned}$$

which is not possible. Examine that $B \subseteq C$. This means that

$$\lim_{t\to\infty} \frac{1}{c_t} \left|\left\{i \le t : \mathrm{Cr}\left\{|\varpi_i - \varpi| \ge \frac{\eta}{2}\right\} \ge \frac{\gamma}{2}\right\}\right| = 0,$$

i.e.,

$$\lim_{t\to\infty} \frac{1}{c_t} \left|\{i \le t : \mathrm{Cr}\{|\varpi_i - \varpi| \ge \eta\} \ge \gamma\}\right| = 0.$$

Therefore, $\varpi_i \overset{q.st.\mathrm{Cr}}{\to} \varpi$. This denotes that credibility space is quasi statistically complete in credibility. □

Now, we put forward the relation between quasi statistical convergence uniformly a.s. and quasi statistical convergence a.s. of fuzzy variable sequence $\{\varpi_i\}$ in credibility space.

Proposition 3.5. *Take $\varpi, \varpi_1, \varpi_2, ...$ as fuzzy variables. Then, $\{\varpi_i\}$ quasi statistically converges a.s. to ϖ iff for any $\eta, \gamma > 0$, we have*

$$\lim_{t\to\infty} \frac{1}{c_t} \left| \left\{ i \le t : \mathrm{Cr}\left(\bigcap_{t=1}^{\infty} \bigcup_{i=t}^{\infty} |\varpi_i(\theta) - \varpi(\theta)| \ge \eta \right) \ge \gamma \right\} \right| = 0.$$

Proof. According to the definition of quasi statistical convergence a.s., we have that there is an $Y \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{Y\} = 1$ so that

$$\lim_{t\to\infty} \frac{1}{c_t} |\{ i \le t : |\varpi_i(\theta) - \varpi(\theta)| \ge \eta \}| = 0$$

for all $\theta \in Y$, all $\eta > 0$. Afterwards, for any $\eta > 0$, there exists t such that $|\varpi_i(\theta) - \varpi(\theta)| < \eta$, where $i > t$ and for any $\theta \in Y$, that is equivalent to

$$\lim_{t\to\infty} \frac{1}{c_t} \left| \left\{ i \le t : \mathrm{Cr}\left(\bigcap_{t=1}^{\infty} \bigcup_{i=t}^{\infty} |\varpi_i(\theta) - \varpi(\theta)| < \eta \right) \ge 1 \right\} \right| = 0.$$

As a result of the duality axiom of crebility measure we get

$$\lim_{t\to\infty} \frac{1}{c_t} \left| \left\{ i \le t : \mathrm{Cr}\left(\bigcap_{t=1}^{\infty} \bigcup_{i=t}^{\infty} |\varpi_i(\theta) - \varpi(\theta)| \ge \eta \right) \ge \gamma \right\} \right| = 0.$$

□

Proposition 3.6. *Take $\varpi, \varpi_1, \varpi_2, ...$ as fuzzy variables. At that time, $\{\varpi_i\}$ quasi statistically converges uniformly a.s. to the fuzzy variable ϖ iff for any η, $\delta > 0$, we obtain*

$$\lim_{t\to\infty} \frac{1}{c_t} \left| \left\{ i \le t : \mathrm{Cr}\left(\bigcup_{i=t}^{\infty} |\varpi_i(\theta) - \varpi(\theta)| \ge \eta \right) \ge \delta \right\} \right| = 0.$$

Proof. When $\varpi_i \overset{q.st.u.a.s.}{\to} \varpi$, then for any $\delta > 0$ there is a Y so that $\mathrm{Cr}\{Y\} < \delta$ and $\{\varpi_i\}$ quasi statistically uniformly converges to ϖ on $\mathcal{P}(\Theta) - Y$. For that reason, for any $\eta > 0$, there is a $t > 0$ so that $|\varpi_i(\theta) - \varpi(\theta)| < \eta$ where $i > t$ and $\theta \in \mathcal{P}(\Theta) - Y$. That is

$$\bigcup_{i=t}^{\infty} \{ |\varpi_i(\theta) - \varpi(\theta)| \ge \eta \} \subset Y.$$

According to the subadditivity axiom that

$$\lim_{t\to\infty} \frac{1}{c_t} \left| \left\{ i \le t : \mathrm{Cr}\left(\bigcup_{i=t}^{\infty} |\varpi_i(\theta) - \varpi(\theta)| \ge \eta \right) \right\} \right| \le \delta\left(Cr\{Y\} \right) < \delta.$$

Then

$$\lim_{t\to\infty} \frac{1}{c_t} \left| \left\{ i \le t : \lim_{t\to\infty} \mathrm{Cr}\left(\bigcup_{i=t}^{\infty} |\varpi_i(\theta) - \varpi(\theta)| \ge \eta \right) \ge \delta \right\} \right| = 0.$$

On the contrary, if

$$\lim_{t\to\infty} \frac{1}{c_t} \left| \left\{ i \le t : \mathrm{Cr}\left(\bigcup_{i=t}^{\infty} |\varpi_i(\theta) - \varpi(\theta)| \ge \eta \right) \ge \delta \right\} \right| = 0,$$

for any η, then for given $\delta > 0$ and $p \geq 1$, there is p_k so that

$$\delta_c \left(\mathrm{Cr} \left(\bigcup_{i=p_k}^{\infty} \left\{ \left| \varpi_i(\theta) - \varpi(\theta) \right| \geq \frac{1}{p} \right\} \right) \right) < \frac{\delta}{2^p}.$$

Take

$$Y = \bigcup_{p=1}^{\infty} \bigcup_{i=p_k}^{\infty} \left\{ \left| \varpi_i(\theta) - \varpi(\theta) \right| \geq \frac{1}{p} \right\}.$$

Then

$$\delta_c \left(\mathrm{Cr}\{Y\} \right) \leq \sum_{p=1}^{\infty} \delta_c \left(\mathrm{Cr} \left(\bigcup_{i=p_k}^{\infty} \left\{ \left| \varpi_i(\theta) - \varpi(\theta) \right| \geq \frac{1}{p} \right\} \right) \right) \leq \sum_{p=1}^{\infty} \frac{\delta}{2^p}.$$

In addition, we obtain

$$\sup_{\theta \in \mathcal{P}(\Theta) - Y} \left| \varpi_i(\theta) - \varpi(\theta) \right| < \frac{1}{p}$$

for any $p = 1, 2, ...$ and $i > p_k$. The proof of the proposition is finalized. □

Theorem 3.7. *When* $\varpi_i \overset{q.st.u.a.s.}{\to} \varpi$*, then* $\varpi_i \overset{q.st.a.s.}{\to} \varpi$.

Proof. By Proposition 3.6 that if $\varpi_i \overset{q.st.u.a.s.}{\to} \varpi$, then

$$\lim_{t \to \infty} \frac{1}{c_t} \left| \left\{ i \leq t : \mathrm{Cr} \left(\bigcup_{i=t}^{\infty} \left| \varpi_i(\theta) - \varpi(\theta) \right| \geq \eta \right) \geq \delta \right\} \right| = 0.$$

Since

$$\delta_c \left(\mathrm{Cr} \left(\bigcap_{t=1}^{\infty} \bigcup_{i=t}^{\infty} \left\{ \left| \varpi_i(\theta) - \varpi(\theta) \right| \geq \eta \right\} \right) \right) \leq \delta_c \left(\mathrm{Cr} \left(\bigcup_{i=t}^{\infty} \left\{ \left| \varpi_i(\theta) - \varpi(\theta) \right| \geq \eta \right\} \right) \right)$$

getting the limit as $t \to \infty$ on both sides of above inequality, we acquire

$$\delta_c \left(\mathrm{Cr} \left(\bigcap_{t=1}^{\infty} \bigcup_{i=t}^{\infty} \left\{ \left| \varpi_i(\theta) - \varpi(\theta) \right| \geq \eta \right\} \right) \right) = 0$$

By Proposition 3.5, we have $\varpi_i \overset{q.st.a.s.}{\to} \varpi$. □

Following theorem gives the relation between quasi statistical convergence uniformly a.s. and quasi statistical convergence in credibility for fuzzy variable sequence $\{\varpi_i\}$ in credibility space.

Theorem 3.8. *When* $\varpi_i \overset{q.st.u.a.s.}{\to} \varpi$*, then* $\varpi_i \overset{q.st.\mathrm{Cr}}{\to} \varpi$.

Proof. If $\varpi_i \overset{q.st.u.a.s.}{\to} \varpi$, then from Proposition 3.6 we obtain

$$\lim_{t \to \infty} \frac{1}{c_t} \left| \left\{ i \leq t : \mathrm{Cr} \left(\bigcup_{i=t}^{\infty} \left| \varpi_i(\theta) - \varpi(\theta) \right| \geq \eta \right) \geq \delta \right\} \right| = 0.$$

and

$$\delta_c \left(\mathrm{Cr} \left\{ \bigcap_{t=1}^{\infty} \bigcup_{i=t}^{\infty} \left| \varpi_i(\theta) - \varpi(\theta) \right| \geq \eta \right\} \right) \leq \delta_c \left(\mathrm{Cr} \left(\bigcup_{i=t}^{\infty} \left\{ \left| \varpi_i(\theta) - \varpi(\theta) \right| \geq \eta \right\} \right) \right).$$

We obtain by letting $i \to \infty$ that we acquire

$$\delta_c \left(\mathrm{Cr} \left\{ \bigcap_{t=1}^{\infty} \bigcup_{i=t}^{\infty} |\varpi_i(\theta) - \varpi(\theta)| \geq \eta \right\} \right) = 0.$$

As a result, we obtain $\varpi_i \overset{q.st.\mathrm{Cr}}{\to} \varpi$. □

Definition 3.8. *The sequence $\{\varpi_i\}$ is called to be quasi statistical $T_n(\mathrm{Cr})$-summable to ϖ iff there is $Y \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{Y\} = 1$ so that*

$$\lim_{t\to\infty} \frac{1}{c_t} \left| \left\{ i \leq t : \sum_{i=m}^{t} \mathrm{Cr}\{|\varpi_i - \varpi| \geq \eta\} \geq \gamma \right\} \right| = 0$$

for any η, $\gamma > 0$, and for any m, where

$$T_n(\mathrm{Cr}) = \frac{1}{t} \sum_{i=m}^{t} \mathrm{Cr}\{|\varpi_i| \geq \eta\}.$$

In that case, we write $\varpi_i \overset{qst-T_n(Cr)}{\to} \varpi$.

Example 3.8. *Take $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ to be $\{\theta_1, \theta_2, ...\}$ with $\mathrm{Cr}(\theta_1) = \mathrm{Cr}(\theta_2) = \frac{1}{2}$ and $\mathrm{Cr}(\theta_s) = \frac{1}{2s}$, for $s = 3, 4, ...$ The fuzzy variables are determined as*

$$\varpi_i(\theta_s) = \begin{cases} 1, & \text{if } i = s \\ 0, & \text{otherwise.} \end{cases}$$

Take $\varpi = 0$. For any $\gamma \in \left[\frac{1}{2}, 1\right)$ and $\eta \in (0, 1)$, we get

$$\sum_{i=1}^{t} \mathrm{Cr}\{|\varpi_i - \varpi| \geq \eta\} \leq \frac{t}{2t} = \frac{1}{2}.$$

So

$$\lim_{t\to\infty} \frac{1}{c_t} \left| \left\{ i \leq t : \sum_{i=m}^{t} \mathrm{Cr}\{|\varpi_i - 0| \geq \eta\} \geq \gamma \right\} \right| = 0.$$

In other words, we obtain $\varpi_i \overset{qst-T_n(Cr)}{\to} \varpi$.

Theorem 3.9. *If $\varpi_i \overset{qst-T_n(Cr)}{\to} \varpi$, then $\varpi_i \overset{q.st.\mathrm{Cr}}{\to} \varpi$.*

Proof. Let η, $\gamma > 0$. For any m, we get

$$\sum_{i=m}^{t} \mathrm{Cr}\{|\varpi_i - \varpi| \geq \eta\} \geq \mathrm{Cr}\{|\varpi_i - \varpi| \geq \eta\}.$$

Thus

$$\left| \left\{ i \leq t : \sum_{i=m}^{t} \mathrm{Cr}\{|\varpi_i - \varpi| \geq \eta\} \geq \gamma \right\} \right| \geq |\{i \leq t : \mathrm{Cr}\{|\varpi_i - \varpi| \geq \eta\} \geq \gamma\}|.$$

It is easy to understand that $\varpi_i \overset{qst-T_n(\mathrm{Cr})}{\to} \varpi$ gives $\varpi_i \overset{q.st.\mathrm{Cr}}{\to} \varpi$. □

Example 3.9. *Quasi statistical convergence in credibility does not imply quasi statistical $T_n\,(\mathrm{Cr})$-summable. To show this, take $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ to be $\{\theta_1, \theta_2, ...\}$ with $\mathrm{Cr}(\theta_s) = \frac{s}{2s+1}$ for $s = 1, 2, ...$ The fuzzy variables are defined by*

$$\varpi_i(\theta_s) = \begin{cases} 1, & \text{if } i = s \\ 0, & \text{otherwise.} \end{cases}$$

Take $\varpi = 0$. For any $\gamma \in \left[\frac{1}{2}, 1\right)$ and $\eta \in (0, 1)$, we get

$$\lim_{t\to\infty} \frac{1}{t} \left|\{i \le t : \mathrm{Cr}\{|\varpi_i - 0| \ge \eta\} \ge \gamma\}\right| = 0.$$

Then, we get $\varpi_i \overset{q.st.\mathrm{Cr}}{\rightarrow} \varpi$. At the same time,

$$\lim_{t\to\infty} \frac{1}{c_t} \left|\left\{i \le t : \sum_{i=m}^{t} \mathrm{Cr}\{|\varpi_i - 0| \ge \eta\} \ge \gamma\right\}\right| = 1.$$

Namely, we obtain $\varpi_i \overset{qst-T_n(Cr)}{\nrightarrow} \varpi$.

Quasi statistical convergence in credibility supplies some usual axioms of convergence in credibility. The known axioms of convergence in credibility are given, as follows:

Take $\varpi, \varpi_1, \varpi_2, ...$ as fuzzy variables identified on credibility space $(\Theta, \mathcal{P}(\Theta), Cr)$.

(**U**) The uniqueness of limit: If $qst\,(\mathrm{Cr})-\lim \varpi_i = \varpi_1$ and $qst\,(\mathrm{Cr})-\lim \varpi_i = \varpi_2$, then $\varpi_1 = \varpi_2$ in credibility.

(**E**) If there exists a subset $T = \{m_1 < m_2 < ...\} \subseteq \mathbb{N}$ such that $\{\varpi_{m_i}\}$ quasi converges in credibility to ϖ, then $qst\,(\mathrm{Cr}) - \lim \varpi_i = \varpi$.

Theorem 3.10. *Let $\varpi, \varpi_1, \varpi_2, ...$ be fuzzy variables determined on credibility space $(\Theta, \mathcal{P}(\Theta), Cr)$. Then, quasi statistical convergence in credibility supplies the axioms* (**U**) *and* (**E**).

Proof. It is obvious that statistical convergence in credibility supplies the axiom (**E**). Suppose that there is a subset $T = \{m_1 < m_2 < ...\} \subseteq \mathbb{N}$ such that $\{\varpi_{m_i}\}$ quasi converges in credibility to ϖ, i.e., for every $\theta \in A$, any $\eta, \gamma > 0$, there exists $Y \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{Y\} = 1$ and $i_0 = i_0(\eta)$ so that

$$\mathrm{Cr}\{|\varpi_{m_i} - \varpi| \ge \eta\} < \gamma$$

for all $i > i_0$. Let $T = \{m_{i_0+1}, m_{i_0+2}, ...\}$. At that time, there is $Y \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{Y\} = 1$ so that

$$\lim_{t\to\infty} \frac{1}{c_t} \left|\{i \in T : \mathrm{Cr}\{|\varpi_i - \varpi| \ge \eta\} \ge \gamma\}\right| = 0$$

for all η, $\gamma > 0$. So, we get

$$\lim_{t\to\infty} \frac{1}{c_t} \left|\{i \le t : \mathrm{Cr}\{|\varpi_i - \varpi| \ge \eta\} \ge \gamma\}\right| = 0$$

for all η, $\gamma > 0$, i.e., $qst\,(\mathrm{Cr}) - \lim \varpi_i = \varpi$.

Now, we demonstrate that quasi statistical convergence in credibility supplies the axiom (**U**). Assume that $qst\,(\mathrm{Cr}) - \lim \varpi_i = \varpi_1$ and $qst\,(\mathrm{Cr}) - \lim \varpi_i = \varpi_2$. At that time, there is $Y \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{Y\} = 1$ so that

$$\lim_{t\to\infty} \frac{1}{c_t} \left|\{i \in T : \mathrm{Cr}\{|\varpi_i - \varpi_1| \ge \eta\} \ge \gamma\}\right| = 0$$

and

$$\lim_{t\to\infty}\frac{1}{c_t}\left|\{i\in T:\mathrm{Cr}\{|\varpi_i-\varpi_2|\geq\eta\}\geq\gamma\}\right|=0$$

for all η, $\gamma>0$. Establish the sets B_1 and B_2 as follows:

$$B_1=\{i\leq t:\mathrm{Cr}\{|\varpi_i-\varpi_1|\geq\eta\}\geq\gamma\}$$

and

$$B_2=\{i\leq t:\mathrm{Cr}\{|\varpi_i-\varpi_2|\geq\eta\}\geq\gamma\}.$$

Now let $i\in B_1\cup B_2$. Then, we obtain

$$\mathrm{Cr}\{|\varpi_i-\varpi_1|\geq\eta\}<\gamma,\ \mathrm{Cr}\{|\varpi_i-\varpi_2|\geq\eta\}<\gamma.$$

Therefore

$$\begin{aligned}\mathrm{Cr}\{|\varpi_1-\varpi_2|\geq\eta\}&=\mathrm{Cr}\{|\varpi_1-\varpi_i+\varpi_i-\varpi_2|\geq\eta\}\\&\leq\mathrm{Cr}\left\{|\varpi_i-\varpi_1|\geq\tfrac{\eta}{2}\right\}+\mathrm{Cr}\left\{|\varpi_i-\varpi_2|\geq\tfrac{\eta}{2}\right\}\\&<2\gamma.\end{aligned}$$

As $\gamma>0$ is arbitrary, we can acquire $\mathrm{Cr}\{|\varpi_1-\varpi_2|\geq\eta\}=0$, which gives $\varpi_1=\varpi_2$ in credibility. □

Theorem 3.11. *Take $\varpi,\varpi_1,\varpi_2,...$ as fuzzy variables and assume $f:\mathbb{R}\to\mathbb{R}$ be a convex function. If $\varpi_i\overset{q.st.a.s.}{\to}\varpi$, then $f(\varpi_i)\overset{q.st.a.s.}{\to}f(\varpi)$.*

Proof. Let f be a convex function. Then, there is a constant c such that

$$|f(q)-f(r)|\leq c\,|q-r|$$

for all $q,r\in\mathbb{R}$. Replacing q with ϖ_i and r with ϖ, we acquire

$$|f(\varpi_i)-f(\varpi)|\leq c\,|\varpi_i-\varpi|.$$

Since $\varpi_i\overset{q.st.a.s.}{\to}\varpi$, there is $Y\in\mathcal{P}(\Theta)$ with $\mathrm{Cr}\{Y\}=1$ so that

$$\lim_{t\to\infty}\frac{1}{c_t}\left|\{i\leq t:|\varpi_i(\theta)-\varpi(\theta)|\geq\eta\}\right|=0$$

for each $\theta\in Y$, each $\eta>0$. Then, for any $\eta>0$, there exists $Y\in\mathcal{P}(\Theta)$ with $\mathrm{Cr}\{Y\}=1$ so that $|\varpi_i(\theta)-\varpi(\theta)|<\frac{\eta}{c}$. Then

$$|f(\varpi_i(\theta))-f(\varpi(\theta))|\leq c\,|\varpi_i(\theta)-\varpi(\theta)|<c\frac{\eta}{c}<\eta,$$

i.e.

$$\lim_{t\to\infty}\frac{1}{c_t}\left|\{i\leq t:|f(\varpi_i(\theta))-f(\varpi(\theta))|\geq\eta\}\right|=0$$

for all $\theta\in Y$, each $\eta>0$. Thus, we obtain $f(\varpi_i)\overset{q.st.a.s.}{\to}f(\varpi)$. □

Theorem 3.12. *If $\varpi_i\overset{q.st.\mathrm{Cr}}{\to}\varpi$ and $f:\mathbb{R}\to\mathbb{R}$ is a convex function, then $f(\varpi_i)\overset{q.st.\mathrm{Cr}}{\to}f(\varpi)$.*

Proof. Since $\varpi_i\overset{q.st.\mathrm{Cr}}{\to}\varpi$, there exists $Y\in\mathcal{P}(\Theta)$ with $\mathrm{Cr}\{Y\}=1$ so that

$$\lim_{t\to\infty}\frac{1}{c_t}\left|\left\{i\leq t:\mathrm{Cr}\left\{|\varpi_i-\varpi|\geq\frac{\eta}{c}\right\}\geq\gamma\right\}\right|=0$$

for all η, $\gamma>0$. So

$$\lim_{t\to\infty}\frac{1}{c_t}\left|\left\{i\leq t:\mathrm{Cr}\left\{|\varpi_i-\varpi|\geq\frac{\eta}{c}\right\}<\gamma\right\}\right|=1.$$

At that time, according to the convexity of f, we get

$$|f(\varpi_i)-f(\varpi)|\leq c\,|\varpi_i-\varpi|<c\frac{\eta}{c}<\eta.$$

Therefore

$$\lim_{t\to\infty}\frac{1}{c_t}\left|\left\{i\leq t:\mathrm{Cr}\left\{|\varpi_i-\varpi|\geq\eta\right\}<\gamma\right\}\right|=1.$$

Hence, we acquire

$$\lim_{t\to\infty}\frac{1}{c_t}\left|\left\{i\leq t:\mathrm{Cr}\left\{|f(\varpi_i)-f(\varpi)|\geq\eta\right\}\geq\gamma\right\}\right|=0$$

for all η, $\gamma>0$ which gives $f(\varpi_i)\overset{q.st.\mathrm{Cr}}{\to}f(\varpi)$. □

Theorem 3.13. *Take $f:\mathbb{R}\to\mathbb{R}$ as a convex function and assume $\varpi,\varpi_1,\varpi_2,...$ be fuzzy variables. If $\varpi_i\overset{q.st.E.}{\to}\varpi$, then $f(\varpi_i)\overset{q.st.E.}{\to}f(\varpi)$.*

Proof. If $\varpi_i\overset{q.st.E.}{\to}\varpi$, then for all $\eta>0$, we have

$$\lim_{t\to\infty}\frac{1}{c_t}\left|\left\{i\leq t:E\left[|\varpi_i-\varpi|\right]\geq\eta\right\}\right|=0.$$

Considering Theorem 2.2, for any η, $\gamma>0$,

$$\lim_{t\to\infty}\frac{1}{c_t}\left|\left\{i\leq t:\mathrm{Cr}\left\{|\varpi_i-\varpi|\geq\eta\right\}\geq\gamma\right\}\right|=0.$$

For the reason that f is a convex function, from Theorem 3.12 we obtain

$$\lim_{t\to\infty}\frac{1}{c_t}\left|\left\{i\leq t:\mathrm{Cr}\left\{|f(\varpi_i)-f(\varpi)|\geq\eta\right\}\geq\gamma\right\}\right|=0.$$

Simultaneously, we can deduce that $|f(\varpi_i)-f(\varpi)|$is bounded. That is $|f(\mu_i)-f(\mu)|$ is uniformly essentially bounded. Therefore, we get

$$\lim_{t\to\infty}\frac{1}{c_t}\left|\left\{i\leq t:E\left[|f(\varpi_i)-f(\varpi)|\right]\geq\eta\right\}\right|=0.$$

which implies that $f(\varpi_i)\overset{q.st.E.}{\to}f(\varpi)$. □

Corollary 3.14. *Take $f:\mathbb{R}\to\mathbb{R}$ as a convex function and let $\varpi,\varpi_1,\varpi_2,...$ be fuzzy variables. If $\varpi_i\overset{q.st.\mathrm{Cr}}{\to}\varpi$, then $f(\varpi_i)\overset{q.st.a.s.}{\to}f(\varpi)$.*

Proof. This is obtained from Theorem 2.1 and Theorem 3.12. □

Corollary 3.15. *Take $f:\mathbb{R}\to\mathbb{R}$ as a convex function and let $\varpi,\varpi_1,\varpi_2,...$ be fuzzy variables. If $\varpi_i\overset{q.st.E.}{\to}\varpi$, then $f(\varpi_i)\overset{q.st.\mathrm{Cr}}{\to}f(\varpi)$.*

Proof. This is obtained from Theorem 2.2 and Theorem 3.13. □

Theorem 3.16. *If $\varpi_i\overset{q.st.\mathrm{Cr}}{\to}\varpi$ and $f:\mathbb{R}\to\mathbb{R}$ be a continuous function, then $f(\varpi_i)\overset{q.st.\mathrm{Cr}}{\to}f(\varpi)$.*

Proof. Considering the fact that f is a continuous function, for every $\eta>0$, there exists $\delta>0$ such that $|\varpi_i-\varpi|<\delta$ means $|f(\varpi_i)-f(\varpi)|<\eta$. Therefore, $|f(\varpi_i)-f(\varpi)|\geq\eta$ implies $|\varpi_i-\varpi|\geq\delta$. For this reason one can write,

$$\left\{|f(\varpi_i)-f(\varpi)|\geq\eta\right\}\subset\left\{|\varpi_i-\varpi|\geq\delta\right\}.$$

Take credibility both sides,

$$\text{Cr}\left\{\left|f\left(\varpi_i\right)-f\left(\varpi\right)\right|\geq\eta\right\}\leq\text{Cr}\left\{\left|\varpi_i-\varpi\right|\geq\delta\right\},$$

which implies

$$\left\{i\leq t:\text{Cr}\left\{\left|f\left(\varpi_i\right)-f\left(\varpi\right)\right|\geq\eta\right\}\geq\gamma\right\}\subset\left\{i\leq t:\text{Cr}\left\{\left|\varpi_i-\varpi\right|\geq\delta\right\}\geq\gamma\right\}.$$

Since $\varpi_i \overset{q.st.\text{Cr}}{\rightarrow} \varpi$, we have

$$\lim_{t\to\infty}\frac{1}{c_t}\left|\left\{i\leq t:\text{Cr}\left\{\left|\varpi_i-\varpi\right|\geq\delta\right\}\geq\gamma\right\}\right|=0.$$

Thus, we obtain

$$\lim_{t\to\infty}\frac{1}{c_t}\left|\left\{i\leq t:\text{Cr}\left\{\left|f\left(\varpi_i\right)-f\left(\varpi\right)\right|\geq\eta\right\}\geq\gamma\right\}\right|=0,$$

which gives that $f\left(\varpi_i\right) \overset{q.st.\text{Cr}}{\rightarrow} f\left(\varpi\right)$. □

Theorem 3.17. *Let $\varpi, \varpi_1, \varpi_2, ...$ be fuzzy variables and $f:\mathbb{R}\to\mathbb{R}$ be a continuous function. If $\varpi_i \overset{q.st.E.}{\rightarrow} \varpi$, then $f\left(\varpi_i\right) \overset{q.st.E.}{\rightarrow} f\left(\varpi\right)$.*

Proof. If $\varpi_i \overset{q.st.E.}{\rightarrow} \varpi$. At that time, for all $\eta>0$, we obtain

$$\lim_{t\to\infty}\frac{1}{c_t}\left|\left\{i\leq t:E\left[\left|\varpi_i-\varpi\right|\right]\geq\eta\right\}\right|=0.$$

By utilizing Theorem 2.1, for any $\gamma>0$,

$$\lim_{t\to\infty}\frac{1}{c_t}\left|\left\{i\leq t:\text{Cr}\left\{\left|\varpi_i-\varpi\right|\geq\eta\right\}\geq\gamma\right\}\right|=0.$$

Since f is continuous function, it is obvious from Theorem 3.16 that

$$\lim_{t\to\infty}\frac{1}{c_t}\left|\left\{i\leq t:\text{Cr}\left\{\left|f\left(\varpi_i\right)-f\left(\varpi\right)\right|\geq\eta\right\}\geq\gamma\right\}\right|=0.$$

Simultaneously, we can conclude that $\left|f\left(\varpi_i\right)-f\left(\varpi\right)\right|$ is bounded. That is $\left|f\left(\varpi_i\right)-f\left(\varpi\right)\right|$ is uniformly essentially bounded. Therefore, we get

$$\lim_{t\to\infty}\frac{1}{c_t}\left|\left\{i\leq t:E\left[\left|f\left(\varpi_i\right)-f\left(\varpi\right)\right|\right]\geq\eta\right\}\right|=0.$$

Hence, $f\left(\varpi_i\right) \overset{q.st.E.}{\rightarrow} f\left(\varpi\right)$. □

Corollary 3.18. *Presume $\varpi, \varpi_1, \varpi_2, ...$ be fuzzy variables, and take $f:\mathbb{R}\to\mathbb{R}$ as a continuous function. If $\varpi_i \overset{q.st.\text{Cr}}{\rightarrow} \varpi$, then $f\left(\varpi_i\right) \overset{q.st.a.s.}{\rightarrow} f\left(\varpi\right)$.*

Proof. This follows from Theorem 2.1 and Theorem 3.16. □

Corollary 3.19. *Assume $\varpi, \varpi_1, \varpi_2, ...$ be fuzzy variables, and take $f:\mathbb{R}\to\mathbb{R}$ as a continuous function. If $\varpi_i \overset{q.st.E.}{\rightarrow} \varpi$, then $f\left(\varpi_i\right) \overset{q.st.\text{Cr}}{\rightarrow} f\left(\varpi\right)$.*

Proof. This follows from Theorem 2.2 and Theorem 3.17. □

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