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GROWTH OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS IN THE UNIT DISC

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Abstract. In this paper, we investigate the growth of solutions of higher order linear differential equations with analytic coefficients in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. We obtain four results which are similar to those in the complex plane.

1. Introduction and statement of results

The growth of solutions of the complex linear differential equation

$$f^{(n)} + a_{n-1}(z) f^{(n-1)} + ... + a_1(z) f^{'} + a_0(z) f = 0 \tag{1.1}$$

is well understood when the coefficients a_j are polynomials [16, 17, 22, 27] or of finite (iterated) order of growth in the complex plane [7, 14, 15, 21]. In particular, Wittich [27] showed that all solutions of (1.1) are entire functions of finite order if and only if all coefficients are polynomials, and Gundersen, Steinbart and Wang [17] listed all possible orders of solutions of (1.1) in terms of the degrees of the polynomial coefficients.

Recently, there has been an increasing interest in studying the growth of analytic solutions of linear differential equations in the unit disc Δ by making use of Nevanlinna theory. The analysis of slowly growing solutions have been studied in [13, 19, 25]. Fast growth of solutions are considered by [8, 9, 11, 12, 19, 20].

In this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's theory on the complex plane and in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ (see [18, 19, 24, 26, 28, 29]).

We need to give some definitions and discussions. Firstly, let us give definition about the degree of small growth order of functions in Δ as polynomials on the complex plane $\mathbb{C}$. There are many types of definitions of small growth order of functions in Δ (i.e., see [12, 13]).

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Definition 1.1. *For a meromorphic function f in Δ let*

$$D(f) := \lim_{r\to 1^-} \sup \frac{T(r,\ f)}{-log\ (1-r)},$$

where $T(r,f)$ is the characteristic function of Nevanlinna of f. If $D(f) < \infty$, we say that f is of finite degree $D(f)$ (or is non-admissible); if $D(f) = \infty$, we say that f is of infinite degree (or is admissible). If f is an analytic function in Δ, and

$$D_M(f) := \lim_{r\to 1^-} \sup \frac{\log^+ M(r,\ f)}{-log\ (1-r)},$$

in which $M(r,\ f) = \max_{|z|=r} |f(z)|$ is the maximum modulus function, then we say that f is a function of finite degree $D_M(f)$ if $D_M(f) < \infty$; otherwise, f is of infinite degree.

Now, we give the definitions of iterated order and growth index to classify generally the functions of fast growth in Δ as those in $\mathbb{C}$ (see [7, 21, 22]). Let us define inductively, for $r \in [0,1)$, $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp\left(\exp_p r\right)$, $p \in \mathbb{N}$. We also define for all r sufficiently large in $(0,1)$, $\log_1 r := \log r$ and $\log_{p+1} r := \log\left(\log_p r\right)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$.

Definition 1.2. [8, 11, 20] *Let f be a meromorphic function in Δ. Then the iterated $p-$order of f is defined by*

$$\rho_p(f) = \lim_{r\to 1^-} \sup \frac{log_p^+ T(r,\ f)}{-log\ (1-r)} \quad (p \geqslant 1 \quad \textit{is an integer}),$$

where $\log_1^+ x = \log^+ x = \max\{\log x, 0\}$, $\log_{p+1}^+ x = \log^+ \log_p^+ x$. For $p=1$, this notation is called order and for $p=2$ hyper order [19, 23]. *If f is analytic in Δ, then the iterated $p-$order of f is defined by*

$$\rho_{M,p}(f) = \lim_{r\to 1^-} \sup \frac{log_{p+1}^+ M(r,\ f)}{-log\ (1-r)} \quad (p \geqslant 1 \ \textit{is an integer}).$$

Remark 1.3. *It follows by M. Tsuji (*[26]*, p. 205) that if f is an analytic function in Δ, then we have the inequalities*

$$\rho_1(f) \leqslant \rho_{M,1}(f) \leqslant \rho_1(f) + 1,$$

which are the best possible in the sense that there are analytic functions g and h such that $\rho_{M,1}(g) = \rho_1(g)$ and $\rho_{M,1}(h) = \rho_1(h) + 1$, see [13]. *However, it follows by Proposition 2.2.2 in* [22] *that $\rho_{M,p}(f) = \rho_p(f)$ for $p \geqslant 2$.*

Definition 1.4. *(see* [11]*) The growth index of the iterated order of a meromorphic function $f(z)$ in Δ is defined by*

$$i(f) = \begin{cases} 0, & \textit{if } f \quad \textit{is non-admissible}, \\ \min\{j \in \mathbb{N} : \rho_j(f) < +\infty\} & \textit{if } f \quad \textit{is admissible}, \\ +\infty, & \textit{if } \rho_j(f) = +\infty \textit{ for all } j \in \mathbb{N}. \end{cases}$$

For an analytic function f in Δ, we also define

$$i_M(f)=\begin{cases}0, & \text{if } f \text{ is non-admissible,}\\ \min\{j\in\mathbb{N}:\rho_{M,j}(f)<+\infty\} & \text{if } f \text{ is admissible,}\\ +\infty, & \text{if } \rho_{M,j}(f)=+\infty \text{ for all } j\in\mathbb{N}.\end{cases}$$

Remark 1.5. *If $\rho_p(f)<\infty$ or $i(f)\leqslant p$, then we say that f is of finite iterated $p-$order; if $\rho_p(f)=\infty$ or $i(f)>p$, then we say that f is of infinite iterated $p-$order. In particular, we say that f is of finite order if $\rho_1(f)<\infty$ or $i(f)\leqslant 1$; f is of infinite order if $\rho_1(f)=\infty$ or $i(f)>1$.*

In [2], the author and Hamouda have proved the following result.

Theorem 1.6. [2] *Let $B_0(z),...,B_{n-1}(z)$ with $B_0(z)\not\equiv 0$ be entire functions. Suppose that there exist a sequence of complex numbers $(z_j)_{j\in\mathbb{N}}$ with $\lim_{j\to+\infty} z_j=\infty$ and three real numbers α,β and μ satisfying $0\leqslant\beta<\alpha$ and $\mu>0$ such that*

$$|B_0(z_j)|\geqslant\exp\{\alpha|z_j|^{\mu}\}$$

and

$$|B_k(z_j)|\leqslant\exp\{\beta|z_j|^{\mu}\}\quad(k=1,2,...,n-1)$$

as $j\to+\infty$. Then every solution $f\not\equiv 0$ of the equation

$$f^{(n)}+B_{n-1}(z)f^{(n-1)}+...+B_1(z)f'+B_0(z)f=0$$

has an infinite order.

It is natural to ask what can be said about similar situations in the unit disc Δ. For $n\geqslant 2$, we consider a linear differential equation of the form

$$f^{(n)}+A_{n-1}(z)f^{(n-1)}+...+A_1(z)f'+A_0(z)f=0 \tag{1.2}$$

where $A_0(z),...,A_{n-1}(z),\ A_0(z)\not\equiv 0$ are analytic functions in the unit disc $\Delta=\{z\in\mathbb{C}:|z|<1\}$. It is well-known that all solutions of the equation (1.2) are analytic functions in Δ and that there are exactly n linearly independent solutions of (1.2), (see [19]).

A question arises : What conditions on $A_0(z),...,A_{n-1}(z)$ will guarantee that every solution $f\not\equiv 0$ of (1.2) has infinite order? Many authors have investigated the growth of the solutions of complex linear differential equations in $\mathbb{C}$, see [2, 3, 4, 5, 6, 10, 15, 21, 22, 28, 29]. For the unit disc, there already exist many results [8, 9, 11, 12, 19, 20, 23], but the study is more difficult than that in the complex plane, because the efficient tool of Wiman-Valiron theory doesn't work in the unit disc.

In the present paper, we investigate the growth of solutions of the equation (1.2). We also study the growth of solutions of non-homogeneous linear differential equations. We give answer to the question posed above and we obtain four theorems which are analogous to those obtained by the author in the complex plane (see [2, 5]).

Theorem 1.7. *Let $A_0(z)\not\equiv 0,...,A_{n-1}(z)$ be analytic functions in the unit disc Δ. Suppose that there exist a sequence of complex numbers $(z_j)_{j\in\mathbb{N}}$ with $\lim_{j\to+\infty}|z_j|=1^-$*

and three real numbers α, β and μ satisfying $0 \leqslant \beta < \alpha$ and $\mu > 0$ such that

$$|A_0(z_j)| \geqslant \exp_p \left\{ \alpha \left(\frac{1}{1 - |z_j|} \right)^{\mu} \right\} \tag{1.3}$$

and

$$|A_k(z_j)| \leqslant \exp_p \left\{ \beta \left(\frac{1}{1 - |z_j|} \right)^{\mu} \right\} \quad (k = 1, ..., n-1) \tag{1.4}$$

as $j \to +\infty$, where $p \geqslant 1$ is an integer. Then every solution $f \not\equiv 0$ of the equation (1.2) satisfies $\rho_p(f) = \rho_{M,p}(f) = +\infty$ and $\rho_{p+1}(f) = \rho_{M,p+1}(f) \geqslant \mu$.

Setting $p = 1$ in Theorem 1.7, we deduce the following result.

Corollary 1.8. *Let $A_0(z) \not\equiv 0, ..., A_{n-1}(z)$ be analytic functions in the unit disc Δ. Suppose that there exist a sequence of complex numbers $(z_j)_{j \in \mathbb{N}}$ with $\lim\limits_{j \to +\infty} |z_j| = 1^-$ and three real numbers α, β and μ satisfying $0 \leqslant \beta < \alpha$ and $\mu > 0$ such that*

$$|A_0(z_j)| \geqslant \exp \left\{ \alpha \left(\frac{1}{1 - |z_j|} \right)^{\mu} \right\}$$

and

$$|A_k(z_j)| \leqslant \exp \left\{ \beta \left(\frac{1}{1 - |z_j|} \right)^{\mu} \right\} \quad (k = 1, ..., n-1)$$

as $j \to +\infty$. Then every solution $f \not\equiv 0$ of the equation (1.2) satisfies $\rho_1(f) = \rho_{M,1}(f) = +\infty$ and $\rho_2(f) = \rho_{M,2}(f) \geqslant \mu$.

Theorem 1.9. *Let $A_0(z) \not\equiv 0, ..., A_{n-1}(z)$ be analytic functions in the unit disc Δ with $i(A_0) = p$ $(1 \leqslant p < +\infty)$ and $\max\{\rho_{M,p}(A_k) : k = 1, ..., n-1\} \leqslant \rho_{M,p}(A_0) = \rho$. Suppose that there exist a sequence of complex numbers $(z_j)_{j \in \mathbb{N}}$ with $\lim\limits_{j \to +\infty} |z_j| = 1^-$ and three real numbers α, β and μ satisfying $0 \leqslant \beta < \alpha$ and $\mu > 0$ such that for any given $\varepsilon > 0$ sufficiently small, we have*

$$|A_0(z_j)| \geqslant \exp_p \left\{ \alpha \left(\frac{1}{1 - |z_j|} \right)^{\rho - \varepsilon} \right\} \tag{1.5}$$

and

$$|A_k(z_j)| \leqslant \exp_p \left\{ \beta \left(\frac{1}{1 - |z_j|} \right)^{\rho - \varepsilon} \right\} \quad (k = 1, ..., n-1) \tag{1.6}$$

as $j \to +\infty$. Then every solution $f \not\equiv 0$ of the equation (1.2) satisfies $\rho_p(f) = \rho_{M,p}(f) = +\infty$ and $\rho_{p+1}(f) = \rho_{M,p+1}(f) = \rho_{M,p}(A_0) = \rho$.

From Theorem 1.9, by setting $p = 1$ we obtain the following result.

Corollary 1.10. *Let $A_0(z) \not\equiv 0, ..., A_{n-1}(z)$ be analytic functions in the unit disc Δ with $\max\{\rho_{M,1}(A_k) : k = 1, ..., n-1\} \leqslant \rho_{M,1}(A_0) = \rho < +\infty$. Suppose that there exist a sequence of complex numbers $(z_j)_{j \in \mathbb{N}}$ with $\lim\limits_{j \to +\infty} |z_j| = 1^-$ and three real numbers α, β and μ satisfying $0 \leqslant \beta < \alpha$ and $\mu > 0$ such that for any given $\varepsilon > 0$ sufficiently small, we have*

$$|A_0(z_j)| \geqslant \exp \left\{ \alpha \left(\frac{1}{1 - |z_j|} \right)^{\rho - \varepsilon} \right\}$$

and

$$|A_k(z_j)| \leqslant \exp\left\{\beta\left(\frac{1}{1-|z_j|}\right)^{\rho-\varepsilon}\right\} \quad (k=1,...,n-1)$$

as $j \to +\infty$. *Then every solution* $f \not\equiv 0$ *of the equation (1.2) satisfies* $\rho_1(f) = \rho_{M,1}(f) = +\infty$ *and* $\rho_2(f) = \rho_{M,2}(f) = \rho_{M,1}(A_0) = \rho$.

In the next, we study the growth of non-homogeneous linear differential equation

$$f^{(n)} + A_{n-1}(z) f^{(n-1)} + ... + A_1(z) f' + A_0(z) f = F(z) \quad (n \geqslant 2), \tag{1.7}$$

where $A_0(z), ..., A_{n-1}(z)$, $F(z)$ are analytic functions in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. We obtain the following results:

Theorem 1.11. *Let* $A_0(z), ..., A_{n-1}(z)$ *and* $F(z)$ *be analytic functions in the unit disc* Δ *such that for some integer* s, $1 \leqslant s \leqslant n-1$, *satisfying* $i(A_s) = p$ $(1 \leqslant p < \infty)$ *and* $\max\{\rho_p(A_j)\ (j \neq s),\ \rho_p(F)\} < \rho_p(A_s)$. *Then every admissible solution* f *of (1.7) with* $\rho_p(f) < +\infty$ *satisfies* $\rho_p(f) \geqslant \rho_p(A_s)$.

Setting $p = 1$ in Theorem 1.11, we deduce the following result.

Corollary 1.12. *Let* $A_0(z), ..., A_{n-1}(z)$ *and* $F(z)$ *be analytic functions in the unit disc* Δ *such that for some integer* s, $1 \leqslant s \leqslant k-1$ *satisfying* $\max\{\rho_1(A_j)\ (j \neq s),\ \rho_1(F)\} < \rho_1(A_s) < +\infty$. *Then every admissible solution* f *of (1.7) with* $\rho_1(f) < +\infty$ *satisfies* $\rho_1(f) \geqslant \rho_1(A_s)$.

Theorem 1.13. *Let* $A_0(z), ..., A_{n-1}(z)$ *and* $F(z)$ *be analytic functions in the unit disc* Δ *such that for some integer* s, $0 \leqslant s \leqslant k-1$, *we have* $\rho_p(A_s) = +\infty$ *and* $\max\{\rho_p(A_j)\ (j \neq s),\ \rho_p(F)\} < +\infty$ $(1 \leqslant p < \infty)$. *Then every solution* f *of (1.7) satisfies* $\rho_p(f) = +\infty$.

From Theorem 1.13, by setting $p = 1$ we obtain the following result.

Corollary 1.14. *Let* $A_0(z), ..., A_{n-1}(z)$ *and* $F(z)$ *be analytic functions in the unit disc* Δ *such that for some integer* s, $0 \leqslant s \leqslant k-1$, *we have* $\rho_1(A_s) = +\infty$ *and* $\max\{\rho_1(A_j)\ (j \neq s),\ \rho_1(F)\} < +\infty$. *Then every solution* f *of (1.7) satisfies* $\rho_1(f) = +\infty$.

Remark 1.15. *A similar results to Theorem 1.11 and Theorem 1.13 in the plane case were obtained by the author in* [5].

2. Some Lemmas

In this section we give some lemmas which are used in the proofs of our theorems.

Lemma 2.1. *(*[13]*, Theorem 3.1) Let* k *and* j *be integers satisfying* $k > j \geqslant 0$, *and let* $\varepsilon > 0$ *and* $d \in (0,1)$. *If* f *is a meromorphic in* Δ *such that* $f^{(j)}$ *does not vanish identically, then*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leqslant \left(\left(\frac{1}{1-|z|}\right)^{2+\varepsilon} \max\left\{\log\frac{1}{1-|z|}, T(s(|z|), f)\right\}\right)^{k-j} \quad (|z| \notin E_1), \tag{2.1}$$

where $E_1 \subset [0,1)$ *is a set with finite logarithmic measure* $\int_{E_1} \frac{dr}{1-r} < +\infty$ *and* $s(|z|) = 1 - d(1-|z|)$. *Moreover, if* $\rho_1(f) < +\infty$, *then*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leqslant \left(\frac{1}{1-|z|}\right)^{(k-j)(\rho_1(f)+2+\varepsilon)} \quad (|z| \notin E_1), \tag{2.2}$$

while if $\rho_p(f) < +\infty$ *for some* $p \geqslant 2$, *then*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leqslant \exp_{p-1}\left\{\left(\frac{1}{1-|z|}\right)^{\rho_p(f)+\varepsilon}\right\} \ (|z| \notin E_1). \tag{2.3}$$

Lemma 2.2. *([19]) Let* f *be a meromorphic function in the unit disc* Δ, *and let* $k \geqslant 1$ *be an integer. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r,f), \tag{2.4}$$

where $S(r,f) = O\left(\log^+ T(r,f) + \log(\frac{1}{1-r})\right)$, *possibly outside a set* $E_2 \subset [0,1)$ *with* $\int_{E_2} \frac{dr}{1-r} < +\infty$. *If* f *is of finite order (namely, finite iterated* $1-$*order) of growth, then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log(\frac{1}{1-r})\right). \tag{2.5}$$

In the next, we give the generalized logarithmic derivative lemma.

Lemma 2.3. *Let* f *be a meromorphic function in the unit disc* Δ *for which* $i(f) = p \geqslant 1$ *and* $\rho_p(f) = \beta < +\infty$, *and let* $k \geqslant 1$ *be an integer. Then for any* $\varepsilon > 0$,

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\exp_{p-2}\left\{\frac{1}{1-r}\right\}^{\beta+\varepsilon}\right) \tag{2.6}$$

holds for all r *outside a set* $E_3 \subset [0,1)$ *with* $\int_{E_3} \frac{dr}{1-r} < +\infty$.

Proof. First for $k=1$. Since $\rho_p(f) = \beta < +\infty$, we have for all $r \to 1^-$

$$T(r,f) \leqslant \exp_{p-1}\left\{\frac{1}{1-r}\right\}^{\beta+\varepsilon}. \tag{2.7}$$

By Lemma 2.2, we have

$$m\left(r, \frac{f'}{f}\right) = O\left(\ln^+ T(r,f) + \ln(\frac{1}{1-r})\right) \tag{2.8}$$

holds for all r outside a set $E_3 \subset [0,1)$ with $\int_{E_3} \frac{dr}{1-r} < +\infty$. Hence, we have

$$m\left(r, \frac{f'}{f}\right) = O\left(\exp_{p-2}\left\{\frac{1}{1-r}\right\}^{\beta+\varepsilon}\right), \ r \notin E_3. \tag{2.9}$$

Next, we assume that we have

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\exp_{p-2}\left\{\frac{1}{1-r}\right\}^{\beta+\varepsilon}\right), \ r \notin E_3 \tag{2.10}$$

for some an integer $k \geqslant 1$. Since $N\left(r, f^{(k)}\right) \leqslant (k+1)\, N(r,f)$, it holds that

$$\begin{aligned}T\left(r,f^{(k)}\right) &= m\left(r,f^{(k)}\right)+N\left(r,f^{(k)}\right)\\ &\leqslant m\left(r,\frac{f^{(k)}}{f}\right)+m\left(r,f\right)+\left(k+1\right)N\left(r,f\right)\\ &\leqslant m\left(r,\frac{f^{(k)}}{f}\right)+\left(k+1\right)T\left(r,f\right)\end{aligned}$$

$$= O\left(\exp_{p-2}\left\{\frac{1}{1-r}\right\}^{\beta+\varepsilon}\right)+\left(k+1\right)T\left(r,f\right)=O\left(\exp_{p-1}\left\{\frac{1}{1-r}\right\}^{\beta+\varepsilon}\right). \tag{2.11}$$

By (2.8) and (2.11), we again obtain

$$m\left(r,\frac{f^{(k+1)}}{f^{(k)}}\right)=O\left(\exp_{p-2}\left\{\frac{1}{1-r}\right\}^{\beta+\varepsilon}\right),\ r\notin E_3 \tag{2.12}$$

and hence,

$$\begin{aligned}m\left(r,\frac{f^{(k+1)}}{f}\right) &\leqslant m\left(r,\frac{f^{(k+1)}}{f^{(k)}}\right)+m\left(r,\frac{f^{(k)}}{f}\right)\\ &= O\left(\exp_{p-2}\left\{\frac{1}{1-r}\right\}^{\beta+\varepsilon}\right),\ r\notin E_3.\end{aligned} \tag{2.13}$$

□

Lemma 2.4. *([1]) Let $g:(0,1)\to\mathbb{R}$ and $h:(0,1)\to\mathbb{R}$ be monotone increasing functions such that $g(r)\leqslant h(r)$ holds outside of an exceptional set $E_4\subset[0,1)$ of finite logarithmic measure. Then there exists a $d\in(0,1)$ such that if $s(r)=1-d(1-r)$, then $g(r)\leqslant h(s(r))$ for all $r\in[0,1)$.*

Lemma 2.5. *([20]) Let $A_0(z),...,A_{n-1}(z)$ be analytic functions in the unit disc Δ. Then every solution $f\not\equiv 0$ of (1.2) satisfies*

$$\rho_{M,p+1}(f)\leqslant\max\left\{\rho_{M,p}(A_k):k=0,1,...,n-1\right\}. \tag{2.14}$$

Proof of Theorem 1.7. Suppose that $f\not\equiv 0$ is a solution of (1.2) with $\rho_p(f)<+\infty$. Then by Lemma 2.1, there exists a set $E_1\subset[0,1)$ with finite logarithmic measure $\int_{E_1}\frac{dr}{1-r}<+\infty$ such that for all z satisfying $|z|\notin E_1$ and for $k=1,2,...,n$, we have

$$\left|\frac{f^{(k)}(z)}{f(z)}\right|\leqslant\left(\frac{1}{1-|z|}\right)^{k(\rho_1(f)+2+\varepsilon)}\qquad(|z|\notin E_1), \tag{2.15}$$

if $\rho_1(f)<+\infty$, while if $\rho_p(f)<+\infty$ for some $p\geqslant 2$, then

$$\left|\frac{f^{(k)}(z)}{f(z)}\right|\leqslant\exp_{p-1}\left\{\left(\frac{1}{1-|z|}\right)^{\rho_p(f)+\varepsilon}\right\}\ (|z|\notin E_1). \tag{2.16}$$

By (1.2), we can write

$$1\leqslant\frac{1}{|A_0(z)|}\left|\frac{f^{(n)}}{f}\right|+\left|\frac{A_{n-1}(z)}{A_0(z)}\right|\left|\frac{f^{(n-1)}}{f}\right|+...+\left|\frac{A_1(z)}{A_0(z)}\right|\left|\frac{f'}{f}\right|. \tag{2.17}$$

Then from (1.3), (1.4), (2.15), (2.16) and (2.17), we next conclude that

$$1 \leqslant \frac{1}{\exp\left\{\alpha\left(\frac{1}{1-|z_j|}\right)^{\mu}\right\}}\left(\frac{1}{1-|z_j|}\right)^{n(\rho_1(f)+2+\varepsilon)}$$
$$+\exp\left\{(\beta-\alpha)\left(\frac{1}{1-|z_j|}\right)^{\mu}\right\}\left(\frac{1}{1-|z_j|}\right)^{(n-1)(\rho_1(f)+2+\varepsilon)}$$
$$+...+\exp\left\{(\beta-\alpha)\left(\frac{1}{1-|z_j|}\right)^{\mu}\right\}\left(\frac{1}{1-|z_j|}\right)^{\rho_1(f)+2+\varepsilon}$$
$$\leqslant n\left(\frac{1}{1-|z_j|}\right)^{n(\rho_1(f)+2+\varepsilon)}\exp\left\{(\beta-\alpha)\left(\frac{1}{1-|z_j|}\right)^{\mu}\right\} \tag{2.18}$$

or

$$1 \leqslant \frac{1}{\exp_p\left\{\alpha\left(\frac{1}{1-|z_j|}\right)^{\mu}\right\}}\exp_{p-1}\left\{\left(\frac{1}{1-|z_j|}\right)^{\rho_p(f)+\varepsilon}\right\}$$
$$+\frac{\exp_p\left\{\beta\left(\frac{1}{1-|z_j|}\right)^{\mu}\right\}}{\exp_p\left\{\alpha\left(\frac{1}{1-|z_j|}\right)^{\mu}\right\}}\exp_{p-1}\left\{\left(\frac{1}{1-|z_j|}\right)^{\rho_p(f)+\varepsilon}\right\}$$
$$+...+\frac{\exp_p\left\{\beta\left(\frac{1}{1-|z_j|}\right)^{\mu}\right\}}{\exp_p\left\{\alpha\left(\frac{1}{1-|z_j|}\right)^{\mu}\right\}}\exp_{p-1}\left\{\left(\frac{1}{1-|z_j|}\right)^{\rho_p(f)+\varepsilon}\right\}$$
$$\leqslant n\frac{\exp_p\left\{\beta\left(\frac{1}{1-|z_j|}\right)^{\mu}\right\}}{\exp_p\left\{\alpha\left(\frac{1}{1-|z_j|}\right)^{\mu}\right\}}\exp_{p-1}\left\{\left(\frac{1}{1-|z_j|}\right)^{\rho_p(f)+\varepsilon}\right\} \tag{2.19}$$

holds all z_j satisfying $|z_j| \notin E_1$ as $|z_j| \to 1^-$, both the limits of the right hand of inequalities (2.18) and (2.19) are zero as $|z_j| \to 1^-$. Thus, we get a contradiction. Hence $\rho_p(f) = \rho_{M,p}(f) = +\infty$.

Now let $f \not\equiv 0$ be a solution of (1.2) with $\rho_p(f) = +\infty$. By Lemma 2.1, there exist $s(|z|) = 1 - d(1-|z|)$ and a set $E_2 \subset [0,1)$ with finite logarithmic measure $\int_{E_2} \frac{dr}{1-r} < +\infty$ such that for $r = |z| \notin E_2$, we have

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leqslant \left(\left(\frac{1}{1-|z|}\right)^{2+\varepsilon}\max\left\{\log\frac{1}{1-|z|}, T(s(|z|), f)\right\}\right)^{k} \quad (k=1,...,n). \tag{2.20}$$

By (1.2), we can write

$$|A_0(z)| \leqslant \left|\frac{f^{(n)}}{f}\right| + |A_{n-1}(z)|\left|\frac{f^{(n-1)}}{f}\right| + ... + |A_0(z)|\left|\frac{f'}{f}\right|. \tag{2.21}$$

Again from the conditions of Theorem 1.7, for all z_j satisfying $|z_j| = r_j \notin E_2$ as $|z_j| \to 1^-$, we get from (1.3), (1.4), (2.20) and (2.21) that

$$\exp_p\left\{\alpha\left(\frac{1}{1-|z_j|}\right)^{\mu}\right\} \leqslant |A_0(z_j)| \leqslant n\exp_p\left\{\beta\left(\frac{1}{1-|z_j|}\right)^{\mu}\right\}$$

$$\times\left(\left(\frac{1}{1-|z_j|}\right)^{2+\varepsilon}\max\left\{\log\frac{1}{1-|z_j|},T(s(|z_j|),f)\right\}\right)^n. \tag{2.22}$$

Noting that $\alpha>\beta\geqslant 0$, it follows from (2.22) that

$$\exp\left\{\alpha(1-\gamma)\exp_{p-1}\left\{\left(\frac{1}{1-|z_j|}\right)^{\mu}\right\}\right\}$$
$$\leqslant n\left(\frac{1}{1-|z_j|}\right)^{n(2+\varepsilon)}T^n(s(|z_j|),f) \tag{2.23}$$

holds for all z_j satisfying $|z_j|=r_j\notin E_2$ as $|z_j|\to 1^-$, where γ $(0<\gamma<1)$ is a real number. Hence by Lemma 2.4 and (2.23), we obtain

$$\rho_{p+1}(f)=\rho_{M,p+1}(f)=\lim_{r_j\to 1^-}\sup\frac{\log^+_{p+1}T(r_j,\ f)}{-\log\ (1-r_j)}\geqslant\mu.$$

Theorem 1.7 is thus proved. □

Proof of Theorem 1.9. Suppose that $f\not\equiv 0$ is a solution of equation (1.2). Using the same proof as in Theorem 1.7, we get $\rho_p(f)=\rho_{M,p}(f)=+\infty$.

Now we prove that $\rho_{p+1}(f)=\rho_{M,p}(A_0)=\rho$. By Theorem 1.7, we have $\rho_{p+1}(f)\geq\rho-\varepsilon$, and since $\varepsilon>0$ is arbitrary, we get $\rho_{p+1}(f)\geq\rho$. On the other hand, from Lemma 2.5, we have

$$\rho_{p+1}(f)=\rho_{M,p+1}(f)\leqslant\max\{\rho_{M,p}(A_k):k=0,1,...,n-1\}$$
$$=\rho_{M,p}(A_0)=\rho.$$

It yields $\rho_{p+1}(f)=\rho_{M,p+1}(f)=\rho_{M,p}(A_0)=\rho$. □

Proof of Theorem 1.11. Let $\max\{\rho_p(A_j)\ (j\neq s),\rho_p(F)\}=\beta<\rho_p(A_s)=\alpha$. Suppose that f is an admissible solution of (1.7) with $\rho=\rho_p(f)<+\infty$. It follows from (1.7) that

$$A_s(z)=\frac{F}{f^{(s)}}-\left(\frac{f^{(n)}}{f^{(s)}}+A_{n-1}(z)\frac{f^{(n-1)}}{f^{(s)}}+...+A_{s+1}(z)\frac{f^{(s+1)}}{f^{(s)}}\right.$$
$$\left.+A_{s-1}(z)\frac{f^{(s-1)}}{f^{(s)}}+...+A_1(z)\frac{f'}{f^{(s)}}+A_0(z)\frac{f}{f^{(s)}}\right)$$
$$=\frac{F}{f^{(s)}}-\frac{f}{f^{(s)}}\left(\frac{f^{(n)}}{f}+A_{n-1}(z)\frac{f^{(n-1)}}{f}+...+A_{s+1}(z)\frac{f^{(s+1)}}{f}\right.$$
$$\left.+A_{s-1}(z)\frac{f^{(s-1)}}{f}+...+A_1(z)\frac{f'}{f}+A_0(z)\right). \tag{2.24}$$

By Lemma 2.3 and (2.24), we obtain

$$T(r,A_s)=m(r,A_s)\leqslant m(r,F)+m\left(r,\frac{1}{f^{(s)}}\right)$$
$$+m\left(r,\frac{f}{f^{(s)}}\right)+\sum_{\substack{j=0\\j\neq s}}^{n-1}m(r,A_j)+O\left(\exp_{p-2}\left\{\frac{1}{1-r}\right\}^{\rho+\varepsilon}\right)$$

$$\leqslant T(r,F)+T\left(r,\frac{1}{f^{(s)}}\right)+T\left(r,\frac{f}{f^{(s)}}\right)+\sum_{\substack{j=0\\ j\neq s}}^{n-1}T(r,A_j)$$

$$+O\left(\exp_{p-2}\left\{\frac{1}{1-r}\right\}^{\rho+\varepsilon}\right) \tag{2.25}$$

holds for all r outside a set $E_3\subset[0,1)$ with $\int_{E_3}\frac{dr}{1-r}<+\infty$. Noting that

$$T\left(r,\frac{f}{f^{(s)}}\right)\leqslant T(r,f)+T\left(r,\frac{1}{f^{(s)}}\right)=T(r,f)+T\left(r,f^{(s)}\right)+O(1)$$

$$\leqslant(s+2)\,T(r,f)+o\,(T(r,f))+O(1)\,. \tag{2.26}$$

Thus, by (2.25), (2.26), we have

$$T(r,A_s)\leqslant T(r,F)+cT(r,f)+\sum_{\substack{j=0\\ j\neq s}}^{n-1}T(r,A_j)$$

$$+O\left(\exp_{p-2}\left\{\frac{1}{1-r}\right\}^{\rho+\varepsilon}\right)(r\notin E_3)\,, \tag{2.27}$$

where $c>0$ is a constant. Since $\rho_p(A_s)=\alpha$, then there exists $\left\{r_n'\right\}$ $\left(r_n'\to1^-\right)$ such that

$$\lim_{r_n'\to1^-}\frac{\log_p^+T\left(r_n',A_s\right)}{-\log\left(1-r_n'\right)}=\alpha. \tag{2.28}$$

Set $\int_{E_3}\frac{dr}{1-r}:=\log\gamma<+\infty$. Since $\int_{r_n'}^{1-\frac{1-r_n'}{\gamma+1}}\frac{dr}{1-r}=\log(\gamma+1)$, then there exists a point $r_n\in\left[r_n',1-\frac{1-r_n'}{\gamma+1}\right]-E_3\subset[0,1)$. From

$$\frac{\log_p^+T(r_n,A_s)}{\log\frac{1}{1-r_n}}\geqslant\frac{\log_p^+T\left(r_n',A_s\right)}{\log\left(\frac{\gamma+1}{1-r_n'}\right)}=\frac{\log_p^+T\left(r_n',A_s\right)}{\log\frac{1}{1-r_n'}+\log(\gamma+1)}, \tag{2.29}$$

it follows that

$$\lim_{r_n\to1^-}\frac{\log_p^+T(r_n,A_s)}{\log\frac{1}{1-r_n}}=\alpha. \tag{2.30}$$

So for any given $\varepsilon\,(0<2\varepsilon<\alpha-\beta)$, and for $j\neq s$

$$T(r_n,A_s)>\exp_{p-1}\left(\frac{1}{1-r_n}\right)^{\alpha-\varepsilon} \tag{2.31}$$

$$T(r_n,A_j)\leqslant\exp_{p-1}\left(\frac{1}{1-r_n}\right)^{\beta+\varepsilon},\ T(r_n,F)\leqslant\exp_{p-1}\left(\frac{1}{1-r_n}\right)^{\beta+\varepsilon} \tag{2.32}$$

hold for $r_n\to1^-$. By (2.27), (2.31) and (2.32) we obtain for $r_n\to1^-$

$$\exp_{p-1}\left(\frac{1}{1-r_n}\right)^{\alpha-\varepsilon}\leqslant n\exp_{p-1}\left(\frac{1}{1-r_n}\right)^{\beta+\varepsilon}+cT(r_n,f)$$

$$+O\left(\exp_{p-2}\left\{\frac{1}{1-r_n}\right\}^{\rho+\varepsilon}\right). \tag{2.33}$$

Therefore,

$$\lim_{r_n\to 1^-}\sup\frac{\log_p^+ T\left(r_n,f\right)}{-\log\left(1-r_n\right)}\geqslant \alpha-\varepsilon \tag{2.34}$$

and since $\varepsilon>0$ is arbitrary, we get $\rho_p\left(f\right)\geqslant\rho_p\left(A_s\right)=\alpha$. This proves Theorem 1.11. □

Proof of Theorem 1.13. Setting $\max\left\{\rho_p\left(A_j\right)\ (j\neq s),\rho_p\left(F\right)\right\}=\beta$, then for a given $\varepsilon>0$, we have

$$T\left(r,A_j\right)\leqslant\exp_{p-1}\left(\frac{1}{1-r}\right)^{\beta+\varepsilon}\quad(j\neq s),\ T\left(r,F\right)\leqslant\exp_{p-1}\left(\frac{1}{1-r}\right)^{\beta+\varepsilon} \tag{2.35}$$

for $r\to 1^-$. Now by (2.24), we have

$$T\left(r,A_s\right)=m\left(r,A_s\right)\leqslant m\left(r,F\right)+m\left(r,\frac{1}{f^{(s)}}\right)$$

$$+m\left(r,\frac{f}{f^{(s)}}\right)+\sum_{\substack{j=0\\ j\neq s}}^{n-1}m\left(r,A_j\right)+\sum_{\substack{j=1\\ j\neq s}}^{n}m\left(r,\frac{f^{(j)}}{f}\right)$$

$$\leqslant T\left(r,F\right)+T\left(r,\frac{1}{f^{(s)}}\right)+T\left(r,\frac{f}{f^{(s)}}\right)+\sum_{\substack{j=0\\ j\neq s}}^{n-1}T\left(r,A_j\right)+\sum_{\substack{j=1\\ j\neq s}}^{n}m\left(r,\frac{f^{(j)}}{f}\right). \tag{2.36}$$

Hence by (2.26) and (2.36) we obtain that

$$T\left(r,A_s\right)\leqslant T\left(r,F\right)+cT\left(r,f\right)+\sum_{\substack{j=0\\ j\neq s}}^{n-1}T\left(r,A_j\right)+\sum_{\substack{j=1\\ j\neq s}}^{n}m\left(r,\frac{f^{(j)}}{f}\right), \tag{2.37}$$

where $c>0$ is a constant. If $\rho=\rho_p\left(f\right)<+\infty$, then by Lemma 2.3

$$m\left(r,\frac{f^{(j)}}{f}\right)=O\left(\exp_{p-2}\left\{\frac{1}{1-r}\right\}^{\rho+\varepsilon}\right)\quad(j=1,...,n;j\neq s) \tag{2.38}$$

holds for all r outside a set $E_3\subset[0,1)$ with $\int_{E_3}\frac{dr}{1-r}<+\infty$. For $r\to 1^-$, we have

$$T\left(r,f\right)\leqslant\exp_{p-1}\left(\frac{1}{1-r}\right)^{\rho+\varepsilon}. \tag{2.39}$$

Thus, by (2.35), (2.37), (2.38) and (2.39), we get

$$T\left(r,A_s\right)\leqslant n\exp_{p-1}\left(\frac{1}{1-r}\right)^{\beta+\varepsilon}+c\exp_{p-1}\left(\frac{1}{1-r}\right)^{\rho+\varepsilon}$$

$$+O\left(\exp_{p-2}\left\{\frac{1}{1-r}\right\}^{\rho+\varepsilon}\right) \tag{2.40}$$

for $r\notin E_3$ and $r\to 1^-$. By Lemma 2.4, we have for any $d\in(0,1)$

$$T\left(r,A_s\right)\leqslant n\exp_{p-1}\left(\frac{1}{d\left(1-r\right)}\right)^{\beta+\varepsilon}+c\exp_{p-1}\left(\frac{1}{d\left(1-r\right)}\right)^{\rho+\varepsilon}$$

$$+ O\left(\exp_{p-2}\left\{\frac{1}{d\left(1-r\right)}\right\}^{\rho+\varepsilon}\right) \tag{2.41}$$

for $r \to 1^-$. Therefore,

$$\rho_p\left(A_s\right) \leqslant \max\left\{\beta+\varepsilon,\ \rho+\varepsilon\right\} < +\infty. \tag{2.42}$$

This contradicts the fact that $\rho_p\left(A_s\right) = +\infty$. Hence $\rho_p\left(f\right) = +\infty$. □

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