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HARMONIC UNIVALENT FUNCTIONS ASSOCIATED WITH GENERALIZED HYPERGEOMETRIC FUNCTIONS

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Abstract. The generalized hypergeometric function is used here to introduce a new class of complex valued harmonic functions which are orientation preserving and univalent in the open unit disc. Among the results presented in this paper include the coefficient bounds, distortion inequality and covering property, extreme points and certain inclusion results for this generalized class of functions.

1. Introduction

A continuous function $f = u + iv$ is a complex- valued harmonic function in a complex domain Ω if both u and v are real and harmonic in Ω. In any simply-connected domain $D \subset \Omega$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [2]).

Denote by $\mathcal{H}$ the family of functions

$$f = h + \overline{g} \tag{1.1}$$

which are harmonic, univalent and orientation preserving in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ so that f is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Thus, for $f = h + \overline{g} \in \mathcal{H}$, the functions h and g analytic $\mathcal{U}$ can be expressed in the following forms:

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ \ g(z) = \sum_{n=1}^{\infty} b_n z^n \ \ (|b_1| < 1),$$

and $f(z)$ is then given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} \quad (|b_1| < 1). \tag{1.2}$$

We note that the family $\mathcal{H}$ of orientation preserving, normalized harmonic univalent functions reduces to the well known class $\mathcal{S}$ of normalized univalent functions if the

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co-analytic part of f is identically zero, i.e. $g \equiv 0$. Also, we denote by $\overline{\mathcal{H}}$ the subfamily of $\mathcal{H}$ consisting of harmonic functions $f = h + \overline{g}$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + \overline{\sum_{n=1}^{\infty} |b_n| z^n}, \ (|b_1| < 1) \tag{1.3}$$

introduced and studied by Silverman[12].

The Hadamard product (or convolution) of two power series

$$\phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n \tag{1.4}$$

and

$$\psi(z) = z + \sum_{n=2}^{\infty} \psi_n z^n \tag{1.5}$$

in S is defined (as usual) by

$$(\phi * \psi)(z) = \phi(z) * \psi(z) = z + \sum_{n=2}^{\infty} \phi_n \psi_n z^n. \tag{1.6}$$

For positive real values of $\alpha_1, \ldots, \alpha_l$ and $\beta_1, \ldots, \beta_m$ ($\beta_j \neq 0, -1, \ldots; j = 1, 2, \ldots, m$) the *generalized hypergeometric function* ${}_lF_m(z)$ is defined by

$$\begin{aligned} {}_lF_m(z) \equiv {}_lF_m(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m; z) \quad &:= \quad \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \, \frac{z^n}{n!} \\ (l \leq m+1; \ l, m \in N_0 \quad &:= \quad N \cup \{0\}; z \in U), \end{aligned} \tag{1.7}$$

where N denotes the set of all positive integers and $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \begin{cases} 1, & n = 0 \\ a(a+1)(a+2)\ldots(a+n-1), & n \in N. \end{cases} \tag{1.8}$$

The notation ${}_lF_m$ is quite useful for representing many well-known functions such as the exponential, the Binomial, the Bessel and Laguerre polynomial. Let

$$H[\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m] : \mathcal{S} \to \mathcal{S}$$

be a linear operator defined by

$$\begin{aligned} H[\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m]\phi(z) \quad &:= \quad z \; {}_lF_m(\alpha_1, \alpha_2, \ldots \alpha_l; \beta_1, \beta_2 \ldots, \beta_m; z) * \phi(z) \\ &= \quad z + \sum_{n=2}^{\infty} \omega_n(\alpha_1; l; m) \; \phi_n z^n, \end{aligned} \tag{1.9}$$

where

$$\omega_n(\alpha_1; l; m) = \frac{(\alpha_1)_{n-1} \ldots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \ldots (\beta_m)_{n-1}} \, \frac{1}{(n-1)!}. \tag{1.10}$$

For notational simplicity, we use a shorter notation $H_m^l[\alpha_1]$ for $H[\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m]$ in the sequel.It follows from (1.9) that

$$H_0^1[1]\phi(z) = \phi(z), H_0^1[2]\phi(z) = z\phi'(z).$$

The linear operator $H_m^l[\alpha_1]$ is called Dziok-Srivastava operator (see [4])introduced by Dziok and Srivastava which was subsequently extended by Dziok and Raina [3] by using the generalized hypergeometric function , recently Srivastava etal.([11]) defined the linear operator $\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}$ as follows:

$$\mathcal{L}^{0}_{\lambda,\alpha_1}\phi(z)=\phi(z),$$

$$\mathcal{L}^{1,\alpha_1}_{\lambda,l,m}\phi(z)=(1-\lambda)H^l_m[\alpha_1]\phi(z)+\lambda(H^l_m[\alpha_1]\phi(z))'=\mathcal{L}^{\alpha_1}_{\lambda,l,m}\phi(z),(\lambda\geq 0),\quad(1.11)$$

$$\mathcal{L}^{2,\alpha_1}_{\lambda,l,m}\phi(z)=\mathcal{L}^{\alpha_1}_{\lambda,l,m}(\mathcal{L}^{1,\alpha_1}_{\lambda,l,m}\phi(z))\quad(1.12)$$

and in general ,

$$\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}\phi(z)=\mathcal{L}^{\alpha_1}_{\lambda,l,m}(\mathcal{L}^{\tau-1,\alpha_1}_{\lambda,l,m}\phi(z)),(l\leq m+1;\ l,m\in N_0=N\cup\{0\};z\in U).\quad(1.13)$$

If the function $\phi(z)$ is given by (1.4), then we see from (1.9), (1.10),(1.11) and(1.13) that

$$\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}\phi(z)\quad:=\quad z+\sum_{n=2}^{\infty}\omega^{\tau}_n(\alpha_1;\lambda;l;m)\ \phi_n\ z^n,\quad(1.14)$$

where

$$\omega^{\tau}_n(\alpha_1;\lambda;l;m)=\left(\frac{(\alpha_1)_{n-1}\ldots(\alpha_l)_{n-1}}{(\beta_1)_{n-1}\ldots(\beta_m)_{n-1}}\ \frac{[1+\lambda(n-1)]}{(n-1)!}\right)^{\tau},(n\in N\setminus\{1\},\tau\in N_0)\quad(1.15)$$

unless otherwise stated. We note that when $\tau=1$ and $\lambda=0$ the linear operator$\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}$ would reduce to the familiar Dziok-Srivastava linear operator given by (see [4]), includes (as its special cases) various other linear operators introduced and studied by Carlson and Shaffer [1], Owa [9]and Ruscheweyh [10] .

In view of the relationship (1.15) and the linear operator (1.14) for the harmonic function $f=h+\overline{g}$ given by (1.1), we define the operator

$$\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}f(z)=\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}h(z)+\overline{\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}g(z)},\quad(1.16)$$

and introduce below a new subclass $\mathcal{L}_H(\tau;\lambda;\gamma)$ of $\mathcal{H}$ in terms of the operator given by (1.16).

Let $\mathcal{L}_H(\tau;\lambda;\gamma)$ denote a subclass of $\mathcal{H}$ consisting of functions of the form $f=h+\overline{g}$ given by (1.2) satisfying the condition that

$$\frac{\partial}{\partial\theta}\left(arg\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}f(z)\right)>\gamma=\mathrm{Re}\left\{\frac{z(\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}h(z))'-\overline{z(\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}g(z))'}}{\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}h(z)+\overline{(\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}g(z))}}\right\}\geq\gamma,\quad(1.17)$$

$$\left(z=re^{i\theta};0\leq\theta<2\pi;0\leq r<1;0\leq\gamma<1;z\in\mathbb{U}\right)$$

where $\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}f(z)$ is given by (1.16). We also let $\mathcal{L}_{\overline{H}}(\tau;\lambda;\gamma)=\mathcal{L}_H(\tau;\lambda;\gamma)\cap\overline{\mathcal{H}}$.

In this paper, we obtain coefficient conditions for the classes $\mathcal{L}_H(\tau;\lambda;\gamma)$ and $\mathcal{L}_{\overline{H}}(\tau;\lambda;\gamma)$. A representation theorem, inclusion properties and distortion bounds for the class $\mathcal{L}_{\overline{H}}(\tau;\lambda;\gamma)$ are also established.

2. Coefficient bounds

Due to Jahangiri [7], we state the following sufficient coefficient bound for the harmonic functions $f\in\mathcal{H}(\gamma)$ the class of harmonic starlike functions of order $\gamma,(0\leq\gamma<1)$ a subclass of $\mathcal{H}$ consisting of functions of the form $f=h+\overline{g}$ given by (1.2) satisfying the condition that $\frac{\partial}{\partial\theta}(argf(z))>\gamma$.

Theorem 2.1. [7]. *Let $f = h + \overline{g}$ be given by (1.2). If*

$$\sum_{n=1}^{\infty}\left[\frac{n-\gamma}{1-\gamma}|a_n| + \frac{n+\gamma}{1-\gamma}|b_n|\right] \leq 2, \tag{2.1}$$

where $a_1 = 1$ and $0 \leq \gamma < 1$, then $f \in \mathcal{H}(\gamma)$.

The following result gives a sufficient coefficient condition for the harmonic functions $f \in \mathcal{L}_H(\tau;\lambda;\gamma)$.

Theorem 2.2. *Let $f = h + \overline{g}$ be given by (1.2). If*

$$\sum_{n=1}^{\infty}\left[\frac{n-\gamma}{1-\gamma}|a_n| + \frac{n+\gamma}{1-\gamma}|b_n|\right]\omega_n^{\tau}(\alpha_1;\lambda;l;m) \leq 2, \tag{2.2}$$

where $a_1 = 1$ and $0 \leq \gamma < 1$, then $f \in \mathcal{L}_H(\tau;\lambda;\gamma)$.

Proof. Since $n \leq min\{\frac{n-\gamma}{1-\gamma}\omega_n^{\tau}(\alpha_1;\lambda;l;m);\ \frac{n+\gamma}{1-\gamma}\omega_n^{\tau}(\alpha_1;\lambda;l;m)\}$, it follows from Theorem2.1 that $f \in \mathcal{H}(\gamma)$ and hence f is harmonic, orientation preserving and univalent in $\mathbb{U}$. Suppose the condition (2.2) holds true. To show that $f \in \mathcal{L}_H(\tau;\lambda;\gamma)$, we show (in view of (1.17)) that

$$\Re\left\{\frac{z\left(\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}h(z)\right)' - \overline{z\left(\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}g(z)\right)'}}{\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}h(z) + \overline{\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}g(z)}}\right\} = \Re\left\{\frac{A(z)}{B(z)}\right\} \geq \gamma \quad (z \in \mathbb{U}),$$

where

$$A(z) = z(\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}h(z))' - \overline{z(\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}g(z))'} = z + \sum_{n=2}^{\infty} n\omega_n^{\tau}(\alpha_1;\lambda;l;m)a_n z^n - \sum_{n=1}^{\infty} n\omega_n^{\tau}(\alpha_1;\lambda;l;m)\overline{b}_n\overline{z}^n$$

and

$$B(z) = z + \sum_{n=2}^{\infty}\omega_n^{\tau}(\alpha_1;\lambda;l;m)a_n z^n + \sum_{n=1}^{\infty}\omega_n^{\tau}(\alpha_1;\lambda;l;m)\overline{b}_n\overline{z}^n.$$

Using the fact that Re $\{w\} \geq \gamma$ if and only if $|1-\gamma+w| \geq |1+\gamma-w|$, it suffices to show that

$$|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)| \geq 0. \tag{2.3}$$

Substituting for $A(z)$ and $B(z)$ in (2.3), and performing elementary calculations, we find that

$$\begin{aligned} &|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)| \\ &\geq 2(1-\gamma)|z|\left\{2 - \sum_{n=1}^{\infty}\left[\frac{n-\gamma}{1-\gamma}|a_n| + \frac{n+\gamma}{1-\gamma}|b_n|\right]\omega_n^{\tau}(\alpha_1;\lambda;l;m)|z|^{n-1}\right\} \\ &> 2(1-\gamma)\left\{2 - \sum_{n=1}^{\infty}\left[\frac{n-\gamma}{1-\gamma}|a_n| + \frac{n+\gamma}{1-\gamma}|b_n|\right]\omega_n^{\tau}(\alpha_1;\lambda;l;m)\right\} \geq 0, \end{aligned}$$

which implies that $f(z) \in \mathcal{L}_H(\tau;\lambda;\gamma)$. □

The harmonic function

$$f(z) = z + \sum_{n=2}^{\infty} \frac{1-\gamma}{(n-\gamma)\omega_n^{\tau}(\alpha_1;\lambda;l;m)} x_n z^n + \sum_{n=1}^{\infty} \frac{1-\gamma}{(n+\gamma)\omega_n^{\tau}(\alpha_1;\lambda;l;m)} \overline{y}_n (\overline{z})^n, \tag{2.4}$$

where $\sum\limits_{n=2}^{\infty} |x_n| + \sum\limits_{n=1}^{\infty} |y_n| = 1$ shows that the coefficient bound given by (2.2) is sharp.

The functions of the form (2.4) are in $\mathcal{L}_H(\tau;\lambda;\gamma)$ because

$$\begin{aligned}&\sum_{n=1}^{\infty} \left(\frac{(n-\gamma)\omega_n^{\tau}(\alpha_1;\lambda;l;m)}{1-\gamma} |a_n| + \frac{(n+\gamma)\omega_n^{\tau}(\alpha_1;\lambda;l;m)}{1-\gamma} |b_n| \right)\\ =&1 + \sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 2.\end{aligned}$$

Our next theorem gives a necessary and sufficient condition for functions of the form (1.3) to be in the class $\mathcal{L}_{\overline{H}}(\tau;\lambda;\gamma)$.

Theorem 2.3. *For $a_1 = 1$ and $0 \le \gamma < 1$, $f = h + \overline{g} \in \mathcal{L}_{\overline{H}}(\tau;\lambda;\gamma)$ if and only if*

$$\sum_{n=1}^{\infty} \left[\frac{n-\gamma}{1-\gamma} |a_n| + \frac{n+\gamma}{1-\gamma} |b_n| \right] \omega_n^{\tau}(\alpha_1;\lambda;l;m) \le 2. \tag{2.5}$$

Proof. Since $\mathcal{L}_{\overline{H}}(\tau;\lambda;\gamma) \subset \mathcal{L}_H(\tau;\lambda;\gamma)$, we only need to prove the "only if" part of the theorem. To this end, for functions f of the form (1.3), we notice that the condition

$$\Re \left\{ \frac{z(\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1} h'(z)) - \overline{z(\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1} g'(z))}}{\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1} h(z) + \overline{(\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1} g(z))}} \right\} \ge \gamma$$

implies that

$$\Re \left\{ \frac{(1-\gamma)z - \sum\limits_{n=2}^{\infty} (n-\gamma)\omega_n^{\tau}(\alpha_1;\lambda;l;m) a_n z^n - \sum\limits_{n=1}^{\infty} (n+\gamma)\omega_n^{\tau}(\alpha_1;\lambda;l;m) \overline{b}_n \overline{z}^n}{z - \sum\limits_{n=2}^{\infty} \omega_n^{\tau}(\alpha_1;\lambda;l;m) a_n z^n + \sum\limits_{n=1}^{\infty} \omega_n^{\tau}(\alpha_1;\lambda;l;m) \overline{b}_n \overline{z}^n} \right\} \ge 0.$$

The above required condition must hold for all values of z in $\mathbb{U}$. Upon choosing the values of z on the positive real axis where $0 \le z = r < 1$, we must have

$$\frac{(1-\gamma) - \sum\limits_{n=2}^{\infty} (n-\gamma)\omega_n^{\tau}(\alpha_1;\lambda;l;m) a_n r^{n-1} - \sum\limits_{n=1}^{\infty} (n+\gamma)\omega_n^{\tau}(\alpha_1;\lambda;l;m) b_n r^{n-1}}{1 - \sum\limits_{n=2}^{\infty} \omega_n^{\tau}(\alpha_1;\lambda;l;m) a_n r^{n-1} + \sum\limits_{n=1}^{\infty} \omega_n^{\tau}(\alpha_1;\lambda;l;m) b_n r^{n-1}} \ge 0. \tag{2.6}$$

If the condition (2.5) does not hold, then the numerator in (2.6) is negative for r sufficiently close to 1. Hence, there exist $z_0 = r_0$ in (0,1) for which the quotient of (2.6) is negative. This contradicts the required condition for $f(z) \in \mathcal{L}_{\overline{H}}(\tau;\lambda;\gamma)$. This completes the proof of the theorem. □

3. Distortion bounds and Extreme Points

By applying the condition (2.5) and employing similar steps of derivation as given in [5, 6, 8, 7], we state the following results without proof.

Theorem 3.1. (**Distortion bounds**) *Let $f \in \mathcal{L}_{\overline{H}}(\tau;\lambda;\gamma)$, then (for $|z| = r < 1$)*

$$(1-b_1)r - \frac{1}{\omega_2^\tau(\alpha_1;\lambda;l;m)}\left(\frac{1-\gamma}{2-\gamma} - \frac{1+\gamma}{2-\gamma}b_1\right)r^2 \leq |f(z)|$$
$$\leq (1+b_1)r + \frac{1}{\omega_2^\tau(\alpha_1;\lambda;l;m)}\left(\frac{1-\gamma}{2-\gamma} - \frac{1+\gamma}{2-\gamma}b_1\right)r^2.$$

Corollary 3.2. *If $f(z) \in \mathcal{L}_{\overline{H}}(\tau;\lambda;\gamma)$, then*

$$\left\{w : |w| < \frac{2\omega_2^\tau(\alpha_1;\lambda;l;m) - 1 - [\omega_2^\tau(\alpha_1;\lambda;l;m)-1]\gamma}{(2-\gamma)\omega_2^\tau(\alpha_1;\lambda;l;m)} - \frac{2\omega_2^\tau(\alpha_1;\lambda;l;m) - 1 - [\omega_2^\tau(\alpha_1;\lambda;l;m)+1]\gamma}{(2-\gamma)\omega_2^\tau(\alpha_1;\lambda;l;m)}b_1\right\} \subset f(U).$$

The extreme points of closed convex hulls of $\mathcal{L}_{\overline{H}}(\tau;\lambda;\gamma)$ denoted by $clco\mathcal{L}_{\overline{H}}(\tau;\lambda;\gamma)$.

Theorem 3.3. (**Extreme Points**) *A function $f(z) \in clco\mathcal{L}_{\overline{H}}(\tau;\lambda;\gamma)$ if and only if $f(z) = \sum\limits_{n=1}^{\infty}\left[X_n h_n(z) + Y_n g_n(z)\right]$, where*

$$h_1(z) = z, h_n(z) = z - \frac{1-\gamma}{(n-\gamma)\omega_n^\tau(\alpha_1;\lambda;l;m)}z^n \ (n \geq 2), \ \ g_n(z) = z + \frac{1-\gamma}{(n+\gamma)\omega_n^\tau(\alpha_1;\lambda;l;m)}\overline{z}^n \ (n \geq 2),$$

$$\sum_{n=1}^{\infty}(X_n + Y_n) = 1, \quad X_n \geq 0 \text{ and } \ Y_n \geq 0.$$

In particular, the extreme points of $\mathcal{L}_{\overline{H}}(\tau;\lambda;\gamma)$ are $\{h_n\}$ and $\{g_n\}$.

4. Inclusion Results

The following result gives the convex combinations of the class $\mathcal{L}_{\overline{H}}(\tau;\lambda;\gamma)$.

Theorem 4.1. *The family $\mathcal{L}_{\overline{H}}(\tau;\lambda;\gamma)$ is closed under convex combinations.*

Proof. Let $f_i \in \mathcal{L}_{\overline{H}}(\tau;\lambda;\gamma)$ $(i = 1, 2, \dots)$,where

$$f_i(z) = z - \sum_{n=2}^{\infty}|a_{i,n}|z^n + \sum_{n=2}^{\infty}|b_{i,n}|\overline{z}^n.$$

The convex combination of f_i may be written as

$$\sum_{i=1}^{\infty}t_i f_i(z) = z - \sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty}t_i|a_{i,n}|\right)z^n + \sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty}t_i|b_{i,n}|\right)\overline{z}^n,$$

provided that $\sum\limits_{i=1}^{\infty}t_i = 1$ $(0 \leq t_i \leq 1)$.

Applying the inequality (2.5) of Theorem 2.3, we obtain

$$\sum_{n=2}^{\infty}\frac{(n-\gamma)\omega_n^\tau(\alpha_1;\lambda;l;m)}{1-\gamma}\left(\sum_{i=1}^{\infty}t_i|a_{i,n}|\right) + \sum_{n=1}^{\infty}\frac{(n+\gamma)\omega_n^\tau(\alpha_1;\lambda;l;m)}{1-\gamma}\left(\sum_{i=1}^{\infty}t_i|b_{i,n}|\right)$$
$$= \sum_{i=1}^{\infty}t_i\left(\sum_{n=2}^{\infty}\frac{(n-\gamma)\omega_n^\tau(\alpha_1;\lambda;l;m)}{1-\gamma}|a_{i,n}| + \sum_{n=1}^{\infty}\frac{(n+\gamma)\omega_n^\tau(\alpha_1;\lambda;l;m)}{1-\gamma}|b_{i,n}|\right)$$
$$\leq \sum_{i=1}^{\infty}t_i = 1,$$

and therefore, $\sum\limits_{i=1}^{\infty}t_i f_i \in \mathcal{L}_{\overline{H}}(\tau;\lambda;\gamma)$. ∎

Theorem 4.2. *Let* $f(z) \in \mathcal{L}_{\overline{H}}(\tau;\lambda;\gamma)$ *and* $F(z) \in \mathcal{L}_{\overline{H}}(\tau;\lambda;\delta)$, where $0 \leq \delta \leq \gamma < 1$, *then*
$f(z) * F(z) \in \mathcal{L}_{\overline{H}}(\tau;\lambda;\gamma) \subset \mathcal{L}_{\overline{H}}(\tau;\lambda;\delta)$.

Proof. Let $f(z) = z - \sum\limits_{n=2}^{\infty} |a_n| z^n + \sum\limits_{n=1}^{\infty} |b_n| \overline{z}^n \in \mathcal{L}_{\overline{H}}(\tau;\lambda;\gamma)$ and $F(z) = z - \sum\limits_{n=2}^{\infty} |A_n| z^n + \sum\limits_{n=1}^{\infty} |B_n| \overline{z}^n \in \mathcal{L}_{\overline{H}}(\tau;\lambda;\delta)$, then

$$f(z) * F(z) = z - \sum_{n=2}^{\infty} |a_n||A_n| z^n + \sum_{n=1}^{\infty} |b_n||B_n| \overline{z}^n.$$

From the assertion that $f(z) * F(z) \in \mathcal{L}_{\overline{H}}(\tau;\lambda;\delta)$, we note that $|A_n| \leq 1$ and $|B_n| \leq 1$. In view of Theorem 2.3 and the inequality $0 \leq \delta \leq \gamma < 1$, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{(n-\delta)\omega_n^{\tau}(\alpha_1;\lambda;l;m)}{1-\delta} |a_n|\ |A_n| + \sum_{n=1}^{\infty} \frac{(n+\delta)\omega_n^{\tau}(\alpha_1;\lambda;l;m)}{1-\delta} |b_n|\ |B_n| \\
\leq\ & \sum_{n=2}^{\infty} \frac{(n-\delta)\omega_n^{\tau}(\alpha_1;\lambda;l;m)}{1-\delta} |a_n| + \sum_{n=1}^{\infty} \frac{(n+\delta)\omega_n^{\tau}(\alpha_1;\lambda;l;m)}{1-\delta} |b_n| \\
\leq\ & \sum_{n=2}^{\infty} \frac{(n-\gamma)\omega_n^{\tau}(\alpha_1;\lambda;l;m)}{1-\gamma} |a_n| + \sum_{n=1}^{\infty} \frac{(n+\gamma)\omega_n^{\tau}(\alpha_1;\lambda;l;m)}{1-\gamma} |b_n| \leq 1,
\end{aligned}$$

by Theorem 2.3, $f(z) \in \mathcal{L}_{\overline{H}}(\tau;\lambda;\gamma)$. Hence $f(z) * F(z) \in \mathcal{L}_{\overline{H}}(\tau;\lambda;\gamma) \subset \mathcal{L}_{\overline{H}}(\tau;\lambda;\delta)$. □

Lastly, we consider the closure property of the class $\mathcal{L}_{\overline{H}}(\tau;\lambda;\gamma)$ under the generalized Bernardi-Libera -Livingston integral operator$\mathcal{L}_c[f(z)]$ which is defined by

$$\mathcal{L}_c[f(z)] = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)dt \ (c > -1).$$

We prove the following result.

Theorem 4.3. *Let* $f(z) \in \mathcal{L}_{\overline{H}}(\tau;\lambda;\gamma)$, *then* $\mathcal{L}_c[f(z)] \in \mathcal{L}_{\overline{H}}(\tau;\lambda;\gamma)$

Proof. Using (1.1) and (1.3), we get

$$\mathcal{L}_c[f(z)] = \frac{c+1}{z^c} \int_0^z t^{c-1} h(t)dt + \overline{\frac{c+1}{z^c} \int_0^z t^{c-1} g(t)}dt.$$

$$\begin{aligned}
=\ & \frac{c+1}{z^c} \int_0^z t^{c-1} \left(t - \sum_{n=2}^{\infty} |a_n| t^n \right) dt + \overline{\frac{c+1}{z^c} \int_0^z t^{c-1} \left(\sum_{n=1}^{\infty} |b_n| t^n \right) dt} \\
& = z - \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} B_n z^n,
\end{aligned}$$

where $A_n = \frac{c+1}{c+n}\ |a_n|$ and $B_n = \frac{c+1}{c+n}\ |b_n|$. Hence

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left[\frac{n-\gamma}{1-\gamma}\left(\frac{c+1}{c+n}|a_n|\right)+\frac{n+\gamma}{1-\gamma}\left(\frac{c+1}{c+n}|b_n|\right)\right]\omega_n^{\tau}(\alpha_1;\lambda;l;m)\\
\leq\ &\sum_{n=1}^{\infty}\left[\frac{n-\gamma}{1-\gamma}|a_n|+\frac{n+\gamma}{1-\gamma}|b_n|\right]\omega_n^{\tau}(\alpha_1;\lambda;l;m)\\
\leq\ &2,
\end{aligned}$$

since $f(z) \in \mathcal{L}_{\overline{H}}(\tau;\lambda;\gamma)$, therefore by Theorem 2.3, $\mathcal{L}_c(f(z)) \in \mathcal{L}_{\overline{H}}(\tau;\lambda;\gamma)$. □

Concluding Remarks: By choosing $\tau = 1$ and $\lambda = 0$ the various results presented in this paper would provide interesting extensions and generalizations of those considered earlier for simpler harmonic function classes (see [5, 6, 8, 12]). The details involved in the derivations of such specializations of the results presented in this paper are fairly straight- forward.

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