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NEW GENERALISATIONS OF GRUSS INEQUALITY USING RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS

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Abstract. In this paper, we use the Riemann-Liouville fractional integrals to establish some new integral inequalities of Gruss type. We give two main results; the first one deals with some inequalities using one fractional parameter. The second result concerns others inequalities using two fractional parameters.

1. Introduction

In 1935, G. Gruss [3] proved the well known inequality:

$$\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\left(\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right) \le \frac{(M-m)(P-p)}{4} \tag{1.1}$$

provided that f and g are two integrable functions on $[a,b]$ and satisfying the conditions

$$m\le f(x)\le M,\ \ p\le g(x)\le P;\quad m,M,p,P\in\mathbb{R},x\in[a,b]. \tag{1.2}$$

The inequality (1.1) has evoked the interest of many researchers and numerous generalizations, variants and extensions have appeared in the literature, to mention a few, see [1, 4, 5, 6, 7] and the references cited therein.
The main aim of this paper is to establish some new generalizations for (1.1) by using the Riemann-Liouville fractional integrals. We give two main results; the first one deals with some inequalities using one fractional parameter. The second result concerns another class of inequalities using two fractional parameters.

2. Basic Definitions of the Fractional Calculus

Definition 1. A real valued function $f(t), t\ge 0$ is said to be in the space $C_\mu, \mu\in\mathbb{R}$ if there exists a real number $p>\mu$ such that $f(t)=t^p f_1(t)$, where $f_1(t)\in C([0,\infty[)$.

Definition 2. A function $f(t), t\ge 0$ is said to be in the space $C_\mu^n, \mu\in\mathbb{R}$, if

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$f^{(n)} \in C_\mu$.

Definition 3. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a function $f \in C_\mu, (\mu \geq -1)$ is defined as

$$\begin{array}{c} J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau; \quad \alpha > 0, t > 0, \\ J^0 f(t) = f(t), \end{array} \tag{2.1}$$

where $\Gamma(\alpha) := \int_0^\infty e^{-u} u^{\alpha-1} du$.

For the convenience of establishing the results, we give the semigroup property:

$$J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t), \alpha \geq 0, \beta \geq 0, \tag{2.2}$$

which implies the commutative property

$$J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t). \tag{2.3}$$

More details, one can consult [2].

3. Main Results

Theorem 3.1. *Let f and g be two integrable functions on $[0, \infty[$ satisfying the condition (1.2) on $[0, \infty.[$ Then for all $t > 0, \alpha > 0$, we have:*

$$\left| \frac{t^\alpha}{\Gamma(\alpha+1)} J^\alpha fg(t) - J^\alpha f(t) J^\alpha g(t) \right| \leq \Big(\frac{t^\alpha}{2\Gamma(\alpha+1)} \Big)^2 (M-m)(P-p). \tag{3.1}$$

We need the following lemma

Lemma 3.2. *Let u be an integrable function on $[0, \infty[$ satisfying the condition (1.2) on $[0, \infty[$. Then for all $t > 0, \alpha > 0$, we have:*

$$\begin{array}{c} \frac{t^\alpha}{\Gamma(\alpha+1)} J^\alpha u^2(t) - \Big(J^\alpha u(t) \Big)^2 \\ = \Big(M \frac{t^\alpha}{\Gamma(\alpha+1)} - J^\alpha u(t) \Big) \Big(J^\alpha u(t) - m \frac{t^\alpha}{\Gamma(\alpha+1)} \Big) \\ - \frac{t^\alpha}{\Gamma(\alpha+1)} J^\alpha (M - u(t))(u(t) - m). \end{array} \tag{3.2}$$

Proof. Let u be an integrable function on $[0, \infty[$ satisfying the condition (1.2) on $[0, \infty[$. For any $\tau, \rho \in [0, \infty[$, we have

$$\begin{array}{c} \Big(M - u(\rho) \Big) \Big(u(\tau) - m \Big) + \Big(M - u(\tau) \Big) \Big(u(\rho) - m \Big) \\ - \Big(M - u(\tau) \Big) \Big(u(\tau) - m \Big) - \Big(M - u(\rho) \Big) \Big(u(\rho) - m \Big) \\ = u^2(\tau) + u^2(\rho) - 2u(\tau)u(\rho). \end{array} \tag{3.3}$$

Multiplying (3.3) by $\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}$; $\tau \in (0,t), t>0$ and integrating the resulting identity with respect to τ from 0 to t, we get

$$\begin{aligned}
&\Big(M-u(\rho)\Big)\Big(J^\alpha u(t)-m\frac{t^\alpha}{\Gamma(\alpha+1)}\Big)+\Big(M\frac{t^\alpha}{\Gamma(\alpha+1)}-J^\alpha u(t)\Big)\Big(u(\rho)-m\Big)\\
&-J^\alpha\Big((M-u(t))(u(t)-m)\Big)-\Big(M-u(\rho)\Big)\Big(u(\rho)-m\Big)\tfrac{t^\alpha}{\Gamma(\alpha+1)}\\
&=J^\alpha u^2(t)+u^2(\rho)\tfrac{t^\alpha}{\Gamma(\alpha+1)}-2u(\rho)J^\alpha u(t).
\end{aligned} \tag{3.4}$$

Now, multiplying (3.4) by $\frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}$; $\rho \in (0,t)$ and integrating the resulting identity with respect to ρ from 0 to t, we have

$$\begin{aligned}
&\Big(J^\alpha u(t)-m\frac{t^\alpha}{\Gamma(\alpha+1)}\Big)\frac{1}{\Gamma(\alpha)}\int_0^t(t-\rho)^{\alpha-1}(M-u(\rho))d\rho\\
&+\Big(M\tfrac{t^\alpha}{\Gamma(\alpha+1)}-J^\alpha u(t)\Big)\tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t(t-\rho)^{\alpha-1}(u(\rho)-m)d\rho\\
&-J^\alpha\Big((M-u(t))(u(t)-m)\Big)\tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t(t-\rho)^{\alpha-1}d\rho\\
&-\tfrac{t^\alpha}{\Gamma(\alpha+1)}\tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t(t-\rho)^{\alpha-1}(M-u(\rho))(u(\rho)-m)d\rho\\
&=\tfrac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha u^2(t)+J^\alpha u^2(t)\tfrac{t^\alpha}{\Gamma(\alpha+1)}-2J^\alpha u(t)J^\alpha u(t)
\end{aligned} \tag{3.5}$$

which gives (3.2) and proves the lemma.

□

Proof of Theorem 3.1. Let f and g be two functions satisfying the conditions of Theorem 3.1.
Define

$$H(\tau,\rho):=(f(\tau)-f(\rho))(g(\tau)-g(\rho));\ \tau,\rho\in(0,t),t>0. \tag{3.6}$$

Then, multiplying (3.6) by $\frac{(t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}}{\Gamma^2(\alpha)}$; $\tau,\rho\in(0,t)$ and integrating with respect to τ and ρ over $(0,t)^2$, we can state that

$$\begin{aligned}
&\frac{1}{\Gamma^2(\alpha)}\int_0^t\int_0^t(t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}H(\tau,\rho)d\tau d\rho\\
&=2\tfrac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha fg(t)-2J^\alpha f(t)J^\alpha g(t).
\end{aligned} \tag{3.7}$$

Applying Cauchy Schwarz inequality, we have

$$\begin{aligned}
&\Big(\frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha fg(t)-J^\alpha f(t)J^\alpha g(t)\Big)^2\\
&\le\Big(\tfrac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha f^2(t)-(J^\alpha f(t))^2\Big)\Big(\tfrac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha g^2(t)-(J^\alpha g(t))^2\Big).
\end{aligned} \tag{3.8}$$

Since $(M-f(x))(f(x)-m)\ge 0$ and $(P-g(x))(g(x)-p)\ge 0$, we have

$$\frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha(M-f(t))(f(t)-m)\ge 0 \tag{3.9}$$

and

$$\frac{t^\alpha}{\Gamma(\alpha+1)} J^\alpha (P-g(t))(g(t)-p) \geq 0. \tag{3.10}$$

Therefore

$$\begin{aligned} &\frac{t^\alpha}{\Gamma(\alpha+1)} J^\alpha f^2(t) - \Big(J^\alpha f(t)\Big)^2 \\ \leq &\Big(M\tfrac{t^\alpha}{\Gamma(\alpha+1)} - J^\alpha f(t)\Big)\Big(J^\alpha f(t) - m\tfrac{t^\alpha}{\Gamma(\alpha+1)}\Big) \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} &\frac{t^\alpha}{\Gamma(\alpha+1)} J^\alpha g^2(t) - \Big(J^\alpha g(t)\Big)^2 \\ \leq &\Big(P\tfrac{t^\alpha}{\Gamma(\alpha+1)} - J^\alpha g(t)\Big)\Big(J^\alpha g(t) - p\tfrac{t^\alpha}{\Gamma(\alpha+1)}\Big). \end{aligned} \tag{3.12}$$

By Lemma 3.2 and the inequalities (3.8), (3.11), (3.12), we deduce that

$$\begin{aligned} &\Big(\frac{t^\alpha}{\Gamma(\alpha+1)} J^\alpha fg(t) - 2J^\alpha f(t) J^\alpha g(t)\Big)^2 \\ \leq &\Big(M\tfrac{t^\alpha}{\Gamma(\alpha+1)} - J^\alpha f(t)\Big)\Big(J^\alpha f(t) - m\tfrac{t^\alpha}{\Gamma(\alpha+1)}\Big)\Big(P\tfrac{t^\alpha}{\Gamma(\alpha+1)} - J^\alpha g(t)\Big)\Big(J^\alpha g(t) - p\tfrac{t^\alpha}{\Gamma(\alpha+1)}\Big). \end{aligned} \tag{3.13}$$

Now using the elementary inequality $4rs \leq (r+s)^2, r, s \in \mathbb{R}$, we can state that

$$4\Big(M\frac{t^\alpha}{\Gamma(\alpha+1)} - J^\alpha f(t)\Big)\Big(J^\alpha f(t) - m\frac{t^\alpha}{\Gamma(\alpha+1)}\Big) \leq \Big(\frac{t^\alpha}{\Gamma(\alpha+1)}(M-m)\Big)^2 \tag{3.14}$$

and

$$4\Big(P\frac{t^\alpha}{\Gamma(\alpha+1)} - J^\alpha g(t)\Big)\Big(J^\alpha g(t) - p\frac{t^\alpha}{\Gamma(\alpha+1)}\Big) \leq \Big(\frac{t^\alpha}{\Gamma(\alpha+1)}(P-p)\Big)^2. \tag{3.15}$$

Using (3.13), (3.14) and (3.15) we get (3.1). □

Remark. Applying Theorem 3.1 for $\alpha = 1$, we obtain the inequality (1.1) on $[0, t]$.

Our next result is the following theorem, in which we use two real positive parameters.

Theorem 3.3. *Let f and g be two integrable functions on $[0, \infty[$ satisfying the condition (1.2) on $[0, \infty[$. Then for all $t > 0, \alpha > 0, \beta > 0$, we have:*

$$\Big(\frac{t^\alpha}{\Gamma(\alpha+1)}J^\beta fg(t)+\frac{t^\beta}{\Gamma(\beta+1)}J^\alpha fg(t)-J^\alpha f(t)J^\beta g(t)-J^\beta f(t)J^\alpha g(t)\Big)^2$$

$$\le\Big[\Big(M\tfrac{t^\alpha}{\Gamma(\alpha+1)}-J^\alpha f(t)\Big)\Big(J^\beta f(t)-m\tfrac{t^\beta}{\Gamma(\beta+1)}\Big)+\Big(J^\alpha f(t)-m\tfrac{t^\alpha}{\Gamma(\alpha+1)}\Big)\Big(M\tfrac{t^\beta}{\Gamma(\beta+1)}-J^\beta f(t)\Big)\Big]$$

$$\times\Big[\Big(P\tfrac{t^\alpha}{\Gamma(\alpha+1)}-J^\alpha g(t)\Big)\Big(J^\beta g(t)-p\tfrac{t^\beta}{\Gamma(\beta+1)}\Big)+\Big(J^\alpha g(t)-p\tfrac{t^\alpha}{\Gamma(\alpha+1)}\Big)\Big(P\tfrac{t^\beta}{\Gamma(\beta+1)}-J^\beta g(t)\Big)\Big]. \tag{3.16}$$

To prove Theorem 3.3 we need the following lemmas:

Lemma 3.4. *Let f and g be two integrable functions on $[0,\infty.[$ Then for all $t>0,\alpha>0,\beta>0$, we have:*

$$\Big(\frac{t^\alpha}{\Gamma(\alpha+1)}J^\beta fg(t)+\frac{t^\beta}{\Gamma(\beta+1)}J^\alpha fg(t)-J^\alpha f(t)J^\beta g(t)-J^\beta f(t)J^\alpha g(t)\Big)^2$$

$$\le\Big(\tfrac{t^\alpha}{\Gamma(\alpha+1)}J^\beta f^2(t)+\tfrac{t^\beta}{\Gamma(\beta+1)}J^\alpha f^2(t)-2J^\alpha f(t)J^\beta f(t)\Big)$$

$$\times\Big(\tfrac{t^\alpha}{\Gamma(\alpha+1)}J^\beta g^2(t)+\tfrac{t^\beta}{\Gamma(\beta+1)}J^\alpha g^2(t)-2J^\alpha g(t)J^\beta g(t)\Big). \tag{3.17}$$

Proof. Multiplying (3.6) by $\frac{(t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}$; $\tau,\rho\in(0,t)$, integrating with respect to τ and ρ over $(0,t)^2$, then applying the Cauchy-Schwarz inequality for double integrals, we obtain (3.17). □

Lemma 3.5. *Let u be an integrable function on $[0,\infty[$ satisfying the condition (1.2) on $[0,\infty[$. Then for all $t>0,\alpha>0,\beta>0$, we have:*

$$\frac{t^\alpha}{\Gamma(\alpha+1)}J^\beta u^2(t)+\frac{t^\beta}{\Gamma(\beta+1)}J^\alpha u^2(t)-2J^\alpha u(t)J^\beta u(t)$$

$$=\Big(M\tfrac{t^\alpha}{\Gamma(\alpha+1)}-J^\alpha u(t)\Big)\Big(J^\beta u(t)-m\tfrac{t^\beta}{\Gamma(\beta+1)}\Big)+\Big(M\tfrac{t^\beta}{\Gamma(\alpha+1)}-J^\beta u(t)\Big)\Big(J^\alpha u(t)-m\tfrac{t^\alpha}{\Gamma(\alpha+1)}\Big)$$

$$-\tfrac{t^\alpha}{\Gamma(\alpha+1)}J^\beta(M-u(t))(u(t)-m)-\tfrac{t^\beta}{\Gamma(\beta+1)}J^\alpha(M-u(t))(u(t)-m). \tag{3.18}$$

Proof. Multiplying (3.4) by $\frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)}$; $\rho \in (0,t)$ and integrating the resulting identity with respect to ρ from 0 to t, we have

$$\begin{aligned}
&\Big(J^{\alpha}u(t) - m\frac{t^{\alpha}}{\Gamma(\alpha+1)}\Big)\frac{1}{\Gamma(\beta)}\int_0^t (t-\rho)^{\beta-1}(M-u(\rho))d\rho \\
&+\Big(M\tfrac{t^{\alpha}}{\Gamma(\alpha+1)} - J^{\alpha}u(t)\Big)\tfrac{1}{\Gamma(\beta)}\int_0^t (t-\rho)^{\beta-1}(u(\rho)-m)d\rho \\
&-J^{\alpha}\Big((M-u(t))(u(t)-m)\Big)\tfrac{1}{\Gamma(\beta)}\int_0^t (t-\rho)^{\beta-1}d\rho \\
&-\tfrac{t^{\alpha}}{\Gamma(\alpha+1)}\tfrac{1}{\Gamma(\beta)}\int_0^t (t-\rho)^{\beta-1}(M-u(\rho))(u(\rho)-m)d\rho \\
&= \tfrac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\beta}u^2(t) + J^{\beta}u^2(t)\tfrac{t^{\alpha}}{\Gamma(\alpha+1)} - 2J^{\alpha}u(t)J^{\beta}u(t).
\end{aligned} \tag{3.19}$$

Lemma 3.5 is thus proved. □

Proof of Theorem 3.3. Since $(M-f(x))(f(x)-m) \geq 0$ and $(P-g(x))(g(x)-p) \geq 0$, then can write

$$-\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\beta}(M-f(t))(f(t)-m) - \frac{t^{\beta}}{\Gamma(\beta+1)}J^{\alpha}(M-f(t))(f(t)-m) \leq 0 \tag{3.20}$$

and

$$-\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\beta}(P-g(t))(g(t)-p) - \frac{t^{\beta}}{\Gamma(\beta+1)}J^{\alpha}(P-g(t))(g(t)-p) \leq 0. \tag{3.21}$$

Applying Lemma 3.5 to f and g, then using Lemma 3.4 and the formulas (3.20), (3.21), we obtain (3.16). □

Remark. (i) Applying Theorem 3.3 for $\alpha = \beta$ we obtain Theorem 3.1.
(ii) Applying Theorem 3.3 for $\alpha = \beta = 1$, we obtain the inequality (1.1) on $[0,t]$.

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