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DISK CHARACTERIZATIONS OF PRINGSHEIM AND STATISTICAL CORES FOR COMPLEX DOUBLE SEQUENCES

SAMI M. HAMID, RICHARD F. PATTERSON

Abstract. In this paper, we prove the equivalence between two characterizations of the Pringsheim core for bounded double sequences using the Hahn-Banach Separation Theorem. We establish that the classical convex hull definition equals the intersection of disk sets determined by Pringsheim limit superior distances. We further extend this approach to the statistical setting, proving an analogous equivalence for the statistical Pringsheim core.

1. Introduction

The core of a sequence, first introduced by Knopp [5] as the intersection of closed convex hulls of sequence tails, provides valuable insights into sequence behavior and convergence properties. Shcherbakov [14] advanced this concept with an alternative characterization using intersections of closed disks.

For double sequences, Patterson [10] introduced core theory in the context of Pringsheim convergence, defining the Pringsheim core as the intersection of closed convex hulls of sequence tails. His subsequent work on four-dimensional matrix characterizations [11] further developed this theory. Sever and Altay [13] provided another significant contribution by demonstrating the equivalence between alternative characterizations of the Pringsheim core using barriers and disks. Comprehensive treatments of these and related developments can be found in Başar [1] and Mursaleen and Mohiuddine [8].

A parallel development in sequence analysis began with Fast [3] and Steinhaus [15], who introduced the concept of statistical convergence. This approach filters out exceptional behavior occurring on sets of indices with zero density. Fridy and Orhan [4] extended this to core theory for statistically convergent sequences, while Morič [6] systematically studied statistical convergence of multiple sequences. Further contributions by Çakan and Altay [2] and Mursaleen and Edely [7] investigated statistical convergence in the context of double sequences.

Core theory for double sequences encompasses several fundamental concepts. The Pringsheim core, defined as the intersection of closed convex hulls of sequence

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tails, characterizes limiting behavior for sequences that may not converge classically. This has been extended to M-cores based on almost convergence, R-cores utilizing Riesz methods, and statistical cores employing natural density. Four-dimensional matrix transformations, including regular, strongly regular, and almost regular matrices, serve as essential tools for studying how these core properties behave under transformation. The theory of α-, β-, and γ-duals provides additional structural understanding of these sequence spaces and their relationships.

In this paper, we accomplish three significant extensions to core theory. First, we extend Shcherbakov's characterization to double sequences, proving the equivalence between convex hull and disk-based definitions of the Pringsheim core using the Hahn-Banach Separation Theorem. Second, following Fridy and Orhan's half-plane approach for single sequences [4], we extend their definition to the statistical Pringsheim core for double sequences using double natural density δ_2, generalizing the convex hull characterization to the statistical setting. Third, we establish an analogous equivalence for this statistical Pringsheim core, demonstrating that our disk-based characterization extends naturally to the statistical framework. Our functional-analytic approach provides direct, unified proofs of both characterizations, bridging classical and statistical convergence theories in the context of double sequences.

The structure of this paper is as follows: Section 2 establishes preliminary concepts and definitions. Section 3 presents our equivalence theorem for Pringsheim core characterizations. Section 4 extends these results to the statistical setting, establishing the equivalence of convex hull and disk-based characterizations for the statistical Pringsheim core. Section 5 provides concluding remarks and identifies directions for future research.

2. Preliminaries

We begin by establishing the mathematical background necessary for proving our main results on the Pringsheim core. We first review basic notions of Pringsheim convergence for double sequences, then introduce the Hahn-Banach Separation Theorem which will serve as our primary functional analysis tool.

2.1. Pringsheim Convergence Concepts.

We begin with the fundamental notion of convergence for double sequences introduced by Pringsheim.

Definition 2.1 (Pringsheim [12]). *A double sequence $x = (x_{k,l})$ has Pringsheim limit L (denoted by $P\text{-}\lim_{k,l} x_{k,l} = L$) provided that given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k, l > N$. We shall describe such a sequence more briefly as "P-convergent."*

For sequences that do not converge, we require the notions of limit superior and limit inferior in the Pringsheim sense.

Definition 2.2 (Patterson [10]). *For a double sequence $x = (x_{k,l})$ of real numbers:*

Let $\alpha_n = \sup\{x_{k,l} : k, l \geq n\}$. The Pringsheim limit superior (denoted by P-$\limsup$) of x is defined as

$$P\text{-}\limsup_{k,l} x_{k,l} = \begin{cases} +\infty & \text{if } \alpha_n = +\infty \text{ for each } n, \\ \inf_n\{\alpha_n\} & \text{if } \alpha_n < \infty \text{ for some } n. \end{cases}$$

Let $\beta_n = \inf\{x_{k,l} : k,l \geq n\}$. *The Pringsheim limit inferior (denoted by* P-$\liminf$*) of* x *is defined as*

$$P\text{-}\liminf_{k,l} x_{k,l} = \begin{cases} -\infty & \text{if } \beta_n = -\infty \text{ for each } n, \\ \sup_n\{\beta_n\} & \text{if } \beta_n > -\infty \text{ for some } n. \end{cases}$$

2.2. **Hahn-Banach Separation Theorem.** For our purposes, we need a specific form of the separation theorem for locally convex spaces. The following theorem is a standard result in functional analysis.

Theorem 2.3 (Hahn-Banach Separation Theorem [9])**.** *Let* X *be a locally convex topological vector space over* $\mathbb{C}$*. If* C *is a non-empty closed convex subset of* X *and* $x_0 \in X \setminus C$*, then there exists a continuous linear functional* $\phi : X \to \mathbb{C}$ *and* $\alpha \in \mathbb{R}$ *such that*

$$\operatorname{Re}(\phi(x_0)) < \alpha < \operatorname{Re}(\phi(y)) \quad \text{for all } y \in C.$$

Local convexity ensures the existence of sufficiently many continuous linear functionals to separate points from closed convex sets. A key consequence of the Hahn-Banach Separation Theorem that we will use in our proof is the following:

Corollary 2.4. *The closed convex hull of a set* S *in a locally convex topological vector space* X *is the intersection of all closed half-planes containing* S*.*

This follows from the fact that any point outside the closed convex hull can be separated from it by a hyperplane, defining a closed half-plane containing the hull but not the point. This corollary is central to our proof of the equivalence between convex hull and disk-based core characterizations in Section 3, as it allows us to represent convex hulls via bounding hyperplanes.

2.3. **Definitions for Pringsheim Core.** Building on the concepts of Pringsheim convergence, we now present the standard definition of the Pringsheim core, followed by the construction of specific disk sets that will lead to an alternative characterization.

Definition 2.5 (Patterson [10])**.** *Let* $x = (x_{k,l})$ *be a bounded double sequence in* $\mathbb{C}$*. For each* $n \in \mathbb{N}$*, let* P-$C_n\{x\}$ *be the least closed convex set containing all points* $x_{k,l}$ *for* $k,l > n$*. The* ***Pringsheim core*** *of* x *is defined as*

$$P\text{-}C\{x\} = \bigcap_{n=1}^{\infty} P\text{-}C_n\{x\}.$$

Definition 2.6 (Sever and Altay [13])**.** *Let* $x = (x_{k,l})$ *be a bounded double sequence in* $\mathbb{C}$*. For each* $z \in \mathbb{C}$*, define*

$$B_x(z) = \left\{ w \in \mathbb{C} : |w - z| \leq P\text{-}\limsup_{k,l} |x_{k,l} - z| \right\},$$

where $P\text{-}\limsup_{k,l} |x_{k,l} - z| = \inf_{n\in\mathbb{N}} \sup_{k,l>n} |x_{k,l} - z|$.

Remark 1. *The boundedness of the sequence ensures that infinite intersections in our characterizations are well-defined in* $\mathbb{C}$*, allowing our equivalence proof to proceed via separation arguments.*

3. Equivalence of Pringsheim Core Characterizations

Having established the necessary background concepts, we now present our first main result. This theorem demonstrates that the Pringsheim core, defined via the intersection of closed convex hulls in Definition 2.5, can be equivalently characterized as the intersection of all disk sets $B_x(z)$ as z ranges over the complex plane.

Theorem 3.1. *For any bounded double sequence $x = (x_{k,l})$ in $\mathbb{C}$,*

$$P\text{-}C\{x\} = \bigcap_{z \in \mathbb{C}} B_x(z).$$

Proof. For necessity part, we show that $P\text{-}C\{x\} \subseteq \bigcap_{z \in \mathbb{C}} B_x(z)$. Let $w \in P\text{-}C\{x\}$ and fix an arbitrary $z \in \mathbb{C}$. Setting $L = P\text{-}\limsup_{k,l} |x_{k,l} - z| = \inf_{m,n} \sup_{k>m,l>n} |x_{k,l} - z|$, we aim to show $|w - z| \leq L$, which implies $w \in B_x(z)$. For any $n \in \mathbb{N}$, the point w belongs to $P\text{-}C_n\{x\}$, the closed convex hull of $\{x_{k,l} : k, l > n\}$. By Carathéodory's theorem in $\mathbb{C}$ (which is isomorphic to $\mathbb{R}^2$), we can approximate w arbitrarily closely using convex combinations of at most three points from this set. Since w belongs to the closed convex hull, for any $\varepsilon > 0$, we can find points $x_{k_1,l_1}, x_{k_2,l_2}, x_{k_3,l_3}$ with $k_i, l_i > n$ for $1 \leq i \leq 3$, and non-negative coefficients $\lambda_1, \lambda_2, \lambda_3$ summing to unity, such that $\left| w - \sum_{i=1}^{3} \lambda_i x_{k_i,l_i} \right| < \varepsilon$. Using the triangle inequality and the condition $\sum_{i=1}^{3} \lambda_i = 1$, we obtain

$$\begin{aligned}
|w - z| &\leq \left| w - \sum_{i=1}^{3} \lambda_i x_{k_i,l_i} \right| + \left| \sum_{i=1}^{3} \lambda_i x_{k_i,l_i} - z \right| \\
&< \varepsilon + \left| \sum_{i=1}^{3} \lambda_i x_{k_i,l_i} - \sum_{i=1}^{3} \lambda_i z \right| = \varepsilon + \left| \sum_{i=1}^{3} \lambda_i (x_{k_i,l_i} - z) \right| \\
&\leq \varepsilon + \sum_{i=1}^{3} \lambda_i \left| x_{k_i,l_i} - z \right| \leq \varepsilon + \max_{1 \leq i \leq 3} \left| x_{k_i,l_i} - z \right| \leq \varepsilon + \sup_{k,l>n} \left| x_{k,l} - z \right|.
\end{aligned}$$

Since this inequality holds for any $\varepsilon > 0$, we conclude that $|w - z| \leq \sup_{k,l>n} |x_{k,l} - z|$. As this bound is valid for every $n \in \mathbb{N}$, we can take the infimum over all n to obtain

$$|w - z| \leq \inf_{n \in \mathbb{N}} \sup_{k,l>n} |x_{k,l} - z| = P\text{-}\limsup_{k,l} |x_{k,l} - z| = L.$$

This establishes that $w \in B_x(z)$ for all $z \in \mathbb{C}$, and consequently, $w \in \bigcap_{z \in \mathbb{C}} B_x(z)$.

To establish the sufficiency part, suppose $w \notin P\text{-}C\{x\}$, then by definition of $P\text{-}C\{x\} = \bigcap_{n=1}^{\infty} P\text{-}C_n\{x\}$, there exists $N \in \mathbb{N}$ such that $w \notin P\text{-}C_N\{x\}$. Since $P\text{-}C_N\{x\}$ is closed and convex in $\mathbb{C}$, and $\{w\}$ is compact, the Hahn-Banach Separation Theorem (Theorem 2.3) yields a continuous linear functional $\phi : \mathbb{C} \to \mathbb{C}$ and a real number α such that

$$\mathrm{Re}(\phi(w)) < \alpha < \mathrm{Re}(\phi(y)) \quad \forall y \in P\text{-}C_N\{x\}.$$

In the finite-dimensional complex space $\mathbb{C}$, every continuous linear functional admits a representation via the standard inner product. Specifically, there exists $u \in \mathbb{C}\backslash\{0\}$ such that $\phi(z) = \langle z, u\rangle$ for all $z \in \mathbb{C}$, where $\langle z, u\rangle = z\overline{u}$ is the standard inner product. Note that the separation inequality applies specifically to the real part of this functional, giving us $\mathrm{Re}(\phi(w)) < \alpha < \mathrm{Re}(\phi(y))$ for all $y \in P\text{-}C_N\{x\}$.

We now construct a special point $z_0 = w + r\dfrac{u}{|u|}$, where $r = \dfrac{\alpha - \mathrm{Re}(\phi(w))}{|u|} > 0$. This point z_0 has the property that $\mathrm{Re}(\phi(z_0)) = \alpha$, which we verify through the following calculation

$$\begin{aligned}
\mathrm{Re}(\phi(z_0)) &= \mathrm{Re}\left(\phi\left(w + r\frac{u}{|u|}\right)\right)\\
&= \mathrm{Re}(\phi(w)) + \mathrm{Re}\left(\phi\left(r\frac{u}{|u|}\right)\right) = \mathrm{Re}(\phi(w)) + r\,\mathrm{Re}\left(\phi\left(\frac{u}{|u|}\right)\right)\\
&= \mathrm{Re}(\phi(w)) + r\,\mathrm{Re}\left(\left\langle \frac{u}{|u|}, u\right\rangle\right) = \mathrm{Re}(\phi(w)) + r\frac{\mathrm{Re}(\langle u, u\rangle)}{|u|}\\
&= \mathrm{Re}(\phi(w)) + r\frac{|u|^2}{|u|} = \mathrm{Re}(\phi(w)) + r\,|u| = \mathrm{Re}(\phi(w)) + \frac{\alpha - \mathrm{Re}(\phi(w))}{|u|}\cdot|u|\\
&= \mathrm{Re}(\phi(w)) + \alpha - \mathrm{Re}(\phi(w)) = \alpha.
\end{aligned}$$

Therefore, z_0 lies precisely on the hyperplane defined by $\mathrm{Re}(\phi(z)) = \alpha$.

Now we define $L = P\text{-}\limsup_{k,l} |x_{k,l} - z_0|$. By the definition of the Pringsheim limit superior (Definition 2.2), for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{k,l>n_0} |x_{k,l} - z_0| < L + \varepsilon.$$

For any $k, l > \max\{n_0, N\}$, we have $x_{k,l} \in P\text{-}C_N\{x\}$ and $|x_{k,l} - z_0| < L + \varepsilon$. Furthermore, from our application of the Hahn-Banach separation theorem, we know that

$$\mathrm{Re}(\phi(x_{k,l})) > \alpha = \mathrm{Re}(\phi(z_0)).$$

These properties allow us to construct a point y in the closure of $\{x_{k,l} : k, l > N\}$ with two properties: $|y - z_0| \le L$ and $\mathrm{Re}(\phi(y)) \ge \alpha$. To construct a point with the desired properties, we first define a sequence of sets. For each positive integer n, let

$$A_n = \left\{x_{k,l} : k, l > \max\{N, n_0, n\} \text{ and } |x_{k,l} - z_0| \le L + \frac{1}{n}\right\}.$$

Let $B_n = \{x_{k,l} \in A_n : \mathrm{Re}(\phi(x_{k,l})) \ge \alpha\}$. We first verify that each B_n is non-empty. From the definition of the Pringsheim limit superior $L = P\text{-}\limsup_{k,l} |x_{k,l} - z_0|$, for any n there exist indices $k, l > \max\{N, n_0, n\}$ such that $|x_{k,l} - z_0| \le L + \dfrac{1}{n}$. Additionally, for all $k, l > N$, our separation result guarantees that $\mathrm{Re}(\phi(x_{k,l})) > \alpha$ since $x_{k,l} \in P\text{-}C_N\{x\}$. Therefore, such points $x_{k,l}$ belong to both A_n and B_n, confirming that B_n is non-empty. Moreover, these sets form a nested sequence $B_{n+1} \subseteq B_n$ because any element $x_{k,l} \in B_{n+1}$ must be some $x_{k,l}$ where $k, l > \max\{N, n_0, n+1\}$, which implies $k, l > \max\{N, n_0, n\}$. Moreover, $|x_{k,l} - z_0| \le$

$L + \dfrac{1}{n+1} < L + \dfrac{1}{n}$ since the upper bound becomes stricter with increasing n. The condition $\text{Re}(\phi(x_{k,l})) \geq \alpha$ holds for elements in both sets. Thus, any element in B_{n+1} satisfies all conditions to be in B_n as well. Each B_n is bounded because the double sequence x is a bounded sequence, and the sequence of sets $\{B_n\}_{n\in\mathbb{N}}$ is nested.

For each n, the closure $\overline{B_n}$ is compact. By the nested compact set theorem, $\bigcap_{n=1}^{\infty} \overline{B_n} \neq \emptyset$. Let $y \in \bigcap_{n=1}^{\infty} \overline{B_n}$. Then y is in the closure of $\{x_{k,l} : k, l > N\}$, so $y \in P\text{-}C_N\{x\}$ since $P\text{-}C_N\{x\}$ is closed. Furthermore, $|y - z_0| \leq L$ follows from the definition of y as an element of $\bigcap_{n=1}^{\infty} \overline{B_n}$. By continuity of ϕ and the fact that $\text{Re}(\phi(x_{k,l})) \geq \alpha$ for all $x_{k,l} \in B_n$, we have $\text{Re}(\phi(y)) \geq \alpha$. Moreover, since $y \in P\text{-}C_N\{x\}$, the Hahn-Banach separation gives us $\text{Re}(\phi(y)) > \alpha$. We now establish the key contradiction that completes our proof. Recalling that

$$|w - z_0| = r = \frac{\alpha - \text{Re}(\phi(w))}{|u|} > 0,$$

we will demonstrate that $|w - z_0| > L$, which implies $w \notin B_x(z_0)$. Assume that $|w - z_0| \leq L$. Consider the line segment from w to y given by $\gamma(t) = (1-t)w + ty$ for $t \in [0, 1]$. Since $\text{Re}(\phi(w)) < \alpha < \text{Re}(\phi(y))$, the intermediate value theorem guarantees the existence of $t_0 \in (0, 1)$ such that $\text{Re}(\phi(\gamma(t_0))) = \alpha$. Define $v = \gamma(t_0) = (1 - t_0)w + t_0 y$. The triangle inequality, combined with our assumptions, yields

$$\begin{aligned} |v - z_0| &= |(1 - t_0)w + t_0 y - z_0| = |(1 - t_0)(w - z_0) + t_0(y - z_0)| \\ &\leq (1 - t_0)\,|w - z_0| + t_0\,|y - z_0| \leq (1 - t_0)L + t_0 L = L. \end{aligned}$$

Thus, we have a point v with $\text{Re}(\phi(v)) = \alpha$ by construction and $|v - z_0| \leq L$. We now demonstrate that $v \notin P\text{-}C_N\{x\}$. Suppose $v \in P\text{-}C_N\{x\}$, then we can express $v = (1 - t_0)w + t_0 y$ where $y \in P\text{-}C_N\{x\}$. Solving for w, we get

$$w = \frac{v}{1 - t_0} - \frac{t_0 y}{1 - t_0}.$$

This gives us $w = \lambda_1 v + \lambda_2 y$ where $\lambda_1 = \dfrac{1}{1 - t_0} > 0$ and $\lambda_2 = \dfrac{-t_0}{1 - t_0} < 0$. Since $\lambda_2 < 0$, this is not a convex combination of v and y. This negative coefficient is crucial because if $v \in P\text{-}C_N\{x\}$, then the convex set $P\text{-}C_N\{x\}$ contains both v and y. Since $P\text{-}C_N\{x\}$ is convex, it contains all convex combinations of these points. However, our expression for w involves $w = \lambda_1 v + \lambda_2 y$ where $\lambda_2 < 0$, which means w lies outside the convex hull of v and y. Since w cannot be represented as a convex combination of points in $P\text{-}C_N\{x\}$, we have contradicted our original assumption that $w \in P\text{-}C_N\{x\}$. Thus, $v \notin P\text{-}C_N\{x\}$. Since $v \notin P\text{-}C_N\{x\}$, by the Hahn-Banach separation theorem, we must have

$$\text{Re}(\phi(v)) < \alpha.$$

But we constructed v precisely to satisfy $\text{Re}(\phi(v)) = \alpha$. This contradiction establishes that our assumption $|w - z_0| \leq L$ must be false. Therefore, $|w - z_0| > L = P\text{-}\limsup_{k,l} |x_{k,l} - z_0|$, which means $w \notin B_x(z_0)$, and consequently, $w \notin \bigcap_{z\in\mathbb{C}} B_x(z)$.

Having established both inclusions, we conclude that $P\text{-}C\{x\} = \bigcap_{z\in\mathbb{C}} B_x(z)$. $\square$

Remark 2. *Theorem 3.1 extends Shcherbakov's work on kernels of complex sequences [14] to double sequences. Our proof uses the Hahn-Banach Separation Theorem to demonstrate equivalence between the convex hull definition and disk intersection characterization, offering complementary analytical perspectives on the Pringsheim core.*

4. Statistical Cores for Double Sequences

We now extend our functional analytic approach to the statistical setting. Statistical convergence, which originated with the work of Fast [3] and Steinhaus [15], offers a more general notion of convergence by effectively ignoring exceptional behavior occurring on sets of indices with zero density. This approach allows us to capture limiting behavior even when traditional convergence fails.

4.1. Definitions for Statistical Convergence.

We begin by establishing the necessary definitions for our treatment of statistical cores.

Definition 4.1 (Natural Density for Double Sequences)**.** *For $E \subseteq \mathbb{N} \times \mathbb{N}$, the natural density of E is defined as*

$$\delta_2(E) = P\text{-}\lim_{m,n\to\infty} \frac{|E \cap (\{1,2,\ldots,m\} \times \{1,2,\ldots,n\})|}{mn},$$

provided the Pringsheim limit exists.

This concept, studied by Mursaleen and Edely [7], extends the natural density from single sequences to double sequences. Our formulation retains the mathematical essence while adopting slightly different notation for clarity in our context.

For convenience, we say that a property holds for "statistically almost all indices" to indicate that it holds for a set of indices with density one.

Building on the work of Çakan and Altay [2] for real double sequences, we extend their statistical concepts to complex double sequences as follows:

Definition 4.2 (Statistical Boundedness)**.** *A double sequence $x = (x_{k,l})$ is said to be statistically bounded if there exists $M > 0$ such that*

$$\delta_2\left(\{(k,l) : |x_{k,l}| > M\}\right) = 0.$$

We denote the space of all statistically bounded double sequences by st_2^∞.

Definition 4.3 (Statistical Limit Superior)**.** *For a statistically bounded double sequence $x = (x_{k,l})$ of complex numbers and any $z \in \mathbb{C}$,*

$$st_2\text{-}P\text{-}\limsup_{k,l} |x_{k,l} - z| = \inf\left\{r \in \mathbb{R} : \delta_2\left(\{(k,l) : |x_{k,l} - z| > r\}\right) = 0\right\}.$$

Following Fridy and Orhan's half-plane approach for single sequences in [4], we extend their definition to the statistical Pringsheim core for double sequences using double natural density δ_2, analogous to the classical Pringsheim core [10]. This generalizes the convex hull characterization of cores to the statistical setting, where closed half-planes contain $x_{k,l}$ for statistically almost all indices (k,l).

Definition 4.4 (Statistical Pringsheim Core - Convex Hull Definition)**.** *For a statistically bounded double sequence $x = (x_{k,l})$ in $\mathbb{C}$, let $\mathcal{H}(x)$ be the collection of all* *closed half-planes that contain $x_{k,l}$ for statistically almost all indices. The statistical Pringsheim core of x is defined as*

$$st_2\text{-}P\text{-}C\{x\} = \bigcap_{H\in\mathcal{H}(x)} H.$$

Definition 4.5 (Statistical Disk Sets)**.** *For a statistically bounded double sequence $x = (x_{k,l})$ in $\mathbb{C}$ and any $z \in \mathbb{C}$, define*

$$B_x^{st}(z) = \left\{ w \in \mathbb{C} : |w - z| \leq st_2\text{-}P\text{-}\limsup_{k,l} |x_{k,l} - z| \right\}.$$

Before proceeding to our main equivalence theorem, we establish several measure-theoretic and structural properties that form the foundation for extending conventional core theory to the statistical setting. The following lemma addresses three essential aspects: the behavior of density-one sets under intersections, the preservation of convex hull properties, and the topological characteristics needed for functional analytic arguments. These properties collectively establish the bridge between traditional Pringsheim core theory and its statistical counterpart, enabling us to apply separation techniques in the measure-theoretic framework of statistical convergence.

Lemma 4.6 (Measure-Theoretic Foundations for Statistical Cores)**.** *Let $x = (x_{k,l})$ be a statistically bounded double sequence in $\mathbb{C}$. The following properties hold:*

(i) ***Finite Intersections:*** *If $E_1, E_2, \ldots, E_n \subseteq \mathbb{N} \times \mathbb{N}$ each having $\delta_2(E_i) = 1$ for all $i \in \{1, 2, \ldots, n\}$, then $\delta_2\left(\bigcap_{i=1}^{n} E_i\right) = 1$.*

(ii) ***Convex Hull Preservation:*** *If $E \subseteq \mathbb{N} \times \mathbb{N}$ satisfies $\delta_2(E) = 1$ and $S \subseteq \mathbb{C}$ contains $\{x_{k,l} : (k,l) \in E\}$, then $st_2\text{-}P\text{-}C\{x\}$ is contained in the closed convex hull $\overline{\mathrm{conv}}(S)$.*

(iii) ***Infinite Intersections:*** *The statistical Pringsheim core $st_2\text{-}P\text{-}C\{x\}$, defined as $\bigcap_{H\in\mathcal{H}(x)} H$, is closed under arbitrary intersections in $\mathbb{C}$.*

Proof. (i) Let $E_1, E_2, \ldots, E_n \subseteq \mathbb{N} \times \mathbb{N}$ be sets each having $\delta_2(E_i) = 1$ for all $i \in \{1, 2, \ldots, n\}$. From the definition of double natural density (Definition 4.1), $\delta_2(E_i^c) = 0$ for each $i \in \{1, 2, \ldots, n\}$. Using the subadditivity of the natural density measure, we obtain

$$\delta_2\left(\bigcup_{i=1}^{n} E_i^c\right) \leq \sum_{i=1}^{n} \delta_2(E_i^c) = 0.$$

Therefore

$$\delta_2\left(\bigcap_{i=1}^{n} E_i\right) = 1 - \delta_2\left(\bigcup_{i=1}^{n} E_i^c\right) = 1.$$

(ii) Suppose, for contradiction, that there exists $p \in st_2\text{-}P\text{-}C\{x\}$ with $p \notin \overline{\mathrm{conv}}(S)$. By Theorem 2.3, there exists a continuous linear functional $\phi : \mathbb{C} \to \mathbb{C}$ and $\alpha \in \mathbb{R}$ such that $\mathrm{Re}(\phi(p)) < \alpha < \mathrm{Re}(\phi(y))$ for all $y \in \overline{\mathrm{conv}}(S)$. Define $H = \{z \in \mathbb{C} : \mathrm{Re}(\phi(z)) \geq \alpha\}$, a closed half-plane containing S but not p. Since S contains $\{x_{k,l} : (k,l) \in E\}$ where $\delta_2(E) = 1$, we have $\delta_2(\{(k,l) : x_{k,l} \in H\}) = 1$. By Definition 4.4, $st_2\text{-}P\text{-}C\{x\}$ is contained in every closed half-plane that contains the sequence values on a set of indices with density one. Since H is such a half-plane but $p \notin H$, we have reached a contradiction, and therefore, st_2-P-$C\{x\} \subseteq \overline{\text{conv}}(S)$.

(iii) We need to prove that the statistical Pringsheim core st_2-P-$C\{x\} = \bigcap_{H \in \mathcal{H}(x)} H$ is a non-empty, closed, and convex subset of $\mathbb{C}$. To establish non-emptiness, we utilize the statistical boundedness of x (Definition 4.2). Since x is statistically bounded, there exists $M > 0$ such that $\delta_2(\{(k,l) : |x_{k,l}| > M\}) = 0$. Consequently, the set $E = \{(k,l) : |x_{k,l}| \leq M\}$ has $\delta_2(E) = 1$. Let $D_M = \{z \in \mathbb{C} : |z| \leq M\}$ be the closed disk of radius M. This disk D_M is compact as it is both closed and bounded in $\mathbb{C}$, and contains $x_{k,l}$ for all $(k,l) \in E$. For any finite collection $\{H_1, H_2, \ldots, H_n\} \subset \mathcal{H}(x)$, each H_i contains $x_{k,l}$ for indices in some set E_i with $\delta_2(E_i) = 1$. By Lemma 4.6(i), the intersection $E' = E \cap E_1 \cap E_2 \cap \ldots \cap E_n$ has $\delta_2(E') = 1$, making E' non-empty. This means there exist indices (k,l) such that $x_{k,l} \in D_M \cap H_1 \cap H_2 \cap \ldots \cap H_n$, proving that any finite intersection of sets from $\{D_M \cap H_\alpha\}_{\alpha \in A}$ is non-empty. Now consider any collection $\{H_\alpha\}_{\alpha \in A}$ in $\mathcal{H}(x)$ and the corresponding family $\{D_M \cap H_\alpha\}_{\alpha \in A}$. Each $D_M \cap H_\alpha$ is closed as the intersection of two closed sets. Moreover, each $D_M \cap H_\alpha$ is compact because it is a closed subset of the compact set D_M. Since we have established that any finite subcollection of $\{D_M \cap H_\alpha\}_{\alpha \in A}$ has non-empty intersection, the finite intersection property of compact spaces guarantees that the infinite intersection $\bigcap_{\alpha \in A} (D_M \cap H_\alpha) = D_M \cap \left(\bigcap_{\alpha \in A} H_\alpha \right)$ is non-empty. This intersection is contained in $\bigcap_{\alpha \in A} H_\alpha = st_2$-$P$-$C\{x\}$, proving that the statistical Pringsheim core is non-empty. For closedness and convexity, we observe that each $H \in \mathcal{H}(x)$ is both closed and convex by definition, being a closed half-plane. Since the intersection of any collection of closed convex sets preserves both properties, st_2-P-$C\{x\}$ is both closed and convex in $\mathbb{C}$. Thus, the statistical Pringsheim core st_2-P-$C\{x\}$ is a non-empty, closed, and convex subset of $\mathbb{C}$. □

4.2. **Equivalence of Statistical Core Characterizations.** We now establish the equivalence between the two characterizations of the statistical Pringsheim core, extending our analysis from the Pringsheim core to the statistical setting.

Theorem 4.7. *For any statistically bounded double sequence $x = (x_{k,l})$ in $\mathbb{C}$,*

$$st_2\text{-}P\text{-}C\{x\} = \bigcap_{z \in \mathbb{C}} B_x^{st}(z).$$

Proof. The proof extends our approach from Theorem 3.1 to the statistical setting, utilizing the measure-theoretic foundations established in Lemma 4.6. The key modification involves replacing tail sets $\{(k,l) : k,l > n\}$ with sets of indices having density one.

For necessity part, we show that st_2-P-$C\{x\} \subseteq \bigcap_{z \in \mathbb{C}} B_x^{st}(z)$. Let $w \in st_2$-P-$C\{x\}$ and fix an arbitrary $z \in \mathbb{C}$. Setting $L = st_2$-P-$\limsup_{k,l} |x_{k,l} - z|$, we aim to show $|w - z| \leq L$. For any $\varepsilon > 0$, the set $E_\varepsilon = \{(k,l) : |x_{k,l} - z| \leq L + \varepsilon\}$ has $\delta_2(E_\varepsilon) = 1$. This follows from the definition of statistical limit superior (Definition 4.3), which gives us $\delta_2(\{(k,l) : |x_{k,l} - z| > L + \varepsilon\}) = 0$, and thus its complement

E_ε has density 1. By Definition 4.4, st_2-P-$C\{x\}$ is contained in every closed half-plane that contains $x_{k,l}$ for statistically almost all indices. The closed disk of radius $L+\varepsilon$ centered at z can be expressed as an intersection of closed half-planes, each containing $x_{k,l}$ for all $(k,l) \in E_\varepsilon$. Since E_ε has density 1, and any finite intersection of density-one sets maintains density one by Lemma 4.6(i), each of these half-planes must contain w. Therefore $|w-z| \leq L+\varepsilon$ for any $\varepsilon > 0$, implying $|w-z| \leq L$.

To establish the sufficiency part, suppose $w \notin st_2$-P-$C\{x\}$, then by Definition 4.4 and Lemma 4.6(ii), there exists a closed half-plane H containing $x_{k,l}$ for statistically almost all indices with $w \notin H$. Applying the Hahn-Banach Separation Theorem (Theorem 2.3), we obtain a continuous linear functional ϕ and construct a point z_0 as in Theorem 3.1. Following the separation procedure and utilizing Lemma 4.6(iii), we establish that $|w-z_0| > st_2$-P-$\limsup_{k,l} |x_{k,l}-z_0|$, which means $w \notin B_x^{st}(z_0)$, and consequently, $w \notin \bigcap_{z\in\mathbb{C}} B_x^{st}(z)$. □

5. Conclusion

In this paper, we have established equivalence theorems for both classical and statistical Pringsheim cores of complex double sequences using the Hahn-Banach Separation Theorem. Our main contributions demonstrate that the traditional convex hull characterizations can be equivalently expressed through intersections of closed disk sets, adapting Shcherbakov's disk-based approach to the double sequence setting and providing the first rigorous disk characterizations for statistical Pringsheim cores. Our approach uses the Hahn-Banach Separation Theorem to connect convex hull and disk-based definitions.

The disk characterization approach may extend to other sequence spaces where geometric separation arguments apply. Recent developments in [16] suggests that disk-based characterizations could potentially be adapted to fuzzy number spaces, though the endograph metric structure and partial ordering of fuzzy numbers would require fundamental modifications to our separation theorem approach.

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Sami M. Hamid
Department of Mathematics & Statistics University of North Florida Jacksonville, FL 32224, USA
E-mail address: shamid@unf.edu

Richard F. Patterson
Department of Mathematics & Statistics University of North Florida Jacksonville, FL 32224, USA
E-mail address: rpatters@unf.edu