Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 17 Issue 3(2025), Pages 96-102
https://doi.org/10.54671/BMAA-2025-3-7

BOUNDED MEASURES ON A LOCALLY COMPACT GROUP WITH REPRODUCING KERNEL

ANOH YANNICK KRAIDI, KINVI KANGNI

Abstract. Let G be a locally compact group, K a compact subset of G, $K(G,K)$ the set of complex functions with support in K. We consider $M_b(G)$, the set of bounded measures on G. In this paper, we investigate a positive definite kernel on $M_b(G)$ and we give some results related to this reproducing kernel. After that, a positive definite kernel obtained by a positive function on G will be considered in order to establish a chain of reproducing kernel spaces.

1. Introduction

The concept of reproducing kernel originated with the works of S. Bergmann and S. Szeg (see [5, 14]). In their work, they presented the reproducing kernels of Szeg and Bergmann. The study of reproducing kernel Hilbert spaces grew out of the work on integral operators by J. Mercer in 1909 and the work of S. Bergman (see [5]) in complex analysis on various domains. The theory of RKHS (reproducing kernel Hilbert space) appears in complex analysis, group representation theory, metric embedding theory, statistics, probability, the study of integral operators, and many other areas of analysis.
Let E be any set, H a Hilbert space of complex valued functions and let's consider the evaluation function on H defined by :

for all $x \in E$,

$$\begin{array}{rcl} \epsilon_x : H & \to & \mathbb{C} \\ f & \mapsto & \epsilon_x(f) = f(x). \end{array}$$

A reproducing kernel Hilbert space H on E is a Hilbert space of complex valued functions on E such that the evaluation function in each point of E is continuous (see [1, 6, 8, 9, 11]). Thanks to $Riesz - Fréchet$ theorem, one can deduce the existence of a kernel K : $E \times E \to \mathbb{C}$ such that for all $x \in E$,

$$f(x) = (f, K(.,x))_H, \text{ for all } f \in H.$$

The kernel verifies

$$K(x,y) = K_y(x) = (K(.,y), K(.,x))_H \text{ for all } x, y \in E.$$

2000 *Mathematics Subject Classification.* 46E27, 46E22, 22D05.
Key words and phrases. Bounded measure, reproducing kernel, locally compact group.

Submitted March 3, 2025. Accepted August 25, 2025. Published September 12, 2025.
Communicated by P. Jorgensen.

Nachman Aronsjan in his work, (see [3]) proved that for every positive definite kernel K, there exists a unique Hilbert space V of complex functions on E where K is the reproducing kernel. Therefore, for all $y \in E$,

$$f(y) = (f, K(., y))_V, \text{ for all } f \in V.$$

The theory of positive definite kernels plays a central role in functional analysis, particularly in the study of reproducing kernel Hilbert spaces (RKHS), group representations, and operator theory. It provides a deep connection between algebraic structures and analytical properties, especially through the construction of function spaces tailored to specific problems and the investigation of duality phenomena. Positive definite kernels play a fundamental role in the study of reproducing kernel Hilbert spaces (RKHS), and many notable examples of such kernels can be found in the literature [1, 5, 10, 11, 14]. These kernels have inspired extensive work in RKHS theory, particularly in the context of positive definite kernels on a group G, with wide-ranging applications across harmonic analysis, probability, and statistics. In this direction, James Tian (see [13]) investigates new duality principles for a class of positive definite kernels defined on locally compact abelian (LCA) groups. By exploiting the harmonic structure of these groups, the author establishes dual relationships between positive definite kernels and their associated representations, thereby extending classical results such as Bochners theorem within the broader framework of abstract harmonic analysis.

The question of extending positive definite functions on a subset of a group to the entire group is closely related to kernel theory. In this regard, the paper by Bakonyi and Timotin(see [4]) presents a significant result: In an amenable group, any operator-valued positive definite function defined on a symmetric subset $S \subset G$ can be extended to a positive definite function on the whole group, provided that the Cayley graph of G with respect to S satisfies the chordality condition. This combinatorial criterion offers a powerful and general tool for kernel extension and unifies various classical results, such as those of Kren or CarathodoryFejr.

Furthermore, kernel theory has recently been extended beyond the classical σ-additive measure-theoretic setting. In [2], Alpay and Jorgensen introduce a generalization of the RadonNikodym derivative operator for finitely additive functions, not necessarily σ-additive, defined on subsets of a σ-algebra. This extension enables the development of a generalized operator-theoretic framework, including the construction of generalized kernels in stochastic or probabilistic settings. Applications include generalized It integrals (stochastic integrals), measure-dependent Brownian motion, and families of transition operators. Taken together, these contributions highlight the richness and versatility of positive definite kernel theory, whether in its classical formulation on topological groups, in extension problems, or in its interaction with non-standard probabilistic structures.

Inspired by these developments, the present work aims to contribute to this theory by studying positive definite kernels in the space $M_b(G)$, the space of bounded measures on a locally compact group G. We explore certain extensions and structural properties of these kernels, and investigate how classical results can be adapted or generalized in this setting. The following section will give the main results.

2. Main results

Let G be a locally compact group, K a compact subset of G, $K(G,K)$ the set of complex functions with support in K. We consider $M_b(G)$, the set of bounded measures on G. Note that $(K(G,K), \|.\|_\infty)$ is a Banach space (see [7]).
Let $\mu \in K(G)^*$, then μ is continuous since it is a Radon measure. The restriction $\mu_{|K(G,K)}$ is continuous and also $\mu \in M_b(G)$. By the Riesz- Frechet Theorem, there exists $\varphi_\mu \in K(G,K)$ with $Supp\varphi_\mu \subset K$ such that for all $f \in K(G,K)$,

$$\mu(f) = \varphi_\mu(f) = (f, \varphi_\mu).$$

For all $\nu \in M_b(G), \nu(\varphi_\mu) = K_\varphi(\mu,\nu)$ where K_φ is the kernel such that $K_\varphi : M_b(G) \times M_b(G) \to \mathbb{C}$ and for all $\mu, \nu \in M_b(G)$,

$$\nu(\varphi_\mu) = K_\varphi(\mu,\nu) = (\varphi_\mu, \varphi_\nu) = (\mu,\nu).$$

Note that $K_\varphi(\mu,\nu) = \nu(\varphi_\mu) = (\varphi_\mu, \varphi_\nu) \geq 0$. Since the kernel K_φ is positive definite, thanks to the theorem of Aronsjan, there exists a reproducing kernel space $\mathcal{H}_K^\varphi$ such that:

$$\text{For all } f \in \mathcal{H}_K^\varphi, \text{ for all } \mu \in M_b(G) \ , \ \mu(f) = (f, K_\varphi(\mu,.))_{\mathcal{H}_K^\varphi}$$

If we consider G a group and $p : G \to \mathbb{C}$ a positive definite function on G. Then, Naimark (see [14]) introduced a reprodcing kernel space denoted by $\mathcal{H}(K_p)$ whose the kernel is defined by:

$$K_p : G \times G \to \mathbb{C} \text{ defined by } K_p(g,h) = p(g^{-1}h) \text{ for all } g,h \in G.$$

Let's consider T a bounded linear operator defined by $T : \mathcal{H}_K^\varphi \longrightarrow M_b(G)$ with T^* its adjoint operator and the kernel k_φ defined by :

$$k_\varphi(\mu,\nu) = (T^*TK_\varphi(.,\nu), T^*TK_\varphi(.,\mu))_{\mathrm{j}^\vee} \text{ on } M_b(G) \times M_b(G).$$

Then, we have

Theorem 2.1. *Let $\theta \in M_b(G)$, there exists $\tilde{\alpha} \in \mathcal{H}_K^\varphi$ such that :*

$$\inf_{\alpha \in \mathcal{H}_K^\varphi} \|T(\alpha) - \theta\|_\infty = \|T(\tilde{\alpha}) - \theta\|_\infty \tag{2.1}$$

if and only if, for the reproducing kernel space h_{k_φ},

$$T^*\alpha \in h_{k_\varphi}$$

Also, if the best approximation $\tilde{\alpha}$ is ensured, then there exists a unique extremal element $\breve{\alpha}$ in $\mathcal{H}_K^\varphi$ where the function $\breve{\alpha}$ has the minimum norm, and

$$\breve{\alpha}(\mu) = (T^*\alpha, T^*TK_\varphi(.,\alpha))_{h_{k_\varphi}}, \alpha \in M_b(G). \tag{2.2}$$

Proof: For any $\alpha \in \mathcal{H}_K^\varphi$, by the reproducing kernel $K_\varphi(\mu,\nu)$ in $\mathcal{H}_K^\varphi$, we have :

$$[T^*T\alpha](\mu) = (T^*T\alpha, K_\varphi(.,\mu))_{\mathcal{H}_K^\varphi} = (\alpha, T^*TK_\varphi(.,\mu))_{\mathcal{H}_K^\varphi}$$

The range of T^*T verifies the equality $R(T^*T) = h_{k_\varphi}$. Let P be the orthogonal projection of $\mathcal{H}_K^\varphi$ onto $\mathcal{H}_K^\varphi \bigoplus Ker(T^*T)$. Then, we have the following equality:

$$\|T^*T\alpha\|_{h_{k_\varphi}} = \|P\alpha\|_{\mathcal{H}_K^\varphi}.$$

Now, we suppose that the best approximation $\tilde{\alpha}$ in (2.1) holds. Then, we have :

$$\|T(\tilde{\alpha}) - \theta\|_\infty \leq \|\theta_0 - \theta\|_\infty$$

for all θ_0 in $\overline{R(T)}$. Hence, $\theta = T\tilde{\alpha} + \theta'$ for some $\theta' \in M_b(G) \bigoplus \overline{R(T)}$. Since $Ker(T^*) = M_b(G) \bigoplus \overline{R(T)}$, $T^*T\tilde{\alpha} = T^*\theta$, and we have $T^*\theta \in h_{k_\varphi}$.

Conversely, let $\alpha_1 \in \mathcal{H}_K^\varphi$ with $T^*T\alpha_1 = T^*\theta$. We choose θ_1 in $\overline{R(T)}$ such that

$$\|\theta_1 - \theta\|_\infty \leq \|\theta_0 - \theta\|_\infty$$

for all θ_0 in $\overline{R(T)}$. Then, $T^*T\alpha_1 = T^*\theta_1$ and $T\alpha_1 = \theta_1$ because T^* is one-to-one on $\overline{R(T)}$. Hence, we have, from the previous inequality :

$$\|T(\alpha_1) - \theta\|_\infty = \inf_{\alpha \in \mathcal{H}_K^\varphi} \|T(\alpha) - \theta\|_\infty$$

Setting $\check{\alpha} = P\alpha_1$, we see that $\check{\alpha}$ is a unique element in $\mathcal{H}_K^\varphi$ verifying

$$\|T(\check{\alpha}) - \theta\|_\infty = \inf_{\alpha \in \mathcal{H}_K^\varphi} \|T(\alpha) - \theta\|_\infty$$

and $\check{\alpha}$ has the minimum norm in $\mathcal{H}_K^\varphi$ because the set of functions α_1 that satisfy (2.1) is $\check{\alpha} + Ker(T^*T)$.

Finally, we determine the expression in (2.2). Since T^*T is an isometry of $\mathcal{H}_K^\varphi \bigoplus Ker(T^*T)$ onto h_{k_φ}, its adjoint S is the inversion of T^*T. Hence, we have

$$\check{\alpha}(\mu) = [ST^*l](\mu) = (ST^*l, K(.,\mu))_{\mathcal{H}_K^\varphi} = (T^*l, T^*TK(.,\mu))_{h_{k_\varphi}}$$

$\square$

3. Chain of reproducing kernel on a locally compact group

Let G be a locally compact group, and let $p : G \to \mathbb{C}$ be a function. The function p is said to be a positive definite function if for every n and every $g_1, ..., g_n \in G$, $p(g_i^{-1}g_j)$ is positive semidefinite. Consider the function K_p where

$$K_p : G \times G \to \mathbb{C} \text{ defined by } K_p(g,h) = p(g^{-1}h) \text{ for all } g, h \in G$$

Saying that p is positive definite implies that K_p is a positive definite kernel function. Since K_p is positive definite, for every positive definite function on G, there corresponds a reproducing kernel Hilbert space $\mathcal{H}(K_p)$.

Let U be a bounded linear operator from the reproducing kernel space $\mathcal{H}_K^\varphi$ into $\mathcal{H}(K_p)$. Let's introduce the following inner products:

$$(\tilde{\beta}_1, \tilde{\beta}_2)_{\mathcal{H}_K^\varphi[\mathcal{H}(K_p)]^+} = (\tilde{\beta}_1, \tilde{\beta}_2)_{\mathcal{H}_K^\varphi} + (U\tilde{\beta}_1, U\tilde{\beta}_2)_{\mathcal{H}(K_p)}$$

and

$$(\tilde{\beta}_1, \tilde{\beta}_2)_{\mathcal{H}_K^\varphi[\mathcal{H}(K_p)]^-} = (\tilde{\beta}_1, \tilde{\beta}_2)_{\mathcal{H}_K^\varphi} - (U\tilde{\beta}_1, U\tilde{\beta}_2)_{\mathcal{H}(K_p)}.$$

We shall determine the reproducing kernels for $\mathcal{H}_K^\varphi[\mathcal{H}(K_p)]^+$ and $\mathcal{H}_K^\varphi[\mathcal{H}(K_p)]^-$ verifying :

$$\mathcal{H}_K^\varphi[H_{\mathcal{H}(K_p)}]^+ \subset \mathcal{H}_K^\varphi \subset \mathcal{H}_K^\varphi[H_{\mathcal{H}(K_p)}]^-.$$

We consider $\mathcal{F}(M_b(G))$, the set of all functions defined on $M_b(G)$ and I, the linear map defined by

$$\begin{aligned} I : \mathcal{H}(K_p) &\longrightarrow \mathcal{F}(M_b(G)) \\ \theta &\longmapsto I(\theta) = \check{\theta} \end{aligned}$$

such that for all $\nu \in M_b(G)$, $\check{\theta}(\nu) = (\theta, U(K_\varphi(.,\nu)))_{\mathcal{H}(K_p)}$.

Theorem 3.1. *For $\nu \in M_b(G)$, we set $\check{\theta} = U^*(\theta)$ and we have :*

$$\|\check{\theta}\|^2_{\mathcal{H}_K^\varphi} = (\theta, U(\theta, UK_\varphi(.,\nu))_{\mathcal{H}(K_p)})_{\mathcal{H}(K_p)}.$$

Furthermore, the following conditions are equivalent:

1- $K_\varphi(\mu,\nu) \geq (UK_\varphi(.,\nu), UK_\varphi(.,\mu))_{\mathcal{H}(K_p)}$.

2- $\|U\| \leq 1$.

3- $\|\check{\theta}\|_{\mathcal{H}_K^{\varphi}} \leq \|\theta\|_{H_{\mathcal{H}(K_p)}}$.

Proof :

Let $\nu \in M_b(G), \theta \in \mathcal{H}(K_p)$. $\check{\theta}(\nu) = (\theta, U(K_\varphi(.,\nu)))_{\mathcal{H}(K_p)} = (U^*(\theta), K_\varphi(.,\nu))_{\mathcal{H}_K^{\varphi}}$. Since $\check{\theta} \in \mathcal{H}_K^{\varphi}, \check{\theta}(\nu) = (\check{\theta}, K_\varphi(.,\nu))_{\mathcal{H}_K^{\varphi}}$, then $\check{\theta} = U^*(\theta)$ and

$$\begin{aligned}\|\check{\theta}\|^2_{\mathcal{H}_K^{\varphi}} &= (\check{\theta}, \check{\theta})_{\mathcal{H}_K^{\varphi}} \\ &= (U^*(\theta), (\theta, U(K_\varphi(.,.)))_{\mathcal{H}(K_p)})_{\mathcal{H}(K_p)} \\ &= (\theta, U(\theta, UK_\varphi(.,.))_{\mathcal{H}(K_p)})_{\mathcal{H}(K_p)}.\end{aligned}$$

Let us prove the equivalences.

$1 \Longrightarrow 3$.

Let us introduce the following positive definite kernel K_U defined by:

$K_U^{\varphi}(\mu,\nu) = (UK_\varphi(.,\nu), UK_\varphi(.,\mu))_{\mathcal{H}(K_p)}$ on $M_b(G) \times M_b(G)$, and we denote by $h_{K_U^{\varphi}}$, the reproducing kernel space where K_U^{φ} is the kernel.

Since the range of I verifies $R(I) = h_{K_U^{\varphi}}$, we get the inequality $\|\check{a}\|_{h_{K_U^{\varphi}}} \leq \|\theta\|_{\mathcal{H}(K_p)}$.

For two positive definite kernels $K^{(1)}(p,q)$ and $K^{(2)}(p,q)$ defined in any set $E \times E$, if $K^{(1)}(p,q) \leq K^{(2)}(p,q)$ on $E \times E$, then for the reproducing kernel Hilbert spaces $H_{K^{(1)}}$ and $H_{K^{(2)}}$, we have $\|f\|_{H_{K^{(1)}}} \geq \|f\|_{H_{K^{(2)}}}$ for any $f \in H_{K^{(1)}}$ (see [3] Theorem I p.354). We deduce that if the condition 1 is true, we have $\|\check{\theta}\|_{\mathcal{H}_K^{\varphi}} \leq \|\check{\theta}\|_{h_{K_U^{\varphi}}}$. So, we get $\|\check{\theta}\|_{\mathcal{H}_K^{\varphi}} \leq \|\check{\theta}\|_{h_{K_T^{\varphi}}} \leq \|\theta\|_{\mathcal{H}_{K_p}}$.

$3 \Longrightarrow 2$ Let us assume 3. For $\theta \in \mathcal{H}_{K_p}$ and $\check{\theta} \in \mathcal{H}_K^{\varphi}$ such that $\|\check{\theta}\|_{\mathcal{H}_K^{\varphi}} \leq \|\theta\|_{\mathcal{H}_{K^{\varphi}}}$, we get $\|U\| \leq 1$.

$2 \Longrightarrow 1$

Let us assume 2. We recall that for two positive definite kernels $K^{(1)}(p,q)$ and $K^{(2)}(p,q)$ defined on any set $E \times E$ with $H_{K^{(1)}}$ and $H_{K^{(2)}}$ as reproducing kernel Hilbert spaces. $H_{K^{(1)}} \subset H_{K^{(2)}}$ if and only if there exists a positive constant γ, such that $K^{(1)}(p,q) \leq \gamma^2 K^{(2)}(p,q)$ and the minimum of such γ coincides with the norm of the inclusion map from $H_{K^{(1)}}$ into $H_{K^{(2)}}$ (See [14] Theorem 5.1 p.66). We can deduce that the map $\check{\theta} = T^*(\theta)$ is into $\mathcal{H}_K^{\varphi}$ and onto $\mathcal{H}_K^{\varphi}$ if and only if $K_U^{\varphi}(\mu,\nu) = (UK_\varphi(.,\nu), UK_\varphi(.,\mu))_{\mathcal{H}(K_p)} \leq K_\varphi(\mu,\nu)$ (See [10] lemna 3 p.51), hence 1 is proved.

□

For $\check{\theta} \in \mathcal{H}_K^{\varphi}[\mathcal{H}(K_p)]^+, \|\check{\theta}\|_{\mathcal{H}_K^{\varphi}[\mathcal{H}(K_p)]^+} \geq \|\check{a}\|_{\mathcal{H}_K^{\varphi}}$, then there exist a reproducing kernel ${K^{\varphi}_{\mathcal{H}(K_p)}}^+$ for the space $\mathcal{H}_K^{\varphi}[H_{\Lambda^{U^1}}]^+$ such that ${K^{\varphi}_{\mathcal{H}(K_p)}}^+(\mu,\nu) \leq K_\varphi(\mu,\nu)$ on $M_b(G) \times M_b(G)$ (See [3] Theorem II p.355). Let's consider the kernel K defined on $M_b(G) \times M_b(G)$ by $K_\varphi(\mu,\nu) = (\varphi_\mu, \varphi_\nu) = (K_\varphi(.,\nu), K_\varphi(.,\nu))_{\mathcal{H}_K^{\varphi}}$.

By the following theorem, we construct the reproducing kernel ${K^{\varphi}_{\mathcal{H}(K_p)}}^+(\mu,\nu)$ using $K_\varphi(\mu,\nu)$ for $\mu,\nu \in M_b(G)$.

Theorem 3.2. *Given that* $\{\tilde{K}_\varphi(.,\nu), \nu \in M_b(G)\} \subset \mathcal{H}_K^\varphi$, *the set of functions that satisfy the following equation*

$$K_\varphi(\mu,\nu) = \tilde{K}_\varphi(\mu,\nu) + (U\tilde{K}_\varphi(.,\nu), UK_\varphi(.,\mu))_{\mathcal{H}_K^\varphi} \tag{3.1}$$

Then, the kernel $\tilde{K}_\varphi(\mu,\nu)$ *is a solution of the equation (3.1) if and only if* $\tilde{K}_\varphi(\mu,\nu)$ *is the reproducing kernel of* $K^\varphi_{\mathcal{H}(K_p)}{}^+(\mu,\nu)$.

Proof:

For any $\mu_1, \mu_2 \in M_b$(G), we have

$$(K_\varphi(.,\mu_2), K_\varphi(.,\mu_1))_{\mathcal{H}_K^\varphi} = (\tilde{K}_\varphi(.,\mu_2), K_\varphi(.,\mu_1))_{\mathcal{H}_K^\varphi} + (U\tilde{K}_\varphi(.,\mu_2), U\tilde{K}_\varphi(.,\mu_1))_{\mathfrak{j}^*}$$

$$\Leftrightarrow K_\varphi(.,\mu_2) = \tilde{K}_\varphi(.,\mu_2) + U^*U\tilde{K}_\varphi(.,\mu_2).$$

For all $\tilde{\beta} \in \mathcal{H}_K^\varphi$,

$$(\tilde{\beta}, K_\varphi(.,\mu_2))_{\mathcal{H}_K^\varphi} = (\tilde{\beta}, \tilde{K}_\varphi(.,\mu_2))_{\mathcal{H}_K^\varphi} + (\tilde{\beta}, T^*T\tilde{K}_\varphi(.,\mu_2))_{\mathcal{H}_K^\varphi}$$

$$\Leftrightarrow \tilde{\beta}(\mu_2) = (\tilde{\beta}, \tilde{K}_\varphi(.,\mu_2))_{\mathcal{H}_K^\varphi[\mathcal{H}(K_p)]^+}.$$

If for any $\tilde{\beta} \in \mathcal{H}_K^\varphi$, $\|\tilde{\beta}\|_{\mathcal{H}_K^\varphi} \geq \|U\tilde{\beta}\|_{\mathcal{H}(K_p)}$ with the equality if and only if $\tilde{\beta} = 0$, then we can introduce the pre-Hilbert space $H^{'}$ where the inner product is defined by:

$$(\tilde{\beta}_1, \tilde{\beta}_2)_{H'} = (\tilde{\beta}_1, \tilde{\beta}_2)_{\mathcal{H}_K^\varphi} - (U\tilde{\beta}_1, U\tilde{\beta}_2)_{\mathcal{H}(K_p)}.$$

Let $\mu \in M_b(G)$, $\tilde{\nu}(\mu)$ the evaluation of $\tilde{\nu}$ at point μ is bounded on $H^{'}$ and for any Cauchy sequence $\{\tilde{\nu}_n\}$ in $H^{'}$, we have $\tilde{\nu}_n \longrightarrow 0$ on $M_b(G)$ implies that $\|\tilde{\nu}_n\|_{H'} \longrightarrow 0$. By the completion, $\overline{H'}$ is unique and admits the reproducing kernel.

If the conditions above are satisfied, we shall denote the completion of $H^{'}$ by $\mathcal{H}_K^\varphi[\mathcal{H}(K_p)]^-$,

Then, for any $\tilde{\beta} \in \mathcal{H}_K^\varphi$, $\|\tilde{\beta}\|_{\mathcal{H}_K^\varphi[\mathcal{H}(K_p)]^-} \leq \|\tilde{\beta}\|_{\mathcal{H}_K^\varphi}$ so $\mathcal{H}_K^\varphi$ is a subspace of $\mathcal{H}_K^\varphi[H_{\Lambda^{U^1}}]^-$. We shall denote the reproducing kernel of $\mathcal{H}_K^\varphi[H_{\Lambda^{U^1}}]^-$ by $K^-_{[H_{\Lambda^{U^1}}]}$. In this case, we obtain a similar result by exchanging + and - in (3.1).

□

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

References

[1] D. Alpay, *Reproducing Kernel Spaces and Applications*, Springer Bassel AG 2000.

[2] D. Alpay ; Jorgensen, Palle . *Finitely additive functions in measure theory and applications.*, Opuscula Math. 44 (2024), no. 3, 323–339.

[3] N. Aronszajn, *Theory of Reproducing Kernels*, Transactions of the American Mathematical Society Vol 68, No.3, May 1950.

[4] M. Bakonyi,; Timotin, D. *Extensions of positive definite functions on amenable groups.*, Canad. Math. Bull. 54 (2011), no. 1, 3–11.

[5] S. Bergmann, *ber die Entwicklung der harmischen Funktionen der Ebene und des Rmesnach Orthogonale Functionen*, Math. Ann 86, 1922.

[6] J.H. Manton, P-O. Amblard, *A Primer on Reproducing Kernel Hilbert Spaces.* NOW the essence of knowledge, 2015, pp. 9-19, pp. 34-44.

[7] K.Kinvi, S.Tour, *Analyse harmonique abstraite.*Cours de troisieme cycle, nouvelle dition, pp. 27-28.

[8] A.Y. Kraidi, K.Kinvi, *Reproducing kernel Cartan subalgebra*. Annales mathmatiques africaines, Volume 8 (2020), pp. 7-14.
[9] A.Y. Kraidi, K.Kinvi, *On a chain of reproducing kernel Cartan subalgebras*. Moroccan Journal of Pure and applied analysis (MJPAA), volume 7(1), 2021, Pages 4349. DOI: 10.2478/mjpaa-2021-0005
[10] S. Saitoh, *Integral transforms, reproducing kernel and their applications*. Pitman research notes in Mathematics, february 1997, pp.34-38.
[11] S. Saitoh, Y.Sawano, *Theory of reproducing kernels and Applications*, Developments in Mathematics Volume 44, Springer, pp. 1-2.
[12] S. Szeg, *Uber orthogonale Polynome, die zu einer gegebenen Kurve der Komplexen Ebene gehoren*. Math. Z. 9, pp. 218-270 (1921)
[13] James F. Tian, *Duality principles for a class of positive definite kernels on LCA groups* . (Pure Appl. Funct. Anal. 9 (2024), no. 6, 17351743)
[14] I. Vern Paulsen and Mrinal Raghupathi, *An introduction to the theory of reproducing kernel Hilbert spaces*, volume 152 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, pp. 173-174, 2016.

Anoh Yannick Kraidi
Department of mathematics and computer science.
25 BP 1245 Abidjan 25, Flix Houphouet Boigny university of Cocody, Abidjan, Cte d'Ivoire
E-mail address: kayanoh2000@yahoo.fr

Kinvi Kangni
Department of mathematics and computer science.
22 BP 1214 Abidjan 22, Flix Houphouet Boigny university of Cocody, Abidjan, Cte d'Ivoire
E-mail address: kinvikangni@yahoo.fr