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# OPTIMAL INEQUALITIES FOR HYPERBOLIC AND TRIGONOMETRIC FUNCTIONS

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Abstract. We determine the best positive constants $p$ and $q$ such that

$$\left(\frac{1}{\cosh x}\right)^p < \frac{\sin x}{x} < \left(\frac{1}{\cosh x}\right)^q$$

as well as $p'$ and $q'$ such that

$$\left(\frac{\sinh x}{x}\right)^{p'} < \frac{2}{\cos x + 1} < \left(\frac{\sinh x}{x}\right)^{q'}.$$

## 1. Introduction

In recent years inequalities involving trigonometric and hyperbolic inequalities have attracted attention of several researchers. For instance, the Huygens, the Cusa-Huygens, and the Wilker inequalities for trigonometric and hyperbolic functions have been studied extensively in numerous papers. For more references the interested reader is referred to [1] and [4]. For example, it was demonstrated in [1] that for all $x \in (0, \pi/2)$ one has

$$\frac{x^2}{\sinh^2 x} < \frac{\sin x}{x} < \frac{x}{\sinh x}, \tag{1.1}$$

$$\frac{1}{\cosh x} < \frac{\sin x}{x} < \frac{x}{\sinh x}, \tag{1.2}$$

and

$$\left(\frac{1}{\cosh x}\right)^{1/2} < \frac{x}{\sinh x} < \left(\frac{1}{\cosh x}\right)^{1/4} \tag{1.3}$$

for $0 < x < 1$.

In the recent paper [5] we have determined the best inequalities of type (1.1). The goal of this paper is to determine optimal inequalities which are similar to (1.1) - (1.3). They are contained in Theorems 2.1 and 2.2.

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## 2. Main Results

The following auxiliary results will be needed in the sequel.

**Lemma 2.1.** *For all $x > 0$ one has*

$$\ln \cosh x > \frac{x}{2} \tanh x. \tag{2.1}$$

**Proof.** Let us define $f_1(x) = \ln \cosh x - \frac{x}{2} \tanh x$, $x \geq 0$.

A simple computation gives

$$2 \cosh^2 x \cdot f_1'(x) = \sinh x \cdot \cosh x - x > 0,$$

where the last inequality follows immediately from $\sinh x > x$ and $\cosh x > 1$ ($x > 0$). Thus $f_1$ is a strictly increasing function. This in turn implies that $f_1(x) \geq f_1(0) = 0$ for $x \geq 0$, with equality if $x = 0$. This completes the proof of inequality (2.1). □

**Lemma 2.2.** *For all $x \in (0, \pi/2)$ one has*

$$\ln \frac{x}{\sin x} < \frac{\sin x - x \cos x}{2 \sin x}. \tag{2.2}$$

**Proof.** Let $f_2(x) = \frac{\sin x - x \cos x}{2 \sin x} - \ln \frac{x}{\sin x}$, $0 < x \leq \frac{\pi}{2}$.

A simple computation gives

$$2x \sin^2 x \cdot f_2'(x) = x^2 + x \cdot \sin x \cdot \cos x - 2 \sin^2 x > 0,$$

where the last inequality is satisfied iff

$$\frac{\sin x}{x} < \frac{\cos x + \sqrt{\cos^2 x + 8}}{4}. \tag{2.3}$$

In order to prove (2.3) it suffices to use the Cusa-Huygens inequality (see, e.g., [4])

$$\frac{\sin x}{x} < \frac{\cos x + 2}{3}, \tag{2.4}$$

together with

$$\frac{\cos x + 2}{3} < \frac{\cos x + \sqrt{\cos^2 x + 8}}{4},$$

where the last inequality is equivalent to

$$(\cos x - 1)^2 > 0.$$

Thus $f_2'(x) > 0$ for $x > 0$, and this implies

$$f_2(x) > f_2(0_+) = \lim_{x \to 0_+} f_2(x) = 0.$$

The proof of inequality (2.2) is complete. □

The main results of this paper are contained in the following two theorems.

**Theorem 2.1.** *The best positive constants $p$ and $q$ in the following inequality*

$$\frac{1}{(\cosh x)^p} < \frac{\sin x}{x} < \frac{1}{(\cosh x)^q}, x \in \left(0, \frac{\pi}{2}\right) \tag{2.5}$$

*are $p = \ln(\pi/2)/\ln \cosh(\pi/2) \approx 0.49$ and $q = \frac{1}{3} = 0.33\ldots$*

**Proof.** Let

$$h_1(x) = \frac{\ln \dfrac{x}{\sin x}}{\ln \cosh x} = \frac{f_1(x)}{g_1(x)}, x \in \left(0, \frac{\pi}{2}\right).$$

Simple computations give

$$f_1'(x) = \frac{\sin x - x\cos x}{x \sin x}, g_1'(x) = \frac{\sinh x}{\cosh x},$$

$$(\ln \cosh x)^2 h_1'(x) = \frac{\sin x - x\cos x}{x\sin x} \ln(\cosh x) - \tanh x \ln \frac{x}{\sin x}. \qquad (2.6)$$

Using the inequalities $\sin x > x\cos x$, $\dfrac{x}{\sin x} > 1$, $\cosh x > 1$, (2.1) and (2.2), we see using (2.6), that $h_1'(x) > 0$ for $x > 0$. This shows that, the function $h_1$ is strictly increasing, so

$$h_1(0_+) < h_1(x) < h_1\left(\frac{\pi}{2}\right) \text{ for any } 0 < x < \frac{\pi}{2}. \qquad (2.7)$$

Elementary computations give

$$h_1(0_+) = \lim_{x\to 0} h_1(x) = \frac{1}{3} h_1(\pi/2) = \frac{\ln(\pi/2)}{\ln\cosh(\pi/2)} \approx 0.49\ldots.$$

Thus by virtue of (2.7) we see that $q = h_1(0_+)$ and $p = h_1(\pi/2)$ are the best possible constants in (2.5). □

**Remark 2.1.** The right side inequality in (2.5) also follows from the inequality

$$\frac{\sinh x}{x} > \sqrt[3]{\cosh x} \qquad (2.8)$$

which has been discovered by I. Lazarević (see [3], [4]). We have shown recently (see [6]) that (2.8) is equivalent to an inequality in the theory of bivariate means [2]:

$$L > \sqrt[3]{G^2 A}, \qquad (2.9)$$

where $L = L(a,b) = (b-a)/(\ln b - \ln a)$ $(a \neq b)$ is the logarithmic mean of $a$ and $b$, while $G = G(a,b) = \sqrt{ab}$, and $A = A(a,b) = \dfrac{a+b}{2}$ are, respectively, the geometric and arithmetic means of $a$ and $b$.

We note that inequality (2.1) of Lemma 2.1 also follows from known results in the theory of means. Let

$$S = S(a,b) = (a^a \cdot b^b)^{1/(a+b)}$$

be a mean which has been studied, e.g., in [7]. It is known that

$$S < \frac{A^2}{G} \qquad (2.10).$$

We let $a = e^x$, $b = e^{-x}$ to obtain $A = A(a,b) = \cosh x$, $G = G(a,b) = 1$, and $S = S(a,b) = e^{x\tanh x}$. It is clear that (2.10) becomes (2.1). From results in [8] we can deduce the following refinement of (2.1):

$$\ln\cosh x > \frac{1}{4}[3(x\coth x - 1) + x\tanh x] > \frac{x}{2}\tanh x. \qquad (2.11)$$

**Theorem 2.2.** *The best positive constants $p'$ and $q'$ for which the following inequality*

$$\left(\frac{\sinh x}{x}\right)^{p'} < \frac{2}{\cos x + 1} < \left(\frac{\sinh x}{x}\right)^{q'} \qquad (2.12)$$

*is valid are* $p' = \frac{3}{2} = 1.5$ *and* $q' = \ln 2/\ln[\sinh(\pi/2)/(\pi/2)] = 1.818\ldots$

**Proof.** In order to obtain the desired result let us introduce

$$h_2(x) = \frac{\ln(2/(\cos x + 1))}{\ln(\sinh x/x)} = \frac{f_2(x)}{g_2(x)}, x \in \left(0, \frac{\pi}{2}\right). \tag{2.13}$$

Easy computations give $f_2'(x) = \frac{\sin x}{\cos x + 1}$ and $g_2'(x) = \frac{x\cosh x - \sinh x}{x \sinh x}$. Hence

$$g_2'(x)^2 \cdot h_2'(x) = -\frac{x\cosh x - \sinh x}{x\sinh x}\left(\ln\frac{2}{\cos x + 1}\right) + \left(\ln\frac{\sinh x}{x}\right)\frac{\sin x}{\cos x + 1}. \tag{2.14}$$

We will need the following inequality:

$$\ln\frac{\sinh x}{x} > \frac{1}{2}\cdot\frac{x\cosh x - \sinh x}{x\sinh x}, x > 0. \tag{2.15}$$

We note that (2.15) follows from [7, 8]:

$$L^2 > G \cdot I, \tag{2.16}$$

where $I = I(a, b)$ is the identric mean of $a$ and $b$, defined by

$$I = e^{-1}(b^b/a^a)^{1/(b-a)} \text{ for } a \neq b.$$

Since $L(e^x, e^{-x}) = \frac{\sinh x}{x}$, $I(e^x, e^{-x}) = e^{x\coth x - 1}$, $G(e^x, e^{-x}) = 1$, (2.16) yields (2.15).

We now prove that

$$a(x) = \frac{x}{2}\cdot\frac{\sin x}{\cos x + 1} - \ln\frac{2}{\cos x + 1} > 0 \text{ for } x \in \left(0, \frac{\pi}{2}\right). \tag{2.17}$$

An easy computation gives

$$a'(x) = \frac{x - \sin x}{2(\cos x + 1)} > 0.$$

This in conjunction with $a(0) = 0$, yields (2.17).

Making use of (2.15) and (2.17), and taking into account (2.14) we get $h_2'(x) > 0$ for $x > 0$. Thus $h_2(x)$ is a strictly increasing function. This in turn yields

$$p' = h_2(0_+) < h_2(x) < h_2(\pi/2) = q'. \tag{2.18}$$

A simple computation, involving application of l'Hospital's rule, together with the use of the well known limits

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\sinh x}{x} = 1$$

implies $p' = \frac{3}{2} = 1.5$ and

$$q' = \frac{\ln 2}{\ln\left(\frac{\sinh(\pi/2)}{(\pi/2)}\right)} \approx 1.818\ldots$$

This finishes the proof of Theorem 2.2. □

**Remark 2.2.** Since $\dfrac{\cos x+1}{2}=\cos^2\dfrac{x}{2}$, $\sin x=2\sin\dfrac{x}{2}\cos\dfrac{x}{2}$, $\sin\dfrac{x}{2}<\dfrac{x}{2}$ and $\tan\dfrac{x}{2}>\dfrac{x}{2}$, one obtains

$$\left(\frac{\sin x}{x}\right)^2<\frac{\cos x+1}{2}<\frac{\sin x}{x}. \tag{2.19}$$

This in conjunction with (1.1) yields

$$\frac{\sinh x}{x}<\frac{2}{\cos x+1}<\left(\frac{\sinh x}{x}\right)^4. \tag{2.20}$$

Comparison with inequality (2.12) reveals superiority of the latter result.

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