Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 17 Issue 3(2025), Pages 126-135
https://doi.org/10.54671/BMAA-2025-3-10

SURFACE TENSION-ASSISTED FOR KORTEWEG FLUID MOTION IN WHOLE-SPACE CASE

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Abstract. Diffuse and sharf-interface models are two separated categories of mathematical models that can be used to describe liquid-vapor fluxes. The interfacial layer where phase changes take place is represented differently in each of them. In sharp-interface models, an infinitesimally thin hypersurface is employed in place of the small, positive thickness that is present in diffuse-interface models. By taking the limit where the interfacial regions thichness goes to zero, the diffuse-intreface model can be connected to the related sharp interface model. This phenomena known as Korteweg model which introduced firstly by Diederik Johannes Korteweg. The purpose of this article is considering the $\mathcal{R}$-boundedness of the solution operator families for compressible Korteweg type in whole space case with surface tension by using Fourier transform. The multipliers which appear from the transformation are estimated using Weis's multiplier theorem. This $\mathcal{R}$-boundedness is an essential result for further research related to half-space case.

1. Introduction

Water can be found in many different forms in daily life, including ice, liquid water, and water vapor. The solid, water, and vapour (or gas) phases of water are the terms used to describe these various physical states. When thinking about water vapor, one would think about the process of heating water to make a cup of tea. But what we see is water steama mixture of air, water vapor, and tiny water dropletsemitting from the kettle and filling the kitchen. However, the gaseous phase of water is referred to as water vapour in the natural sciences. Both liquids and gases have the capacity to flow, in contrast to solids. They collectively make up the fluids class. Their mass densities, however, differ considerably. There is an increasing number of literature on fluid motion in recent years. Numerous researchers looked into this topic. But most of them focused on numerical analysis rather than using a mathematical analytic approach to study fluid motion. There are some researchers who consider the Korteweg type. For example, some critical space of strong solution

2000 *Mathematics Subject Classification.* 35Q35, 76N10.
Key words and phrases. Korteweg type; Surface tension; Whole-space; $\mathcal{R}$-Boundedness.

Submitted March 4, 2025. Accepted July 28, 2025. Published September 14, 2025.
Communicated by Peiguang Wang.
S. M. was supported by grant 054/E5/PG.02.00.PL/2024 from Ministry of Education, Culture, Research and Technology, Republic of Indonesia.

global in time for initial data close enough to equilibrium point in whole space has been studied by Danchin and Desjardins [2]. Hattori and Li considered for robust solutions that demand more regularity from the starting data [4]. Meanwhile, Haspot [1] considered the existence of global weak solution.

A mathematical model known as the Korteweg type of fluid is used to explain how compressible fluids behave, especially two-phase liquid-vapor mixes. Korteweg type of fluid motion can be described in non-linear partial differential equations (PDE) [14]. To characterize fluid capillarity effects, the model incorporates the Korteweg stress tensor, which considers higher-order derivatives of the fluid density. The Korteweg type of fluid model is a dissipative system, and its solution exhibit optimal decay of higher-order derivatives. Recent research has focused on establishing the global in time existence of strong solutions to the Korteweg type of fluid model in hybrid Besov spaces.

In this paper, we consider the $\mathcal{R}$-boundedness of the solution operator families of the Korteweg type with surface tension in whole space. The R-boundedness of the solution operator of the Korteweg model without surface tension has been investigated by Saito [5]. He consider the R-bounded solution operator in half-space case. In contrast, Inna et.al [9] investigated slip BC of Korteweg type in half-space. Besides that, for another model fluid flows, such as Oldroyd-B model, in 2016 Maryani [6, 7] studied free boundary problem for Oldroyd-B model and global well-posedness of the same problem, respectively. On the other hand, [8]. investigated the $\mathcal{R}$-boundedness of the solution operator families for two-phase Stokes resolvent equation.

The Korteweg model fluid motion with surface tension can be written in the following equation system:

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho\mathbf{u}) = 0 & \text{in } \Omega_t, \\ \rho(\partial_t\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{u}) - \operatorname{Div}(\mathbf{S}(\mathbf{u}) - P(\rho)\mathbf{I}) = \operatorname{Div}\mathbf{K}(\rho) & \text{in } \Omega_t, \\ \{\mathbf{S}(\mathbf{u}) + \mathbf{K}(\rho) - P(\rho)\mathbf{I})\mathbf{n}_t = -P(\rho*)\mathbf{n}_t + \sigma(\mathcal{H}(\Gamma) - \mathcal{H}(\Gamma_0))\mathbf{n}_t & \text{on } \Gamma_t, \\ \mathbf{n}_t\cdot\nabla\rho = 0 & \text{on } \Gamma_t, \\ V_N = \mathbf{n}_t\cdot\mathbf{u} & \text{on } \Gamma_t, \\ \lambda h - \mathbf{u}\cdot\mathbf{n} = \zeta & \text{on } \mathbb{R}^N_0, \end{cases} \tag{1.1}$$

with the initial data

$$(\rho, \mathbf{u})\mid_{t=0} = (\rho_0, \mathbf{u}_0) \quad \text{in } \Omega_t.$$

Here, $0 < t < T$, $\rho*$ is a positive constant describing the mass density of the reference domain Ω, $\mathbf{S}(\mathbf{u})$ and $\mathbf{K}(\rho)$ are defined by

$$\mathbf{S}(\mathbf{u}) = \mu\mathbf{D}(\mathbf{u}) + (\nu - \mu)\operatorname{div}\mathbf{u}\mathbf{I}, \quad \mathbf{K}(\rho) = \frac{\kappa}{2}(\Delta\rho^2 - |\nabla\rho|^2)\mathbf{I} - \kappa\nabla\rho\otimes\nabla\rho$$

$\mathbf{D}(\mathbf{u})$, $\mathbf{u} = (u_1, \ldots, u_N)$, the doubled deformation tensor whose (i, j)-th components are $D_{ij}(\mathbf{u}) = \partial_i u_j + \partial_j u_i$, $(\partial_i = \partial/\partial x_i)$, $\mathbf{I}$ the $N \times N$ identity matrix, μ, ν are positive constants (μ and ν are the first and second viscosity coefficients, respectively).

In this paper, we discuss the existence of the $\mathcal{R}$-bounded operator families for the resolvent problem (1.1) in whole space case. Once we obtain $\mathcal{R}$-boundedness for the solution operator families, we can consider the maximal L_p-L_q regularity for the linearized problem by the Weis operator valued Fourier multiplier theorem [3], which is the key estimate when we consider the local solvability for the nonlinear problem in the maximal L_p-L_q regularity class. Here we introduce the definition of $\mathcal{R}$-boundedness of operator families.

Definition 1.1. *Let X and Y be Banach spaces, and let $\mathcal{L}(X,Y)$ be the set of all bounded linear operators from X into Y. A family of operators $\mathcal{T} \subset \mathcal{L}(X,Y)$ is called $\mathcal{R}$-bounded on $\mathcal{L}(X,Y)$, if there exist constants $C>0$ and $p \in [1,\infty)$ such that for any $n \in \mathbb{N}$, $\{T_j\}_{j=1}^n \subset \mathcal{T}$, $\{f_j\}_{j=1}^n \subset X$ and sequences $\{r_j\}_{j=1}^n$ of independent, symmetric, $\{-1,1\}$-valued random variables on $[0,1]$, we have the inequality:*

$$\left\{ \int_0^1 \| \sum_{j=1}^n r_j(u) T_j f_j \|_Y^p \, du \right\}^{1/p} \leq C \left\{ \int_0^1 \| \sum_{j=1}^n r_j(u) f_j \|_X^p \, du \right\}^{1/p}.$$

The smallest such C is called $\mathcal{R}$-bound of $\mathcal{T}$, which is denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$.

Concerning $\mathcal{R}$-boundedness, we introduce the following lemma proved by [10, Proposition 3.4].

Lemma 1.2. (1) *Let X and Y be Banach spaces, and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families in $\mathcal{L}(X,Y)$. Then $\mathcal{T}+\mathcal{S} = \{T+S \mid T \in \mathcal{T}, S \in \mathcal{S}\}$ is also $\mathcal{R}$-bounded family in $\mathcal{L}(X,Y)$ and*

$$\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T}+\mathcal{S}) \leq \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T}) + \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{S}).$$

(2) *Let X, Y and Z be Banach spaces and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families in $\mathcal{L}(X,Y)$ and $\mathcal{L}(Y,Z)$, respectively. Then $\mathcal{S}\mathcal{T} = \{ST \mid T \in \mathcal{T}, S \in \mathcal{S}\}$ is also an $\mathcal{R}$-bounded family in $\mathcal{L}(X,Z)$ and*

$$\mathcal{R}_{\mathcal{L}(X,Z)}(\mathcal{S}\mathcal{T}) \leq \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T}) \mathcal{R}_{\mathcal{L}(Y,Z)}(\mathcal{S}).$$

To prove Theorem 2.1, we use the following technical lemma

Lemma 1.3. *Let $1<q<\infty$ and let Λ be a set in $\mathbb{C}$. Let $m(\lambda,\epsilon)$ be a function defined on $\Lambda \times (\mathbb{R}^N \setminus \{0\})$ such that for any multi-index $\alpha \in \mathbb{N}_0^N$ ($\mathbb{N}_0 = \mathbb{N} \cup \{0\}$) there exists a constant C_α depending on α and Λ such that*

$$|\partial_\xi^\alpha m(\lambda,\epsilon)| \leq C_\alpha |\xi|^{-\alpha}$$

for any $(\lambda,\epsilon) \in \Lambda \times (\mathbb{R}^N \setminus \{0\})$. Let K_λ be an operator defined by $K_\lambda f = \mathcal{F}^{-1}[m(\lambda,\epsilon)\hat{f}(\xi)]$. Then, the set $\{K_\lambda \mid \lambda \in \Lambda\}$ is $\mathcal{R}$-bounded on $\mathcal{L}(L_q(\mathbb{R}^N))$ and

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N))}(\{K_\lambda \mid \lambda \in \Lambda\}) \leq C_{q,N} \max_{|\alpha| \leq N+1} C_\alpha$$

with some constant $C_{q,N}$ that depends solely on q and N.

The proof of the Lemma 1.3 can be seen in [11, Theorem 3.3].

1.1. **Notation.**

We summarize several symbols and functional spaces used throughout the paper. Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of all natural numbers, real numbers, and complex numbers, respectively. We use boldface letters, e.g. $\mathbf{u}$ to denote vector-valued functions.

For scalar function f and N-vector functions $\mathbf{g}$, we set

$$\nabla f = (\partial_1 f, \ldots, \partial_N f)^\mathsf{T}, \qquad \nabla^2 f = (\partial_i \partial_j f)_{1 \leq i,j \leq N},$$
$$\nabla^3 f = \{\partial_i \partial_j \partial_k f \mid i,j,k = 1,\ldots,N\} \qquad \nabla \mathbf{g} = (\partial_i g_j)_{1 \leq i,j \leq N},$$
$$\nabla^2 \mathbf{g} = \{\partial_i \partial_j g_k \mid i,j,k = 1,\ldots,N\},$$

where $\partial_i = \partial/\partial x_i$.

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For multi-index $\alpha' = (\alpha_1, \dots, \alpha_{N-1}) \in \mathbb{N}_0^{N-1}$ and scalar function $f = f(\xi_1, \dots, \xi_{N-1})$,

$$\partial_{\xi'}^{\alpha'} f = \frac{\partial^{|\alpha'|}}{\partial \xi_1^{\alpha_1} \cdots \partial \xi_{N-1}^{\alpha_{N-1}}} f, \quad |\alpha'| = \alpha_1 + \cdots + \alpha_{N-1}.$$

For complex valued functions $f = f(x)$ and $g = g(x)$; N-vector functions $\mathbf{f} = (f_1(x), \dots, f_N(x))$ and $\mathbf{g} = (g_1(x), \dots, g_N(x))$, the inner products $(f, g)_{\mathbb{R}^N_+}$, $(f, g)_{\mathbb{R}^N_0}$, $(\mathbf{f}, \mathbf{g})_{\mathbb{R}^N_+}$, and $(\mathbf{f}, \mathbf{g})_{\mathbb{R}^N_0}$ are defined by

$$(f,g)_{\mathbb{R}^N_+} = \int_{\mathbb{R}^N_+} f(x)\overline{g(x)}\,dx, \qquad (f,g)_{\mathbb{R}^N_0} = \int_{\mathbb{R}^N_0} f(x)\overline{g(x)}\,d\omega,$$

$$(\mathbf{f},\mathbf{g})_{\mathbb{R}^N_+} = \sum_{j=1}^N (f_j, g_j)_{\mathbb{R}^N_+}, \qquad (\mathbf{f},\mathbf{g})_{\mathbb{R}^N_0} = \sum_{j=1}^N (f_j, g_j)_{\mathbb{R}^N_0},$$

where $d\omega$ denotes the surface element of $\mathbb{R}^N_0$ and $\overline{g(x)}$ is the complex conjugate of $g(x)$.

The Laplace transform and its inverse are formulated by

$$\mathcal{L}[f](\lambda) := \int_{\mathbb{R}} e^{-\lambda t} f(t)\,dt, \quad \mathcal{L}^{-1}[f](\lambda) := \int_{\mathbb{R}} e^{\lambda t} g(\tau)\,d\tau.$$

Let us define the Fourier transform and its inverse transform as

$$\mathcal{F}[\mathbf{u}] = \hat{\mathbf{u}}(\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} \mathbf{u}(x)\,dx, \quad \mathcal{F}^{-1}_\xi[\mathbf{u}](x) := \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{ix\cdot\xi} \mathbf{u}(\xi)\,d\xi.$$

For Banach spaces X and Y, $\mathcal{L}(X, Y)$ denotes the set of all bounded linear operators from X into Y, $\mathcal{L}(X)$ is the abbreviation of $\mathcal{L}(X, X)$, and $\mathrm{Hol}\,(\mathrm{U}, \mathcal{L}(\mathrm{X}, \mathrm{Y}))$ denotes the set of all $\mathcal{L}(X, Y)$ valued holomorphic functions defined on a domain U in $\mathbb{C}$.

For any $1 < q < \infty$, $m \in \mathbb{N}$, $L_q(\mathbb{R}^N_+)$ and $H^m_q(\mathbb{R}^N_+)$ denote the usual Lebesgue space and Sobolev space; while $\|\cdot\|_{L_q(\mathbb{R}^N_+)}$, $\|\cdot\|_{H^m_q(\mathbb{R}^N_+)}$ denote their norms, respectively; $W^{m+s}_q(\mathbb{R}^N_0) = (H^m_q(\mathbb{R}^N_0), H^{m+1}_q(\mathbb{R}^N_0))_{s,q}$ for $m \in \mathbb{N}_0$ and $0 < s < 1$, where $(\cdot,\cdot)_{s,q}$ denotes the real interpolation functor; $C^\infty((a, b))$ denotes the set of all C^∞ functions defined on (a, b). The d-product space of X is defined by $X^d = \{f = (f, \dots, f_d) \mid f_i \in X\,(i = 1, \dots, d)\}$, while its norm is denoted by $\|\cdot\|_X$ instead of $\|\cdot\|_{X^d}$ for the sake of simplicity. The letter C denotes generic constants and the constant $C_{a,b,\dots}$ depends on $a, b, \dots$. The values of constants C and $C_{a,b,\dots}$ may change from line to line.

2. MAIN RESULTS

2.1. **Reduced resolvent problem.** Now, let us begin with the following resolvent problem of the equation system (1.1)

$$\begin{cases} \lambda\rho + \operatorname{div}\mathbf{u} = f & \text{in } \mathbb{R}^N, \\ \lambda\mathbf{u} - \mu\Delta\mathbf{u} - \nu\nabla\operatorname{div}\mathbf{u} - \kappa\nabla\Delta\rho + \gamma\nabla\rho = \mathbf{g} & \text{in } \mathbb{R}^N, \end{cases} \tag{2.1}$$

with $\rho = \rho(\mathbf{x}, t)$ is a density, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) = (u_1(\mathbf{x}, t), \dots, u_N(\mathbf{x}, t))$ is a velocity respect to $\mathbf{x} \in \mathbb{R}^N$ at $t > 0$, λ is eigen value, $\gamma \geq 0$, $\mu > 0$, $\mu + \nu > 0$, $\kappa > 0$, $f = (f_1, f_2, \dots, f_N)$ and $\mathbf{g} = \mathbf{g}(\mathbf{x}) = g_1(\mathbf{x}), g_2(\mathbf{x}), \dots, g_N(\mathbf{x})$.

Now, we state the main result of this paper

Theorem 2.1. *Let* $1 < q < \infty$, $0 < \epsilon < \pi/2$, *and* $\lambda_0 > 0$. *Setting* $\mathrm{Y_q}(\mathbb{R}^\mathrm{N}) = \mathrm{W^1_q}(\mathbb{R}^\mathrm{N}) \times \mathrm{L_q}(\mathbb{R}^\mathrm{N})$ *Then, there exists an operator family*

$$\mathcal{A}_0(\lambda) \in \mathrm{Hol}(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathrm{Y_q}(\mathbb{R}^\mathrm{N}), \mathrm{W^3_q}(\mathbb{R}^\mathrm{N})))$$
$$\mathcal{A}_1(\lambda) \in \mathrm{Hol}(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathrm{Y_q}(\mathbb{R}^\mathrm{N}), \mathrm{W^2_q}(\mathbb{R}^\mathrm{N})^\mathrm{N}))$$

such that for $\lambda = \gamma + i\tau \in \Sigma_{\epsilon,\lambda_0}$ *and* $\mathbf{F} = (f, \mathbf{g}) \in \mathrm{Y_q}(\mathbb{R}^\mathrm{N})$, $(\rho, \mathbf{u}) = (\mathcal{A}_0(\lambda)\mathbf{F}, \mathcal{A}_1(\lambda)\mathbf{F})$ *is a unique solution of equation* (2.1) *and there exists a positive constant* r *such that*

$$\begin{aligned}
&\mathcal{R}_{\mathcal{L}(\mathrm{Y_q}(\mathbb{R}^\mathrm{N}),\mathfrak{A}_\mathrm{q}(\mathbb{R}^\mathrm{N}))}(\{(\tau\partial_\tau)^n \mathcal{K}_\lambda \mathcal{A}_0(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le r,\\
&\mathcal{R}_{\mathcal{L}(\mathrm{Y_q}(\mathbb{R}^\mathrm{N}),\mathfrak{B}_\mathrm{q}(\mathbb{R}^\mathrm{N}))}(\{(\tau\partial_\tau)^n \mathcal{S}_\lambda \mathcal{A}_1(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le r,
\end{aligned} \tag{2.2}$$

for $n = 0, 1$. *Here, above constants* λ_0 *and* r *depend solely on* N, q, ϵ, μ, ν, κ, *and* σ.

Remark. $\mathfrak{A}_q(\mathbb{R}^N) = L_q(\mathbb{R}^N)^{N^3+N^2} \times W^1_q(\mathbb{R}^N)$, $\quad \mathfrak{B}_q(\mathbb{R}^N) = L_q(\mathbb{R}^N)^{N^3+N^2+N}$
$\mathcal{K}_\lambda \rho = (\nabla^3\rho, \lambda^{1/2}\nabla^2\rho, \lambda\rho)$, $\quad \mathcal{S}_\lambda \mathbf{u} = (\nabla^2\mathbf{u}, \lambda^{1/2}\nabla\mathbf{u}, \lambda\mathbf{u})$

The remaining part of this section is the proof of the Theorem 2.1. Applying Fourier transform to the equation (2.1), we have

$$\begin{cases} \lambda\hat{\rho}(\xi) + \hat{\phi}(\xi) = \hat{f}(\xi) & \text{in } \mathbb{R}^N, \\ \lambda\hat{\mathbf{u}}(\xi) + \mu|\xi|^2\hat{\mathbf{u}}(\xi) - \nu i\xi\hat{\phi}(\xi) + \kappa i\xi|\xi|^2\hat{\rho}(\xi) + \gamma i\xi\hat{\rho}(\xi) = \hat{\mathbf{g}}(\xi) & \text{in } \mathbb{R}^N, \end{cases} \tag{2.3}$$

Let $\phi = \operatorname{div}\mathbf{u}$, by using Fourier transform, we have

$$\hat{\phi}(\xi) = i\xi \cdot \hat{\mathbf{u}}(\xi). \tag{2.4}$$

For the simplicity, the first equation of (2.3) can be written as

$$\hat{\rho}(\xi) = \frac{\hat{f}(\xi) - \hat{\phi}(\xi)}{\lambda}. \tag{2.5}$$

Then, for $\lambda \neq 0$, substituting equation (2.5) to the second equation of (2.3), we have

$$\lambda\hat{\mathbf{u}}(\xi) + \mu|\xi|^2\hat{\mathbf{u}}(\xi) - \nu i\xi\hat{\phi}(\xi) + \kappa i\xi|\xi|^2\left(\frac{\hat{f}(\xi) - \hat{\phi}(\xi)}{\lambda}\right) + \gamma i\xi\left(\frac{\hat{f}(\xi) - \hat{\phi}(\xi)}{\lambda}\right) = \hat{\mathbf{g}}(\xi). \tag{2.6}$$

Multiplying equation (2.6) by λ and $i\xi$, we have

$$P(\lambda,\xi)\hat{\phi}(\xi) = |\xi|^2(\kappa|\xi|^2 + \gamma)\hat{f}(\xi) + \lambda i\xi \cdot \hat{\mathbf{g}}(\xi). \tag{2.7}$$

with $P(\lambda,\xi) = \lambda^2 + \lambda(\mu+\nu)|\xi|^2 + |\xi|^2(\kappa|\xi|^2 + \gamma)$. By equation (2.7), we have

$$\hat{\phi}(\xi) = \frac{(|\xi|^2(\kappa|\xi|^2 + \gamma)}{P(\lambda,\xi)}\hat{f}(\xi) + \sum_{j=1}^N \frac{\lambda i\xi_j}{P(\lambda,\xi)}\hat{g}_j(\xi), \tag{2.8}$$

which combined with (2.5) furnishes

$$\hat{\rho}(\xi) = \frac{(\lambda + |\xi|^2(\mu+\nu))}{P(\lambda,\xi)}\hat{f}(\xi) - \sum_{j=1}^N \frac{i\xi_j}{P(\lambda,\xi)}\hat{g}_j(\xi), \tag{2.9}$$

Substituting equation (2.8) to equation (2.6), we get the solution formula of $\hat{\mathbf{u}}(\xi)$ as follows

$$\begin{aligned}\hat{\mathbf{u}}(\xi) = &- \frac{(i\xi(\kappa|\xi|^2+\gamma)}{P(\lambda,\xi)}\hat{f}(\xi)\\ &+ \frac{1}{(\lambda+\mu|\xi|^2)}\Big(\hat{\mathbf{g}}(\xi) - \sum_{j=1}^{N}\frac{\xi\xi_j(\lambda\nu+\kappa|\xi|^2+\gamma)}{P(\lambda,\xi)}\hat{g}_j(\xi)\Big). \qquad (2.10)\end{aligned}$$

Applying Fourier transform and inverse Fourier transform to equation (2.9) and (2.10), we have

$$\begin{aligned}\rho(\xi) &= \mathcal{F}_\xi^{-1}\Big[\frac{(\lambda+|\xi|^2(\mu+\nu))}{P(\lambda,\xi)}\hat{f}(\xi)\Big](x) - \sum_{j=1}^{N}\mathcal{F}_\xi^{-1}\Big[\frac{i\xi_j}{P(\lambda,\xi)}\hat{g}_j(\xi)\Big](x),\\ &:= \mathcal{A}_0(\lambda)\mathbf{F} \qquad (2.11)\end{aligned}$$

and

$$\begin{aligned}\mathbf{u}(\xi) = &- \mathcal{F}_\xi^{-1}\Big[\frac{(i\xi(\kappa|\xi|^2+\gamma)}{P(\lambda,\xi)}\hat{f}(\xi)\Big](x)\\ &+ \mathcal{F}_\xi^{-1}\Big[\frac{1}{(\lambda+\mu|\xi|^2)}\hat{\mathbf{g}}(\xi)\Big](x) - \sum_{j=1}^{N}\mathcal{F}_\xi^{-1}\Big[\frac{\xi\xi_j(\lambda\nu+\kappa|\xi|^2+\gamma)}{P(\lambda,\xi)}\hat{g}_j(\xi)\Big](x)\\ :=&\mathcal{A}_1(\lambda)\mathbf{F}. \qquad (2.12)\end{aligned}$$

respectively. Furthermore, we consider the estimation of $P(\lambda,\xi)$

Lemma 2.2. *Let μ, ν and κ are constants satisfying*

$$\mu > 0, \quad \mu+\nu > 0, \quad \kappa > 0.$$

and $\gamma \geq 0$, $\delta > 0$ Then, for any $0 < \epsilon < \pi/2$, $\lambda \in \Sigma_{\epsilon,\lambda_0}$ with $|\lambda| \geq \delta$ and $\xi \in \mathbb{R}^N$, we have the following assertions hold

(1)

$$|P(\lambda,\xi)| \geq C_{\epsilon,\mu,\nu,\kappa}\{(|\lambda|^{1/2}+|\xi|)^2 + \sqrt{\gamma}|\xi|\}^2$$

(2) *Let $\alpha = (\alpha_1,\alpha_2,\cdots,\alpha_N) \in \mathbb{N}_0^N$. Then there exists a positive constant. $C = C(\alpha,\delta)$ such that*

$$|\partial_\xi^\alpha P(\lambda,\xi)^{-1}| \geq C_{\epsilon,\mu,\nu,\kappa}\{(|\lambda|^{1/2}+|\xi|)^2 + \sqrt{\gamma}|\xi|\}^{-2}(|\lambda|^{1/2}+|\xi|)^{-|\alpha|}$$

Proof. Proof of the Lemma 2.2 for $\gamma = 0$ can be seen in [5, Lemma 2.2]. Meanwhile, for $\gamma > 0$ has been proven by [12, Lemma 5.4, Proposition 5.8]. □

Furthermore, it follows from (2.11) and (2.12) that for $k,l,m = 1,\dots,N$, we have

$$\begin{aligned}\partial_k\partial_l\partial_m\mathcal{A}_0(\lambda)\mathbf{F} = &- \mathcal{F}_\xi^{-1}\Big[\frac{i\xi_k\xi_l\xi_m(\lambda+|\xi|^2(\mu+\nu))}{P(\lambda,\xi)}\hat{f}(\xi)\Big](x)\\ &- \sum_{j=1}^{N}\mathcal{F}_\xi^{-1}\Big[\frac{\xi_j\xi_k\xi_l\xi_m}{P(\lambda,\xi)}\hat{g}_j(\xi)\Big](x), \qquad (2.13)\end{aligned}$$

$$\lambda^{1/2}\partial_k\partial_l\mathcal{A}_0(\lambda)\mathbf{F} =\mathcal{F}_\xi^{-1}\left[\frac{\xi_k\xi_l\lambda^{1/2}(\lambda+|\xi|^2(\mu+\nu))}{P(\lambda,\xi)}\hat{f}(\xi)\right](x) + \sum_{j=1}^N \mathcal{F}_\xi^{-1}\left[\frac{i\xi_j\xi_k\xi_l\lambda^{1/2}}{P(\lambda,\xi)}\hat{g}_j(\xi)\right](x), \tag{2.14}$$

$$\lambda\mathcal{A}_1(\lambda)\mathbf{F} = -\mathcal{F}_\xi^{-1}\left[\frac{(i\xi\lambda(\kappa|\xi|^2+\gamma)}{P(\lambda,\xi)}\hat{f}(\xi)\right](x) + \mathcal{F}_\xi^{-1}\left[\frac{\lambda}{(\lambda+\mu|\xi|^2)}\hat{\mathbf{g}}(\xi)\right](x) - \sum_{j=1}^N \mathcal{F}_\xi^{-1}\left[\frac{\xi\xi_j\lambda(\lambda\nu+\kappa|\xi|^2+\gamma)}{P(\lambda,\xi)}\hat{g}_j(\xi)\right](x) \tag{2.15}$$

$$\lambda^{1/2}\partial_k\mathcal{A}_1(\lambda)\mathbf{F} =\mathcal{F}_\xi^{-1}\left[\frac{(\xi\xi_k\lambda^{1/2}(\kappa|\xi|^2+\gamma)}{P(\lambda,\xi)}\hat{f}(\xi)\right](x) + \mathcal{F}_\xi^{-1}\left[\frac{i\xi_k\lambda^{1/2}}{(\lambda+\mu|\xi|^2)}\hat{\mathbf{g}}(\xi)\right](x) - \sum_{j=1}^N \mathcal{F}_\xi^{-1}\left[\frac{i\xi\xi_j\xi_k\lambda^{1/2}(\lambda\nu+\kappa|\xi|^2+\gamma)}{P(\lambda,\xi)}\hat{g}_j(\xi)\right](x) \tag{2.16}$$

$$\partial_k\partial_l\mathcal{A}_1(\lambda)\mathbf{F} =\mathcal{F}_\xi^{-1}\left[\frac{(i\xi\xi_k\xi_l(\kappa|\xi|^2+\gamma)}{P(\lambda,\xi)}\hat{f}(\xi)\right](x) - \mathcal{F}_\xi^{-1}\left[\frac{\xi_k\xi_l}{(\lambda+\mu|\xi|^2)}\hat{\mathbf{g}}(\xi)\right](x) + \sum_{j=1}^N \mathcal{F}_\xi^{-1}\left[\frac{\xi\xi_j\xi_k\xi_l(\lambda\nu+\kappa|\xi|^2+\gamma)}{P(\lambda,\xi)}\hat{g}_j(\xi)\right](x) \tag{2.17}$$

Lemma 2.3. *Let $1 < q < \infty$, $\delta > 0$ and $0 < \epsilon < \pi/2$. Assume that $k(\lambda,\epsilon)$, $l(\lambda,\epsilon)$ and $m(\lambda,\epsilon)$ are functions on $\mathbb{R}^N \setminus \{0\} \times \Sigma_{\epsilon,0}$ such that for any multi-index $\alpha \in \mathbb{N}_0^N$ there exists a positive constant $M_{\alpha,\epsilon}$ such that*

$$|\partial_\xi^\alpha k(\xi,\lambda)| \le M_{\alpha,\epsilon}|\xi|^{1-|\alpha|}, \quad |\partial_\xi^\alpha l(\xi,\lambda)| \le M_{\alpha,\epsilon}|\xi|^{-|\alpha|},$$
$$|\partial_\xi^\alpha m(\xi,\lambda)| \le M_{\alpha,\epsilon}(|\lambda|^{1/2}+|\xi|)^{-1}|\alpha|^{-|\alpha|},$$

for any $(\xi,\lambda) \in \mathbb{R}^N \setminus \{0\} \times \Sigma_{\epsilon,0}$. Let $K(\lambda)$, $L(\lambda)$, $M(\lambda)$ be operators given for

$$[K(\lambda)f](x) = \mathcal{F}^{-1}[k(\xi,\lambda)\hat{f}(\xi)](x) \quad (\lambda \in \Sigma_{\epsilon,\lambda_0}),$$
$$[L(\lambda)f](x) = \mathcal{F}^{-1}[l(\xi,\lambda)\hat{f}(\xi)](x) \quad (\lambda \in \Sigma_{\epsilon,\lambda_0}),$$
$$[M(\lambda)f](x) = \mathcal{F}^{-1}[m(\xi,\lambda)\hat{f}(\xi)](x) \quad (\lambda \in \Sigma_{\epsilon,\lambda_0}).$$

Then the following assertions hold true:

(1) *The set $\{K(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}$ is $\mathcal{R}$-bounded on $\mathcal{L}(W_q^1(\mathbb{R}^N), L_q(\mathbb{R}^N))$ and there exists a positive constant $C_{N,q}$ such that*

$$\mathcal{R}_{\mathcal{L}(W_q^1(\mathbb{R}^N),L_q(\mathbb{R}^N))}(\{K(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le C_{q,N} \max_{|\alpha|\le N+1} M_{\alpha,\lambda_0}$$

(2) *Let $n = 0, 1$. Then the set $\{L(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}$ is $\mathcal{R}$-bounded on $\mathcal{L}(W_q^n(\mathbb{R}^N))$ and there exists a positive constant $C_{N,q}$ such that*

$$\mathcal{R}_{\mathcal{L}(W_q^n(\mathbb{R}^N))}(\{L(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le C_{q,N} \max_{|\alpha|\le N+1} M_{\alpha,\lambda_0}$$

(3) *The set* $\{M(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}$ *is* $\mathcal{R}$*-bounded on* $\mathcal{L}(L_q(\mathbb{R}^N), W_q^1(\mathbb{R}^N))$ *and there exists a positive constant* $C_{N,q,\delta}$ *such that*

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N),W_q^1(\mathbb{R}^N))}(\{M(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le C_{q,N,\delta} \max_{|\alpha|\le N+1} M_{\alpha,\lambda_0}.$$

Proof. Lemma 2.3 has been proven by Saito [5, Lemma 2.5]. □

Lemma 2.4. *For any* $0 < \epsilon < \pi/2$ *and* $s \in \mathbb{R}$*. Set* $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$*. Then, for* $\lambda \in \Sigma_{\epsilon,\lambda_0}$ *and multi-index* $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$*, there is a positive constant* $C_{a,b,s,\alpha,\epsilon}$ *such that*

$$|\partial_\xi^\alpha (a\lambda + b|\xi|^2)^s| \le C_{a,b,s,\alpha,\epsilon}(|\lambda|^{1/2} + |\xi|)^{2s-|\alpha|},$$

for any $\lambda \in (0, \pi/2)$ *and* $\xi \in \mathbb{R}^N$

Proof. Proof of the Lemma 2.4 has been proven by Shibata [13, Lemma 3.4]. □

Furthermore, we consider the estimation of the formula (2.13). By using Lemma 2.2, Lemma 2.4 and Leibniz's rule, for $(\xi, \lambda) \in \mathbb{R}^N \setminus \{0\} \times \Sigma_{\epsilon,\lambda_0}$ we have

$$\left|\partial_\xi^\alpha\left\{\left(\lambda\frac{d}{d\lambda}\right)^n\left(\frac{i\xi_k\xi_l\xi_m(\lambda + |\xi|^2(\mu+\nu))}{P(\lambda,\xi)}\hat{f}(\xi)\right)\right\}\right| \le C_{a,b,s,\alpha,\epsilon}|\xi|^{1-|\alpha|},$$
$$\left|\partial_\xi^\alpha\left\{\left(\lambda\frac{d}{d\lambda}\right)^n\left(\frac{\xi_j\xi_k\xi_l\xi_m}{P(\lambda,\xi)}\hat{g}_j(\xi)\right)\right\}\right| \le C_{a,b,s,\alpha,\epsilon}|\xi|^{-|\alpha|},$$

which combined with Lemma 2.3, furnishes

$$\mathcal{R}_{\mathcal{L}(\mathrm{Y_q(\mathbb{R}^N)},\mathrm{L_q(\mathbb{R}^N)^{N^3}})}(\{(\lambda\partial_\lambda)^n(\nabla^3\mathcal{A}_0(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le C_{a,b,s,\alpha,\epsilon}. \tag{2.18}$$

Moreover, the estimation of equation (2.14), we have

$$\left|\partial_\xi^\alpha\left\{\left(\lambda\frac{d}{d\lambda}\right)^n\left(\frac{\xi_k\xi_l\lambda^{1/2}(\lambda + |\xi|^2(\mu+\nu))}{P(\lambda,\xi)}\hat{f}(\xi)\right)\right\}\right| \le C_{a,b,s,\alpha,\epsilon}(|\lambda|^{1/2}+|\xi|)^{-1}|\xi|^{-|\alpha|},$$
$$\left|\partial_\xi^\alpha\left\{\left(\lambda\frac{d}{d\lambda}\right)^n\left(\frac{i\xi_j\xi_k\xi_l\lambda^{1/2}}{P(\lambda,\xi)}\hat{g}_j(\xi)\right)\right\}\right| \le C_{a,b,s,\alpha,\epsilon}(|\lambda|^{1/2}+|\xi|)^{-1}|\xi|^{-|\alpha|},$$

the same manner combined with Lemma 2.3, furnishes, we have

$$\mathcal{R}_{\mathcal{L}(\mathrm{Y_q(\mathbb{R}^N)},\mathrm{L_q(\mathbb{R}^N)^{N^2}})}(\{(\lambda\partial_\lambda)^n(\lambda^{1/2}\nabla^2\mathcal{A}_0(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le C_{a,b,s,\alpha,\epsilon}. \tag{2.19}$$

Moreover, the estimation of equation (2.15), we have

$$\left|\partial_\xi^\alpha\left\{\left(\lambda\frac{d}{d\lambda}\right)^n\left(\frac{(i\xi\lambda(\kappa|\xi|^2+\gamma)}{P(\lambda,\xi)}\hat{f}(\xi)\right)\right\}\right| \le C_{a,b,s,\alpha,\epsilon}(|\lambda|^{1/2}+|\xi|)^{-1}|\xi|^{-|\alpha|},$$
$$\left|\partial_\xi^\alpha\left\{\left(\lambda\frac{d}{d\lambda}\right)^n\left(\frac{\lambda}{(\lambda+\mu|\xi|^2)}\hat{\mathbf{g}}(\xi)\right)\right\}\right| \le C_{a,b,s,\alpha,\epsilon}(|\lambda|^{1/2}+|\xi|)^{1-|\alpha|},$$
$$\left|\partial_\xi^\alpha\left\{\left(\lambda\frac{d}{d\lambda}\right)^n\left(\frac{\xi\xi_j\lambda(\lambda\nu+\kappa|\xi|^2+\gamma)}{P(\lambda,\xi)}\hat{g}_j(\xi)\right)\right\}\right| \le C_{a,b,s,\alpha,\epsilon}(|\lambda|^{1/2}+|\xi|)^{2-|\alpha|},$$

with similar technique and thanks to Lemma 2.3 which furnishes,

$$\mathcal{R}_{\mathcal{L}(\mathrm{Y_q(\mathbb{R}^N)},\mathrm{L_q(\mathbb{R}^N)^{N^3}})}(\{(\lambda\partial_\lambda)^n(\lambda\mathcal{A}_1(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le C_{a,b,s,\alpha,\epsilon}. \tag{2.20}$$

Next, the estimation of equation (2.16), we have also

$$\left|\partial_\xi^\alpha\left\{\left(\lambda\frac{d}{d\lambda}\right)^n\left(\frac{(\xi\xi_k\lambda^{1/2}(\kappa|\xi|^2+\gamma)}{P(\lambda,\xi)}\hat{f}(\xi)\right)\right\}\right|\le C_{a,b,s,\alpha,\epsilon}(|\lambda|^{1/2}+|\xi|)^{-1}|\xi|^{|\alpha|},$$

$$\left|\partial_\xi^\alpha\left\{\left(\lambda\frac{d}{d\lambda}\right)^n\left(\frac{i\xi_k\lambda^{1/2}}{(\lambda+\mu|\xi|^2)}\hat{\mathbf{g}}(\xi)\right)\right\}\right|\le C_{a,b,s,\alpha,\epsilon}(|\lambda|^{1/2}+|\xi|)^{-1}|\xi|^{-1-|\alpha|},$$

$$\left|\partial_\xi^\alpha\left\{\left(\lambda\frac{d}{d\lambda}\right)^n\left(\frac{i\xi\xi_j\xi_k\lambda^{1/2}(\lambda\nu+\kappa|\xi|^2+\gamma)}{P(\lambda,\xi)}\hat{g}_j(\xi)\right)\right\}\right|\le C_{a,b,s,\alpha,\epsilon}(|\lambda|^{1/2}+|\xi|)^{-1}|\xi|^{1-|\alpha|}.$$

Then by Lemma 2.3, we have

$$\mathcal{R}_{\mathcal{L}(\mathrm{Y_q}(\mathbb{R}^\mathrm{N}),\mathrm{L_q}(\mathbb{R}^\mathrm{N})^{\mathrm{N}^3})}(\{(\lambda\partial_\lambda)^n(\lambda^{1/2}\nabla\mathcal{A}_1(\lambda))\mid\lambda\in\Sigma_{\epsilon,\lambda_0}\})\le C_{a,b,s,\alpha,\epsilon}. \tag{2.21}$$

Lastly, we estimate the equation (2.17). By using Lemma 2.2, Lemma 2.4 and Leibniz's rule, we have

$$\left|\partial_\xi^\alpha\left\{\left(\lambda\frac{d}{d\lambda}\right)^n\left(\frac{(i\xi\xi_k\xi_l(\kappa|\xi|^2+\gamma)}{P(\lambda,\xi)}\hat{f}(\xi)\right)\right\}\right|\le C_{a,b,s,\alpha,\epsilon}|\xi|^{1-|\alpha|},$$

$$\left|\partial_\xi^\alpha\left\{\left(\lambda\frac{d}{d\lambda}\right)^n\left(\frac{\xi_k\xi_l}{(\lambda+\mu|\xi|^2)}\hat{\mathbf{g}}(\xi)\right)\right\}\right|\le C_{a,b,s,\alpha,\epsilon}|\xi|^{-|\alpha|},$$

$$\left|\partial_\xi^\alpha\left\{\left(\lambda\frac{d}{d\lambda}\right)^n\left(\frac{\xi\xi_j\xi_k\xi_l(\lambda\nu+\kappa|\xi|^2+\gamma)}{P(\lambda,\xi)}\hat{g}_j(\xi)\right)\right\}\right|\le C_{a,b,s,\alpha,\epsilon}|\xi|^{-|\alpha|},$$

Then by Lemma 2.3, we have also

$$\mathcal{R}_{\mathcal{L}(\mathrm{Y_q}(\mathbb{R}^\mathrm{N}),\mathrm{L_q}(\mathbb{R}^\mathrm{N})^\mathrm{N})}(\{(\lambda\partial_\lambda)^n(\nabla^2\mathcal{A}_1(\lambda))\mid\lambda\in\Sigma_{\epsilon,\lambda_0}\})\le C_{a,b,s,\alpha,\epsilon}. \tag{2.22}$$

Therefore, equations (2.18), (2.19), (2.20), (2.21) and (2.22) imply the $\mathcal{R}$-boundedness of $\mathcal{K}_\lambda\mathcal{A}_0$ and $\mathcal{S}_\lambda\mathcal{A}_1$ in Theorem 2.1.

The proof of uniquenes property of the solution (2.3) follows similar technique as in [5, Sec.2]. This completes the proof of Theorem 2.1.

Acknowledgments. The first author thank to Ministry of Education, Culture, Research and Technology, Republic of Indonesia 054/E5/PG.02.00.PL/2024 for financial support under the Fundamental Research Grant Scheme 2024 and Institute of Research and Community Services (LPPM) Jenderal Soedirman University for the facilities provided.

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