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ON DIFFERED COPSON I-CONVERGENT SEQUENCE SPACES

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Abstract. Present work is an investigation of some new sequence spaces $c_0^I(C\Delta)$, $c^I(C\Delta)$, $\ell_\infty^I(C\Delta)$ and $\ell_\infty(C\Delta)$ as a domain of triangle Copson matrix via difference sequences $(\Delta^n x)$ over an admissible ideal of $\mathbb{N}$. Also, we investigate some algebraic, topological properties and give inclusion relations concerning these spaces.

1. Introduction

The hypothesis of sequence spaces has an important role in the diverse fields of analysis. Sequence spaces have various applications in different branches of functional analysis, in particular, the theory of functions, the theory of locally convex spaces, matrix transformations etc. The set of all real (or complex) valued sequences is symbolized by ω which turns out to be a vector space under pointwise addition and scalar multiplication. Any vector subspace of ω is called a sequence space. Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of natural, real and complex numbers, respectively. Throughout the paper, by ℓ_∞, c, and c_0 we denote the spaces of all sequences which are bounded, convergent, and convergent to zero (null sequences), respectively. After an extensive research about usual convergence of sequences in point set topology with respect to usual metrics, the conception of Ideal convergence or I-convergence came into existence by well-known author Kostyrko et al. [14]. Ideal convergence is a gereralization of statistical convergence which was introduced by well-known authors Fast and Steinhaus ([8, 28]), independently. Young researchers or scholars are suggested to go through deep analysis about the concept of usual convergence and Ideal convergence as both the concepts are independent. There are many sequences that are convergent but may not I-convergent. A large number of research work has been surfaced in the field of ideal convergent sequence spaces by many researchers, for further details on fundamental theorems in functional analysis, summability theory, sequence spaces and related topics the reader can refer to the recent textbooks [25] and [2] and, to the articles [16, 17, 18, 10, 11, 12, 13, 31, 32, 33, 1, 27] and [3]. Furter, Ideal convergence likned with summability theory by Šalát et al. [29, 30] and develop some new ideas from the perspective of sequence spaces. To know more about this concept one may refer to [19, 20, 21, 15, 22]. An ideal I is defined to be

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a family of a non-empty set X i.e., $I \subseteq 2^{\mathbf{X}}$ if $I_1, I_2 \in I$ implies that their union is in I i.e., $I_1 \cup I_2 \in I$, and $I_1 \in I, I_2 \subseteq I_1$ implies that $I_2 \in I$ whereas a filter is a family of sets $F \subseteq 2^{\mathbf{X}}$ if and only if $\emptyset \notin F$, $F_1, F_2 \in F$ implies that their intersection is in F i.e., $F_1 \cap F_2 \in F$ and $F_1 \subseteq F_2$ implies that $F_2 \in F$. If $I \neq \emptyset$ and $X \notin I$ then I is said to be non-trivial, admissible if and only if $\{\{x\} : x \in X\} \subseteq I$ and maximal if there is no ideal $J \neq I$ that contains I. For every I to be a non-trivial ideal there must corresponds a filter $F = F(I) = \{Y : X - Y \in I\}$.

Let, $T = (t_{nk})$ be an infinite matrix of real or complex numbers t_{nk} where $n, k \in \mathbb{N}$, the sequence defined as

$$T_n(x) = \sum_{k=0}^{\infty} t_{nk} x_k, \quad \text{for each } n \in \mathbb{N} \tag{1.1}$$

which is a $T-$transform of the sequence $x = (x_k) \in \omega$ by a matrix T also assuming that the right of series (1) converges for each $n \in \mathbb{N}$. The mapping of convergent sequences into another convergent sequences is given in Kojima-Schur Theorem [7] whose necessary and sufficient conditions are as follows:

(i) $\sum_{k=1}^{\infty} |t_{nk}| \leq N$ for every $n > m$;
(ii) $\lim_{n\to\infty} t_{nk} = \beta_k$ for every fixed k;
(iii) $\sum_{k=1}^{\infty} t_{nk} = T_n \to \beta$ as $n \to \infty$.

Also, if $t_{nk} = 0$ for $k > n$ and $t_{nn} \neq 0$ for all $n \in \mathbb{N}$, then $T = (t_{nk})$ is known to be trangle matrix which has a unique inverse T^{-1} for $|T| \neq 0$ and T^{-1} is a triangle matrix. The domain λ_T of an infinite matrix T in a sequence space λ is defined by

$$\lambda_T := \{x = (x_k) \in \omega : Tx \in \lambda\}$$

which is a sequense space. λ_T is a BK-space normed by $\|x\|_{\lambda_T} = \|Tx\|_{\lambda}$ for $x \in \lambda_T$ [5] only if λ is a BK-space and T is triangle matrix. A number of research papers have been published on this idea and its generalization [4, 6, 23, 24] which motivated us to investigate some of the new sequence spaces by using a well-known Copson matrix via diference sequences over admissible ideals.
Recalling, the Copson matrix of order n, $C^n = (c^n_{j,k})$ is an infinite matrix defined as,

$$c^n_{j,k} = \begin{cases} \frac{\binom{n+k-j-1}{k-j}}{\binom{n+k}{k}}, & \text{if } j \leq k, \\ 0, & \text{otherwise} \end{cases}$$

where all subscripts in $\mathbb{N}$. The Copson matrix has been used in the analysis of sequence spaces and also over ℓ_p-space it is contemplated as bounded linear operator. According to Helinger-Toeplitz theorem, the ℓ_p-norm is defined as $\|C^n\|_{\ell_p} = \frac{\Gamma(n+1)\Gamma(1/p)}{\Gamma(n+1/p)}$. Recently, by using the Copson square matrix of order n, Roopaei [26] has investigated $c_0(C^n)$, $c(C^n)$ and $\ell_p(C^n)$ as sequence spaces with all sequences whose Copson-transform of the sequence $x = (x_k)$ are in c_0, c, ℓ_p which are also sequence spaces respectively i.e.,

$$\lambda(C^n) = \left\{ x = (x_j) \in \omega : \lim_{j\to\infty} \left(\sum_{k=j}^{\infty} \frac{\binom{n+k-j-1}{k-j}}{\binom{n+k}{k}} \right) x_k \in \lambda \right\}$$

for $\lambda \in \{c_0, c, \ell_p\}$. Throughout the paper c_0^I, c^I and ℓ_∞^I denote the sequence spaces of all sequences which are null, convergent and bounded via an ideal I.

In this paper, by using Copson matrix of order n and via difference sequences $(\Delta^n x)$ through ideal convergence, we investigated $c_0^I(C\Delta)$, $c^I(C\Delta)$, $\ell_\infty^I(C\Delta)$ and $\ell_\infty(C\Delta)$ as sequences spaces with all sequences whose $C-$transform of $(\Delta^n x)$ are in c_0^I, c^I, ℓ_∞^I and ℓ_∞, respectively. We define the sequence $C^n(\Delta^n x)$ as $C-$transform of a sequence $(\Delta^n x)$ as follows:

$$C^n(\Delta^n x) = \lim_{j\to\infty}\left(\sum_{k=j}^{\infty}\frac{\binom{n+k-j-1}{k-j}}{\binom{n+k}{k}}\right)\Delta^n x_k$$

where, $\Delta^0 x = (x_k), \Delta x = (x_k - x_{k+1}), \Delta^n x = (\Delta^{n-1}x_k - \Delta^{n-1}x_{k+1})$ and this generalized difference notion has the following binomial representation:

$$\Delta^n x_k = \sum_{i=0}^{n}(-1)^i\binom{n}{i}x_{k+i}, \quad k \in \mathbb{N}.$$

In order to define main results, we recall some useful definitions and lemmas related to this investigation.

Definition 1.1 ([28]). *If $E = \{s \in E : s \leq n\} \subset \mathbb{N}$, then the natural density of the set E is defined as*

$$d(E) = \lim_{n\to\infty}\frac{1}{n}|E| \text{ exists}$$

where, $|E|$ is the cardinality of pre-defined set E.

Definition 1.2 ([9]). *A sequence $x = (x_k)$ is said t be Δ^n-statistically converges to a number $\zeta \in \mathbb{R}$ if, for every $\varepsilon > 0$ however small, $d(\{k \in \mathbb{N} : |\Delta^n x_k - \zeta| \geq \varepsilon\}) = 0$ and represented as Δ^n-st$-\lim x_k = \zeta$. In case $\zeta = 0$, then the sequence $x = (x_k)$ is said to be Δ^n-st-null.*

Definition 1.3 ([9]). *A sequence $x = (x_k)$ is said to be Δ^n-I-Cauchy if, for every $\varepsilon > 0$ however small, $\exists$ a number $m = m(\varepsilon)$ such that the set*
$\{k \in \mathbb{N} : |\Delta^n x_k - x_m| \geq \varepsilon\}$ belongs to an ideal I.

Definition 1.4 ([9]). *A sequence $x = (x_k)$ is said to be Δ^n-I-convergent to a number $\zeta \in \mathbb{R}$ if, for every $\varepsilon > 0$ however small, the set $\{k \in \mathbb{N} : |\Delta^n x_k - \zeta| \geq \varepsilon\}$ belongs to an ideal I and represented as Δ^n-$I-\lim x_k = \zeta$. In case $\zeta = 0$, then (x_k) is said to be Δ^n-I-null.*

Definition 1.5 ([9]). *A sequence $x = (x_k)$ is said to be Δ^n-I-bounded if there exists a positive real number $M > 0$ however large, such that, the set $\{k \in \mathbb{N} : |\Delta^n x_k| > M\}$ belongs to an ideal I.*

Definition 1.6 ([9]). *A sequence space S is said to be normal or solid, if the Cauchy product $(\alpha_k \Delta^n x_k)$ belongs to S, whenever $(\Delta^n x_k) \in S$ and for any sequence of scalars (α_k) with the condition $|\alpha_k| < 1$, for every $k \in \mathbb{N}$.*

Definition 1.7 ([9]). *Let $S = \{s_i \in \mathbb{N} : s_1 < s_2 < \cdots\} \subseteq \mathbb{N}$ and K be a sequence space. A S-step space of K is a sequence space*

$$\lambda_S^K = \{(x_{s_i}) \in \omega : (x_s) \in K\}.$$

A canonical pre-image of a sequence $(x_{s_i}) \in \lambda_S^K$ *is a sequence* $(y_s) \in \omega$ *defined as follows:*

$$y_s = \begin{cases} \Delta^n x_s, & \text{if } s \in S, \\ 0, & \text{otherwise.} \end{cases}$$

A canonical pre-image of a step space λ_S^K *is a set of canonical pre-images of all elements in* λ_S^K*, i.e., y is in canonical pre-image of* λ_S^K *iff y is canonical pre-image of some element* $x \in \lambda_S^K$.

Definition 1.8 ([29])**.** *A sequence space S is said to be monotone, if it contains the canonical pre-images of its step space.*

Lemma 1.1 ([29])**.** *Every solid space* $\implies$ *monotone space.*

Lemma 1.2 ([30])**.** *Let* $K_1 \in \mathcal{F}(I)$ *and* $K_2 \subseteq \mathbb{N}$*. If* $K_2 \notin I$*, then* $K_1 \cap K_2 \notin I$.

2. Main results

In this section, we investigated some new sequence spaces $c_0^I(C\Delta)$, $c^I(C\Delta)$, $\ell_\infty^I(C\Delta)$ and $\ell_\infty(C\Delta)$ defined by a Copson transformation via difference sequences $(\Delta^n x)$ over an admissible ideal I of subsets of $\mathbb{N}$ and study some algebraic, topological properties and prove some inclusion relations on these spaces.

$$\begin{aligned} c_0^I(C\Delta) &:= \{x = (x_k) \in \omega : \{n \in \mathbb{N} : |C^n(\Delta^n x)| \geq \epsilon\} \in I\}, \\ c^I(C\Delta) &:= \{x = (x_k) \in \omega : \{n \in \mathbb{N} : |C^n(\Delta^n x) - \zeta| \geq \epsilon, \text{ for some } \zeta \in \mathbb{R}\} \in I\}, \\ \ell_\infty^I(C\Delta) &:= \{x = (x_k) \in \omega : \exists M > 0 \text{ s.t } \{n \in \mathbb{N} : |C^n(\Delta^n x)| \geq M\} \in I\}, \\ \ell_\infty(C\Delta) &:= \{x = (x_k) \in \omega : \sup_n |C^n(\Delta^n x)| < \infty\}. \end{aligned}$$

Also,

$$m_0^I(C\Delta) := c_0^I(C\Delta) \cap \ell_\infty(C\Delta) \quad \text{and} \quad m^I(C\Delta) := c^I(C\Delta) \cap \ell_\infty(C\Delta).$$

Sequence spaces $c_0^I(C\Delta)$, $c^I(C\Delta)$, $\ell_\infty^I(C\Delta)$, $m^I(C\Delta)$, and $m_0^I(C\Delta)$ can be redefined as follows:

$$\begin{aligned} c_0^I(C\Delta) = (c_0^I)_{C\Delta}, c^I(C\Delta) = (c^I)_{C\Delta}, \ell_\infty^I(C\Delta) = (\ell_\infty^I)_{C\Delta}, \\ m^I(C\Delta) = (m^I)_{C\Delta} \quad \text{and} \quad m_0^I(C\Delta) = (m_0^I)_{C\Delta}. \end{aligned} \tag{2.1}$$

Definition 2.1 ([14])**.** *A sequence* $x = (x_k)$ *is said to be Copson* Δ^n*-I-convergent to a number* $\zeta \in \mathbb{R}$ *if, for every* $\varepsilon > 0$ *however small, the set* $\{k \in \mathbb{N} : |C^n(\Delta^n x_k) - \zeta| \geq \varepsilon\}$ *belongs to an ideal I and represented as* C_Δ^n*-I*$-\lim x_k = \zeta$*. In case* $\zeta = 0$*, then* (x_k) *is said to be* C_Δ^n*-I-null.*

Definition 2.2. *A sequence* $x = (x_k)$ *is said to be Copson* Δ^n*-I-Cauchy if for each* $\varepsilon > 0$*, however small, there exists a positive integer* $m_{(\varepsilon)} \in \mathbb{N}$ *such that*

$$\{n \in \mathbb{N} : |C^n(\Delta^n x) - C^m(\Delta^n x)| \geq \varepsilon\}$$

belongs to I, where $I \subseteq \mathbb{N}$ *be an admissible ideal.*

Example 2.1. *Define a class of finite subsets of* $\mathbb{N}$ *i.e* $I^f = \{N \subseteq \mathbb{N} : N \text{ is finite}\}$ *is an admissible ideal in* $\mathbb{N}$ *and* $c^{I^f}(C\Delta) = C_{c\Delta}$.

Example 2.2. *Let S_c denotes the space of all Copson Δ^n-statistically convergent sequences i.e.,*

$$S_c := \left\{x = (x_k) : d\big(\{n \in \mathbb{N} : |C^n(\Delta^n x) - \zeta| \geq \varepsilon\}\big) = 0,\ \textit{for any real}\ \zeta\right\}.$$

We define $I^d = \{N \subseteq \mathbb{N} : d(N) = 0\}$ a non trivial ideal that imples that $c^{I^d}(C\Delta) = S_{c\Delta}$, where $d(N)$ represents natural density of the set N.

This follows from the following example:

Example 2.3. *Every usual Copson difference sequence converges Copson Δ^n-statistically but the converse may not be true. To prove this result we consider the sequence $x = (x_k)$ defined by:*

$$\{C^n(\Delta^n x)\}_n = \begin{cases} 1, & \textit{if } n \textit{ is a square,} \\ 0, & \textit{otherwise.} \end{cases}$$

That is $C^n(\Delta^n x) = \{1, 0, 0, 1, 0, 0, 0, 0, 1, 0, \dots\}$ and taking the limit $\zeta = 0$. Then we have the inclusion

$$\{n \in \mathbb{N} : |C^n(\Delta^n x) - \zeta| \geq \varepsilon\} \subset \{1, 4, 9, 16, \dots, m^2, (m+1)^2 \dots\}$$

Since, Natural density of the set on right of above equation is zero i.e., the set of squares of natural numbers, so as a result we get,

$$d(\{n \in \mathbb{N} : |C^n(\Delta^n x) - \zeta| \geq \varepsilon\}) = 0.$$

This implies that, the difference sequence is Copson statistically convergent, but is not usual Copson convergent.

Theorem 2.1. *The spaces $c^I(C\Delta)$, $c_0^I(C\Delta)$, $\ell_\infty^I(C\Delta)$, $m_0^I(C\Delta)$, and $m^I(C\Delta)$ are linear spaces over the real field $\mathbb{R}$.*

Proof. Let $x = (\Delta^n x_k)$, $y = (\Delta^n y_k) \in c^I(C\Delta)$ be two arbitrary sequences and α_1, α_2 are scalars. Now, since $x, y \in c^I(C\Delta)$, then for given $\varepsilon > 0$, there exist $\zeta_1, \zeta_2 \in \mathbb{R}$, such that

$$\left\{n \in \mathbb{N} : |C^n(\Delta^n x) - \zeta_1| \geq \frac{\varepsilon}{2}\right\} \in I \quad \text{and} \quad \left\{n \in \mathbb{N} : |C^n(\Delta^n y) - \zeta_2| \geq \frac{\varepsilon}{2}\right\} \in I.$$

Now, let

$$A_1 = \left\{n \in \mathbb{N} : |C^n(\Delta^n x) - \zeta_1| < \frac{\varepsilon}{2|\alpha_1|}\right\} \in \mathcal{F}(I),$$

$$A_2 = \left\{n \in \mathbb{N} : |C^n(\Delta^n y) - \zeta_2| < \frac{\varepsilon}{2|\alpha_2|}\right\} \in \mathcal{F}(I)$$

be such that $A_1^c, A_2^c \in I$. Then

$$\begin{aligned} A_3 &= \{n \in \mathbb{N} : |C^n(\alpha_1 x + \alpha_2 y) - (\alpha_1\zeta_1 + \alpha_2\zeta_2)| < \varepsilon\} \\ &\supseteq \left\{\left\{n \in \mathbb{N} : |C^n(\Delta^n x) - \zeta_1| < \frac{\varepsilon}{2|\alpha_1|}\right\} \cap \left\{n \in \mathbb{N} : |C^n(\Delta^n y) - \zeta_2| < \frac{\varepsilon}{2|\alpha_2|}\right\}\right\}. \end{aligned} \tag{2.2}$$

As a result, we see that the right side of above equality (2.2) belongs to the to the filter $\mathcal{F}(I)$ associated with I over the set of natural numbers which implies that its complement set always belongs to Ideal I and hence we get $(\alpha_1 x + \alpha_2 y) \in c^I(C)$. Hence, $c^I(C\Delta)$ is a linear space over the field $\mathbb{R}$.

The proof for the remaining spaces $c_0^I(C\Delta)$, $\ell_\infty^I(C\Delta)$, $m_0^I(C\Delta)$, and $m^I(C\Delta)$ can be proven by the similar way used, above.

□

Theorem 2.2. *$\lambda(C\Delta)$ are normed spaces with respect to the sup-norm defined by*

$$\|x\|_{\lambda(C\Delta)} = \sup_n |C^n(\Delta^n x)|, \quad \textit{where } \lambda \in \{c^I, c_0^I, \ell_\infty^I, \ell_\infty\}. \tag{2.3}$$

Theorem 2.3. *A sequence $x = (x_k)$ is said to be Copson Δ^n-I-convergent if and only if for every $\varepsilon > 0$, $\exists\ m = m(\varepsilon) \in \mathbb{N}$, such that*

$$\{n \in \mathbb{N} : |C^n(\Delta^n x) - C^m(\Delta^n x)| < \varepsilon\} \in \mathcal{F}(I). \tag{2.4}$$

Proof. Let, the sequence $x = (x_k)$ is Copson Δ^n-I-convergent to some number $\zeta \in \mathbb{R}$, then for a given $\varepsilon > 0$ however small, we have

$$A_\varepsilon = \left\{n \in \mathbb{N} : |C^n(\Delta^n x) - \zeta| < \frac{\varepsilon}{2}\right\} \in \mathcal{F}(I).$$

Fix an integer $m = m(\varepsilon) \in A_\varepsilon$. Then we have

$$|C^n(\Delta^n x) - C^m(\Delta^n x)| \le |C^n(\Delta^n x) - \zeta| + |\zeta - C^m(\Delta^n x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $n \in A_\varepsilon$. Hence (2.4) holds.

Conversely, suppose that (2.4) holds for all $\varepsilon > 0$. Then

$$B_\varepsilon = \{n \in \mathbb{N} : C^n(\Delta^n x) \in [C^n(\Delta^n x) - \varepsilon, C^n(\Delta^n x) + \varepsilon]\} \in \mathcal{F}(I), \text{ for all } \varepsilon > 0.$$

Let $J_\varepsilon = [C^n(\Delta^n x) - \varepsilon, C^n(\Delta^n x) + \varepsilon]$. Fixing $\varepsilon > 0$, we have $B_\varepsilon \in \mathcal{F}(I)$ and $B_{\frac{\varepsilon}{2}} \in \mathcal{F}(I)$. Hence $B_\varepsilon \cap B_{\frac{\varepsilon}{2}} \in \mathcal{F}(I)$ provided

$$J = J_\varepsilon \cap J_{\frac{\varepsilon}{2}} \neq \emptyset,$$

which implies that,

$$\{n \in \mathbb{N} : C^n(\Delta^n x) \in J\} \in \mathcal{F}(I)$$

and hence

$$\text{diam } (J) \le \frac{1}{2} \text{ diam } (J_\epsilon),$$

where, diam(J) denotes the length of the interval J and by induction we get sequence of closed intervals as follows: $J_\epsilon = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n \supseteq \cdots$ s.t

$$\text{diam } (I_n) \le \frac{1}{2} \text{ diam } (I_{n-1}), \text{ for } n = (2, 3, \dots).$$

As a result we get,

$$\{n \in \mathbb{N} : C^n(\Delta^n x) \in I_n\} \in \mathcal{F}(I).$$

This implies that $\exists\ \zeta \in \bigcap_{n\in\mathbb{N}} I_n$ and it is a routine work to verify that $\zeta = I\text{-}\lim C^n(x)$ which shows that $x = (x_k)$ is Copson Δ^n-I-convergent. This completes the proof. □

Theorem 2.4. *The inclusions $\ell_\infty^I(C\Delta) \supset c^I(C\Delta) \supset c_0^I(C\Delta)$ strictly hold.*

Proof. The inclusion $c^I(C\Delta) \supset c_0^I(C\Delta)$ is obviously true. We show only its strictness. To do this, let us consider the sequence $x = (\Delta^n x_k)$ such that $C^n(\Delta^n x) = 2$ which implies that $C^n(\Delta^n x)$ belongs to c^I but not belongs to c_0^I.

Moreover, if $x = (\Delta^n x_k) \in c^I(C\Delta)$ then there exists a real number ζ such that

$$I\text{-}\lim C^n(\Delta^n x) = \zeta$$

i.e.,

$$\{n \in \mathbb{N} : |C^n(\Delta^n x) - \zeta| \geq \varepsilon\} \in I.$$

We have

$$|C^n(\Delta^n x)| = |C^n(\Delta^n x) - \zeta + \zeta| \leq |C^n(\Delta^n x) - \zeta| + |\zeta|.$$

From the above result we can say that the sequence $(\Delta^n x_k)$ must be an element of $\ell_\infty^I(C\Delta)$.
Also to show the strictness of the inclusion $\ell_\infty^I(C\Delta) \supset c^I(C\Delta)$, we give the following example:

Example 2.4. *Define the sequence $x = (x_k)$ such that*

$$\{C^n(\Delta^n x)\}_n = \begin{cases} 1, & \text{if } n \text{ is odd non-square,} \\ 0, & \text{if } n \text{ is even non-square} \\ \sqrt{n}, & \text{if } n \text{ is square.} \end{cases}$$

Although the sequence $C^n(\Delta^n x)$ belongs to ℓ_∞^I, the $C^n(\Delta^n x)$ does not belongs to c^I which means that the sequence $x \in \ell_\infty^I(C\Delta) \backslash c^I(C\Delta)$.

As a result, we get that the inclusion relations $\ell_\infty^I(C\Delta) \supset c^I(C\Delta) \supset c_0^I(C\Delta)$ strictly hold. □

This follows from the following example:

Example 2.5. *Every Copson bounded difference sequence is Copson Δ^n-I-bounded but the converse may not be true. To prove this, we consider the following example. For this, we define the sequence $x = (x_k)$ by*

$$\{C^n(\Delta^n x)\}_n = \begin{cases} \frac{n^2}{n+1}, & \text{for prime } n, \\ 0, & \text{otherwise} \end{cases}$$

which proves that $C^n(\Delta^n x)$ is not bounded sequence but the set $\{n \in \mathbb{N} : |C^n(\Delta^n x)| \geq 1\}$ belongs to ideal. Hence the sequence (x_k) is Copson Δ^n-I-bounded.

Theorem 2.5. *The following statements are satisfied:*

(a) *The sequence spaces $c^I(C\Delta)$ and $\ell_\infty(C\Delta)$ are overlap but neither contains the other.*
(b) *The sequence spaces $c_0^I(C\Delta)$ and $\ell_\infty(C\Delta)$ are overlap but neither contains the other.*

Proof.
(a) We shall prove the spaces $c^I(C\Delta)$ and $\ell_\infty(C\Delta)$ are not disjoint spaces for this we consider a sequence $x = (x_k)$ s.t $C^n(\Delta^n x) = \frac{1}{n}$ for n belongs to $\mathbb{N}$ then $x \in c^I(C\Delta^n)$ and $x \in \ell_\infty(C\Delta^n)$ both. Moreover, we define the sequence $x = (x_k)$ by

$$\{C^n(\Delta^n x)\}_n = \begin{cases} \sqrt{n}, & n \text{ is a square,} \\ 0, & \text{otherwise} \end{cases}$$

so that, $x \in c^I(C\Delta)$ but $x \notin \ell_\infty(C\Delta)$. Moreover, we again define the sequence $x = (x_k)$ as follows

$$\{C^n(\Delta^n x)\}_n = \begin{cases} n, & n \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, we conclude that $(x) \in \ell_\infty(C\Delta)$ but $x \notin c^I(C\Delta)$.
Since the proof of Part (b) is similar to that of Part (a), we omit detail. □

Theorem 2.6. *The spaces* $m_0^I(C\Delta)$ *and* $m^I(C\Delta)$ *are defined, as follows:*

$$m_0^I(C\Delta) := c_0^I(C\Delta) \cap \ell_\infty(C\Delta) \quad \text{and} \quad m^I(C\Delta) := c^I(C\Delta) \cap \ell_\infty(C\Delta).$$

are closed in $\ell_\infty(C\Delta)$ *as a subspace.*

Proof. We consider a Cauchy sequence $(\Delta^n x_k^{(i)})$ in $m^I(C\Delta) \subset \ell_\infty(C\Delta)$. Then $(\Delta^n x_k^{(i)})$ converges to a point in $\ell_\infty(C\Delta)$ and $\lim_{i\to\infty} C^{n(i)}(\Delta^n x) = C^n(\Delta^n x)$. Let $I - \lim C^{n(i)}(\Delta^n x) = \zeta_i$ for every $i \in \mathbb{N}$. Then we only need to show that

(a) Sequence (ζ_i) converges to ζ;
(b) The limit, $I - \lim C^n(\Delta^n x) = \zeta$ exists.

(a) As $(\Delta^n x_k^{(i)})$ is a Cauchy sequence then for each $\varepsilon > 0$ however small, there always exists a positive integer $m \in \mathbb{N}$ such that

$$\left|C^{n(i)}(\Delta^n x) - C^{n(j)}(\Delta^n x)\right| < \frac{\varepsilon}{3}, \text{ for all } i, j \geq m. \tag{2.5}$$

Now, consider two sets A_i and A_j in an ideal I defined as:

$$A_i = \left\{n \in \mathbb{N} : |C^{n(i)}(\Delta^n x) - \zeta_i| \geq \frac{\varepsilon}{3}\right\} \tag{2.6}$$

and

$$A_j = \left\{n \in \mathbb{N} : |C^{n(j)}(\Delta^n x) - \zeta_j| \geq \frac{\varepsilon}{3}\right\}. \tag{2.7}$$

Moreover, let us suppose that $n \notin A_i \cap A_j$ for all $i, j \geq m$. Then we get

$$|\zeta_i - \zeta_j| \leq |C^{n(i)}(\Delta^n x) - \zeta_i| + |C^{n(j)}(\Delta^n x) - \zeta_j| + |C^{n(i)}(\Delta^n x) - C^{n(j)}(\Delta^n x)| < \varepsilon$$

by (2.5), (2.6),and(2.7). Thus (ζ_i) is a Cauchy sequence and hence converges to $\zeta \in \mathbb{R}$ i.e., $\lim_{i\to\infty} \zeta_i = \zeta$.

(b) Let $\delta > 0$ however small, be given. Then, we have a positive integer n_0 such that

$$|\zeta_i - \zeta| < \frac{\delta}{3}, \text{ for every } i > n_0. \tag{2.8}$$

This implies that $(\Delta^n x_k^{(i)}) \to \Delta^n x_k$ as $i \to \infty$. Thus

$$|C^{n(i)}(\Delta^n x) - C^n(\Delta^n x)| < \frac{\delta}{3}, \text{ for every } i > n_0. \tag{2.9}$$

Since $(C^{n(j)}(\Delta^n x))$ is I-convergent to $\zeta_j \in \mathbb{R}$ then $\exists\ \mathcal{E} \in I$ such that for every $n \notin \mathcal{E}$ so we get,

$$|C^{n(j)}(\Delta^n x) - \zeta_j| < \frac{\delta}{3}. \tag{2.10}$$

Moreover, let $j > n_0$ then $\forall\ n \notin \mathcal{E}$, we get by (2.8), (2.9), and (2.10) such that

$$|C^n(\Delta^n x) - \zeta| \leq |C^n(\Delta^n x) - C^{n(j)}(\Delta^n x)| + |C^{n(j)}(\Delta^n x) - \zeta_j| + |\zeta_j - \zeta| < \delta.$$

Therefore, $(\Delta^n x_k)$ is Copson I-convergent to $\zeta \in \mathbb{R}$. Thus $m^I(C\Delta)$ is closed in $\ell_\infty(C\Delta)$ as a subspace. The other part can be established by the following similar technique. □

Theorem 2.7. *Sequence spaces* $c^I(C\Delta), c_0^I(C\Delta)$, *and* $\ell_\infty^I(C\Delta)$ *are BK-spaces with respect to the sup-norm as follows:*

$$\|x\|_{\lambda(C\Delta^n)} = \sup_n |C^n(\Delta^n x)|, \quad \text{where } \lambda \in \left\{c^I, c_0^I, \ell_\infty^I, \ell_\infty\right\}.$$

Proof. It is known that the spaces c^I, c_0^I, and ℓ_∞^I are BK-spaces. Moreover, (2.1) satisfies and the Copson matrix is a triangle matrix. Now, by considering all these three facts and also by Theorem 4.3.12 of Wilansky [34], we conclude that the spaces are BK-spaces. This completes the proof. □

In the view of Theorem 2.6 and since the inclusions $m^I(C\Delta) \subset \ell_\infty(C\Delta)$ and $m_0^I(C\Delta) \subset \ell_\infty(C\Delta)$ are strict, we formulate the following result without proof.

Theorem 2.8. *Spaces $m^I(C\Delta)$ and $m_0^I(C\Delta)$ are nowhere dense in $\ell_\infty(C\Delta)$ as a subset.*

Theorem 2.9. *Spaces $c_0^I(C\Delta)$ and $m_0^I(C\Delta)$ are monotone and solid respectively.*

Proof. First we shall prove the result only for $c_0^I(C\Delta)$.
Let $x = (\Delta^n x_k) \in c_0^I(C\Delta)$. For $\varepsilon > 0$ however small, we have

$$\{n \in \mathbb{N} : |C^n(x)| \geq \varepsilon\} \in I. \tag{2.11}$$

Let $a = (\Delta^n a_k)$ be a scalar sequence satisfies $|\Delta^n a| \leq 1$ for all $k \in \mathbb{N}$. Then, we have

$$|C^n(ax)| = |aC^n(x)| \leq |a|\,|C^n(x)| \leq |C^n(x)|\,, \forall n \in \mathbb{N}. \tag{2.12}$$

From the relations (2.11), (2.12), we conclude that

$$\{n \in \mathbb{N} : |C^n(ax)| \geq \varepsilon\} \subseteq \{n \in \mathbb{N} : |C^n(x)| \geq \varepsilon\} \in I$$

This implies that the set

$$\{n \in \mathbb{N} : |C^n(\alpha x)| \geq \varepsilon\} \text{ belongs to ideal I.}$$

Therefore as a result we get the sequence $(ax) \in c_0^I(C\Delta)$.
$\Longrightarrow$ Space $c_0^I(C)$ is a solid space.
Also, as we know that every solid space is monotone by Lemma 1.1. This implies that the space $c_0^I(C\Delta)$ is monotone.

□

Corollary 2.10. *If the ideal I is neither maximal nor $I = I_f$, then the sequence spaces $c^I(C\Delta^n)$ and $m^I(C\Delta^n)$ are neither solid nor monotone.*

Proof. We shall prove the above corollary by introducing an example as follows: □

Example 2.6. *Let $I = I_f$ and let $S = \{n \in \mathbb{N} : n \text{ is an odd integer}\}$. Consider the S-step space S_K of K as:*

$$S_K = \{(x_k) \in \omega : (x_k) \in S\}.$$

Let us define the sequence $(y_k) \in S_K$ by

$$C^n(y) = \begin{cases} C^n(\Delta^n x), & \text{if } n \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, we consider the sequence $(\Delta^n x_k)$ defined by $\{C^n(\Delta^n x)\}_n = 3$ for all $n \in \mathbb{N}$. Then, the sequence $(\Delta^n x_k) \in E(C\Delta)$, but its S-step space preimage does not belongs to $E(C\Delta)$, where $E = c^I$ and m^I.
In this way, as a result we find $E(C\Delta)$ are not monotone and by following lemma 1.1, the space $E(C\Delta)$ is not solid.

Theorem 2.11. *Let, for a sequence $x = (\Delta^n x_k)$ and a non-trivial admissible ideal I in $\mathbb{N}$ if there exists a sequence $y = (\Delta^n y_k) \in c^I(C)$ such that $C^n(x) = C^n(y)$ for almost all n relative to I, then $x \in c^I(C\Delta)$.*

Proof. Suppose that $C^n(x) = C^n(y)$ for almost all n relative to I i.e.,

$$\{n \in \mathbb{N} : C^n(x) \neq C^n(y)\} \in I.$$

Consider the sequence $(\Delta^n y_k)$ which is Copson $\Delta^n - I$-convergent to ζ. Then, for every $\varepsilon > 0$ however small, we have the following set belongs to ideal I i.e.,

$$\{n \in \mathbb{N} : |C^n(y) - \zeta| \geq \varepsilon\} \in I.$$

Since, we have considered I as an admissible ideal of set of natural numbers so we derive the following inclusion relation

$$\{n \in \mathbb{N} : |C^n(x) - \zeta| \geq \varepsilon\} \subseteq \{n \in \mathbb{N} : C^n(x) \neq C^n(y)\} \cup \{n \in \mathbb{N} : |C^n(y) - \zeta| \geq \varepsilon\}.$$

□

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