Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 3 Issue 3(2011), Pages 200-208.

UNIQUENESS OF MEROMORPHIC FUNCTIONS SHARING SETS

(COMMUNICATED BY R. K. RAINA)

SUBHAS S BHOOSNURMATH AND RENUKADEVI S DYAVANAL

Abstract. In this article, we investigate the problem of uniqueness of meromorphic functions sharing one set and having deficient values, and obtain a result which provides an answer to a question of F.Gross [2] and H.X.Yi [9].

1. Introduction

In this paper, by a meromorphic function we always mean a function which is meromorphic in the whole complex plane. Let $f(z)$ be a non-constant meromorphic function. We use the following standard notations of the value distributions theory,

$$T(r,f), m(r,f), N(r,f), \overline{N}(r,f), N\left(r,\frac{1}{f}\right),\cdots$$

(See Hayman [3], Yang [7], Yi [8]]). We denote by $S(r,f)$ any function satisfying

$$S(r,f)=o\{T(r,f)\},$$

as $r\to\infty$, possibly outside of a set E with finite measure not necessarily the same at each occurrence.

Let S be a subset of $\bar{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$. Define

$$E_f(S)\ =\ E(S,f)\ =\ \bigcup_{a\in S}\{z\,:\,f(z)-a=0\},$$

where each zero is counted according to its multiplicity.

Let f and g be two nonconstant meromorphic functions. We say that f and g share the set S CM if

$$E(S,f)=E(g,S).$$

2000 *Mathematics Subject Classification.* 30D35.

Key words and phrases. Nevanlinna theory; sharing sets; uniqueness .

Submitted August 13, 2010. Published June 21, 2011.

S.S.B. was supported by the DST grant of India, Project no. SR/S4/MS:520/08.

R.S.D. was partially supported by the UGC grant of India, Project No.F.39-934/2010(SR) and UGC-SAP-DRS-II, No.F.510/1/DRS/2010(SAP-I),India.

Define

$$N_2\left(r,\frac{1}{f-a}\right) = \overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}_{(2}\left(r,\frac{1}{f-a}\right).$$

$$\Theta(\infty,f) = 1-\overline{\lim_{r\to\infty}}\frac{\overline{N}(r,f)}{T(r,f)}.$$

It is assumed that the reader is familiar with the notations of the Nevanlinna Theory that can be found, for instance in [3], [7] and [8]

In 1977, Gross [2] posed the following question.
Question 1.1. Does there exist a finite set S such that, for any pair of nonconstant entire functions f and g, $E(S,f)=E(S,g)$ implies $f=g$?.

If such a finite set exists, a natural problem is the following
Question 1.2. What is the smallest cardinality for such a finite set?.

The best answer to question 1.2 for meromorphic functions was obtained by Frank and Reinders [1]. They proved the following result

Theorem 1.A. *There exists a set S with 11 elements such that $E_f(S)=E_g(S)$ implies $f\equiv g$ for any pair of nonconstant meromorphic functions f and g.*

Question 1.3. If nonconstant meromorphic functions f and g have few poles, can the numbers of elements of the set S in Theorem 1.A be reduced to seven?.

Regarding question 1.3, Xu [5] proved the following result.

Theorem 1.B. *Let f and g be two nonconstant meromorphic functions. If $\Theta(\infty,f)>3/4$ and $\Theta(\infty,g)>3/4$, then there exists a set S with seven elements such that $E_f(S)=E_g(S)$ implies $f\equiv g$*

Regarding question 1.1 and question 1.2, Yi [9] proved the following theorem

Theorem 1.C. *Let $S=\{z:z^n+az^{n-m}+b=0\}$, where n and m are two positive integers such that $m\geq 2, n\geq 2m+7$ with n and m having no common factor, a and b be two nonzero constants such that $z^n+az^{n-m}+b=0$ has no multiple root. If f and g are non-constant meromorphic functions satisfying $E_f(S)=E_g(S)$ and $E_f(\infty)=E_g(\infty)$, then $f\equiv g$.*

Yi asked the following question
Question 1.4. What can be said if $m=1$ in the Theorem 1.C ?

Recently, using the notion of weighted sharing Lahiri [4] proved the following result which provides an answer to the question of Yi.

Theorem 1.D. *Let $S=\left\{z:z^n+az^{n-1}+b=0\right\}$, where $n(\geq 7)$ be a positive integer and a,b be two nonzero constants such that $z^n+az^{n-1}+b=0$ has no multiple root. If $\Theta(\infty,f)+\Theta(\infty,g)>1$ and $E_f(S,2)=E_g(S,2)$, $E_f(\{\infty\},\infty)=E_g(\{\infty\},\infty)$, then $f\equiv g$.*

In this paper, we have reduced the number of elements of S to 5 by proving the following theorem.

Theorem 1.1. *Let* $S = \{z : z^n + az^{n-1} + b = 0\}$*, where* $n \geq 5$ *be a positive integer and* a, b *be two nonzero constants such that* $z^n + az^{n-1} + b = 0$ *has no repeated root. If* f *and* g *are two non constant meromorphic functions satisfying* $N_{1)}\left(r, \frac{1}{f}\right) = S(r, f)$, $N_{1)}\left(r, \frac{1}{f}\right) = S(r, f)$, $\Theta(\infty, f) > \frac{2}{n-1}$, $\Theta(\infty, g) > \frac{2}{n-1}$, *and* $E(S, f) = E(S, g)$, $E(\{\infty\}, f) = E(\{\infty\}, g)$*. Then* $f \equiv g$.

2. **Lemmas**

In order to prove Theorem 1.1, we need the following lemmas.

Lemma 2.1. *(See* [3], [7] *and* [8]*)* *Let* $f(z)$ *be a meromorphic function. Then*

$$(i) \quad T\left(r, \frac{1}{f-a}\right) = T(r, f) + O(1), \qquad a \in \mathbb{C}$$

$$(ii) \quad m\left(r, \frac{f^{(k)}}{f^{(l)}}\right) = S(r, f), \qquad k > l \geq 0$$

$$(iii) \quad T(r, f) \leq \overline{N}(r, f) + N\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f^{(k)} - c}\right) - N_0\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f),$$

$$(iv) \quad T(r, f) \leq \sum_{j=1}^{3} \overline{N}\left(r, \frac{1}{f - a_j}\right) + S(r, f).$$

where a_1, a_2, a_3 *are three distinct small functions,* $c \in C - \{0\}$ *and where in* $N_o\left(r, \frac{1}{f^{(k+1)}}\right)$ *only zeros of* $f^{(k+1)}(z)$ *not corresponding to the repeated roots of* $f^{(k)}(z) = c$ *are to be considered.*

In Lemma 2.1, the four conclusions are called ; The First Fundamental Theorem, The Lemma of Logarithmic Derivative, The Milloux's inequality and The Second Fundamental Theorem, respectively.

Lemma 2.2. *(*[8]*)* *Let* $a_1, a_2, \cdots, a_n$ *be finite complex numbers,* $a_n \neq 0$*, and let* f *be a non-constant meromorphic function. Then*

$$T\left(r, a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f\right) = nT(r, f) + S(r, f).$$

Lemma 2.3. *Let* f *and* g *be two non-constant meromorphic functions and* k *is a positive integer. If* $E\left(1, f^{(k)}\right) = E\left(1, g^{(k)}\right)$, $E(\infty, f) = E(\infty, g)$ *and*

$$\lim_{\substack{r \to \infty \\ r \notin E}} \frac{\overline{N}\left(r, f^{(k)}\right) + N_2\left(r, \frac{1}{f^{(k)}}\right) + \overline{N}\left(r, g^{(k)}\right) + N_2\left(r, \frac{1}{g^{(k)}}\right)}{T\left(r, f^{(k)}\right) + T\left(r, g^{(k)}\right)} < \frac{1}{2}$$

Then either, $f^{(k)} = g^{(k)}$ *or* $f^{(k)} g^{(k)} \equiv 1$.

Proof: Set

$$\Theta(z) = \frac{f^{(k+2)}(z)}{f^{(k+1)}(z)} - \frac{2f^{(k+1)}(z)}{f^{(k)} - 1} - \frac{g^{(k+2)}(z)}{g^{(k+1)}(z)} + \frac{2g^{(k+1)}(z)}{g^{(k)}(z) - 1} \tag{2.1}$$

We consider the cases, $\Theta(z) \not\equiv 0$ and $\Theta(z) \equiv 0$.

Let $\Theta(z) \not\equiv 0$, then if z_0 is a common simple 1-point $f^{(k)}(z)$ and $g^{(k)}(z)$, substituting their Taylor series at z_0 into (2.1), we see that z_0 is a zero of $\Theta(z)$. Thus by the first fundamental theorem, we have

$$\overline{N}_{1)}\left(r,\frac{1}{f^{(k)}-1}\right)=\overline{N}_{1)}\left(r,\frac{1}{g^{(k)}-1}\right)\le N\left(r,\frac{1}{\Theta}\right)\le T(r,\Theta)+O(1)$$

Here $\overline{N}_{1)}\left(r,\frac{1}{f^{(k)}-1}\right)$ is the counting function which only counts those simple zeros of $f^{(k)}-1$.
By the above inequality and the lemma of logarithmic derivative, we have

$$\overline{N}_{1)}\left(r,\frac{1}{f^{(k)}-1}\right)\le N(r,\Theta)+S(r,f)+S(r,g) \tag{2.2}$$

Since $f^{(k)}$ and $g^{(k)}$ share $1,\infty$ CM, from (2.1) we derive

$$N(r,\Theta)\le \overline{N}_{(2}\left(r,\frac{1}{f^{(k)}}\right)+\overline{N}_{(2}\left(r,\frac{1}{g^{(k)}}\right)+N_o\left(r,\frac{1}{f^{(k+1)}}\right)+N_o\left(r,\frac{1}{g^{(k+1)}}\right), \tag{2.3}$$

where $\overline{N}_{(2}\left(r,\frac{1}{f^{(k)}}\right)$ is the counting function of the zeros of $f^{(k)}$ whose multiplicities are greater than or equal to 2 and counted only once.
Substituting above inequality (2.3) into (2.2), we have

$$\begin{aligned}\overline{N}_{1)}\left(r,\frac{1}{f^{(k)}-1}\right)\le\ & \overline{N}_{(2}\left(r,\frac{1}{f^{(k)}}\right)+\overline{N}_{(2}\left(r,\frac{1}{g^{(k)}}\right)+N_o\left(r,\frac{1}{f^{(k+1)}}\right)\\ & +N_o\left(r,\frac{1}{g^{(k+1)}}\right)+S(r,f)+S(r,g).\end{aligned} \tag{2.4}$$

By the Second Fundamental Theorem, we have

$$\begin{aligned}T\left(r,f^{(k)}\right)\le\ & \overline{N}\left(r,f^{(k)}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)\\ & -N_o\left(r,\frac{1}{f^{(k+1)}}\right)+S(r,f)\\ T\left(r,g^{(k)}\right)\le\ & \overline{N}\left(r,g^{(k)}\right)+\overline{N}\left(r,\frac{1}{g^{(k)}}\right)+\overline{N}\left(r,\frac{1}{g^{(k)}-1}\right)\\ & -N_o\left(r,\frac{1}{g^{(k+1)}}\right)+S(r,g).\end{aligned} \tag{2.5}$$

Using (2.4) in (2.5), we obtain

$$\begin{aligned}T\left(r,f^{(k)}\right)+T\left(r,g^{(k)}\right)\ \le\ & \overline{N}\left(r,f^{(k)}\right)+\overline{N}\left(r,g^{(k)}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}}\right)+\overline{N}\left(r,\frac{1}{g^{(k)}}\right)\\ & +\overline{N}_{1)}\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}_{(2}\left(r,\frac{1}{g^{(k)}-1}\right)+\overline{N}\left(r,\frac{1}{g^{(k)}-1}\right)\\ & -N_o\left(r,\frac{1}{f^{(k+1)}}\right)-N_o\left(r,\frac{1}{g^{(k+1)}}\right)+S(r,f)+S(r,g)\\ \le\ & \overline{N}\left(r,f^{(k)}\right)+\overline{N}\left(r,g^{(k)}\right)+N_2\left(r,\frac{1}{f^{(k)}}\right)+N_2\left(r,\frac{1}{g^{(k)}}\right)\\ & +N\left(r,\frac{1}{g^{(k)}-1}\right)+S(r,f)+S(r,g).\end{aligned}$$

Therefore,

$$T\left(r,f^{(k)}\right)\le\overline{N}\left(r,f^{(k)}\right)+\overline{N}\left(r,g^{(k)}\right)+N_2\left(r,\frac{1}{f^{(k)}}\right)+N_2\left(r,\frac{1}{g^{(k)}}\right)+S(r,f)+S(r,g).$$

Since, $E(\infty,f)=E(\infty,g)$ implies $E(\infty,f^{(k)})=E(\infty,g^{(k)})$, we get

$$T\left(r,f^{(k)}\right)\le 2\overline{N}\left(r,f^{(k)}\right)+N_2\left(r,\frac{1}{f^{(k)}}\right)+N_2\left(r,\frac{1}{g^{(k)}}\right)+S(r,f)+S(r,g) \quad (2.6)$$

Similarly,

$$T\left(r,g^{(k)}\right)\le 2\overline{N}\left(r,g^{(k)}\right)+N_2\left(r,\frac{1}{g^{(k)}}\right)+N_2\left(r,\frac{1}{f^{(k)}}\right)+S(r,f)+S(r,g) \quad (2.7)$$

From (2.6) and (2.7), we obtain

$$\lim_{\substack{r\to\infty\\ r\notin E}}\frac{\overline{N}\left(r,f^{(k)}\right)+N_2\left(r,\frac{1}{f^{(k)}}\right)+\overline{N}\left(r,g^{(k)}\right)+N_2\left(r,\frac{1}{g^{(k)}}\right)}{T\left(r,f^{(k)}\right)+T\left(r,g^{(k)}\right)}\ge\frac{1}{2}$$

which is contradiction to our hypothesis.
Hence, $\Theta(z)\equiv 0$. That is

$$\frac{f^{(k+2)}(z)}{f^{(k+1)}(z)}-\frac{2f^{(k+1)}(z)}{f^{(k)}-1}=\frac{g^{(k+2)}(z)}{g^{(k+1)}(z)}-\frac{2g^{(k+1)}(z)}{g^{(k)}(z)-1}$$

Solving above equation, we obtain

$$f^{(k)}=\frac{ag^{(k)}+b}{cg^{(k)}+d}, \quad (2.8)$$

where a,b,c,d are complex numbers such that $ad-bc\neq 0$.
From (2.8), we get

$$T\left(r,f^{(k)}\right)=T\left(r,g^{(k)}\right)+O(1). \quad (2.9)$$

We now consider the following cases
Case 1: Let $ac\neq 0$, then from (2.8), we have

$$f^{(k)}-\frac{a}{c}=\frac{b-\frac{ad}{c}}{cg^{(k)}+d}.$$

By the second fundamental theorem, we have

$$\begin{aligned}T\left(r,f^{(k)}\right)&\le\overline{N}\left(r,f^{(k)}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-a/c}\right)+S(r,f)\\&\le\overline{N}\left(r,f^{(k)}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}}\right)+\overline{N}\left(r,g^{(k)}\right)+S(r,f) \quad (2.10)\\&\le\overline{N}\left(r,f^{(k)}\right)+\overline{N}\left(r,g^{(k)}\right)+N_2\left(r,\frac{1}{f^{(k)}}\right)+S(r,f).\end{aligned}$$

Similarly

$$T\left(r,g^{(k)}\right)\le\overline{N}\left(r,g^{(k)}\right)+\overline{N}\left(r,f^{(k)}\right)+N_2\left(r,\frac{1}{g^{(k)}}\right)+S(r,g). \quad (2.11)$$

From (2.10) and (2.11), we obtain

$$\lim_{\substack{r\to\infty\\ r\notin E}}\frac{\overline{N}\left(r,f^{(k)}\right)+N_2\left(r,\frac{1}{f^{(k)}}\right)+\overline{N}\left(r,g^{(k)}\right)+N_2\left(r,\frac{1}{g^{(k)}}\right)}{T\left(r,f^{(k)}\right)+T\left(r,g^{(k)}\right)}\ge\frac{1}{2},$$

which is contradiction to our hypothesis.
Case 2: Let $ac=0$. Since $ad-bc\neq 0$, it follows that a and c are not simultaneously zero.
Let $a=0$. Then from (2.8), we get

$$g^{(k)}+\frac{d}{c}=\frac{b}{cf^{(k)}}, \tag{2.12}$$

where $bc\neq 0$.
If $d\neq 0$, from (2.12) we get by the Second Fundamental Theorem

$$\begin{aligned} T\left(r,g^{(k)}\right) &\leq \overline{N}\left(r,g^{(k)}\right)+\overline{N}\left(r,\frac{1}{g^{(k)}}\right)+\overline{N}\left(r,\frac{1}{g^{(k)}+d/c}\right)+S(r,g)\\ &\leq \overline{N}\left(r,g^{(k)}\right)+\overline{N}\left(r,\frac{1}{g^{(k)}}\right)+\overline{N}\left(r,f^{(k)}\right)+S(r,g)\\ &\leq \overline{N}\left(r,f^{(k)}\right)+\overline{N}\left(r,g^{(k)}\right)+N_2\left(r,\frac{1}{g^{(k)}}\right)+S(r,f). \end{aligned}$$

Similarly

$$T\left(r,f^{(k)}\right)\leq \overline{N}\left(r,f^{(k)}\right)+\overline{N}\left(r,g^{(k)}\right)+N_2\left(r,\frac{1}{f^{(k)}}\right)+S(r,g).$$

We get a contradiction as in case 1.
Let $d=0$. Then from (2.8), we get

$$g^{(k)}f^{(k)}=\frac{b}{c} \tag{2.13}$$

Since $E(\infty,f)=E(\infty,g)$, we get $E\left(\infty,f^{(k)}\right)=E\left(\infty,g^{(k)}\right)$, it follows from (2.13) that $f^{(k)}$ has no zero and pole. Hence there exists $z_0\in\mathbb{C}$ such that $f^{(k)}(z_0)=g^{(k)}(z_0)=1$, since $E\left(1,f^{(k)}\right)=E\left(1,g^{(k)}\right)$. So from (2.13), we get $b/c=1$ and so $f^{(k)}g^{(k)}\equiv 1$.
Let $c=0$. Then from (2.8), we get

$$f^{(k)}=\frac{a}{d}g^{(k)}+\frac{b}{d}, \tag{2.14}$$

where $ad\neq 0$.
If $b\neq 0$, from (2.14), we get, by the Second Fundamental Theorem

$$\begin{aligned} T\left(r,f^{(k)}\right) &\leq \overline{N}\left(r,f^{(k)}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-b/d}\right)+S(r,f)\\ &\leq \overline{N}\left(r,f^{(k)}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}}\right)+\overline{N}\left(r,\frac{1}{g^{(k)}}\right)+S(r,f)\\ &\leq \overline{N}\left(r,f^{(k)}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}}\right)+N_2\left(r,\frac{1}{g^{(k)}}\right)+S(r,f). \end{aligned}$$

Similarly

$$T\left(r,g^{(k)}\right)\leq \overline{N}\left(r,g^{(k)}\right)+\overline{N}\left(r,\frac{1}{g^{(k)}}\right)+N_2\left(r,\frac{1}{f^{(k)}}\right)+S(r,g).$$

We get a contradiction as in case 1.
Let $b=0$. Then from (2.14), we get

$$f^{(k)}=\frac{a}{d}g^{(k)} \tag{2.15}$$

If $f^{(k)}$ has no 1 - point, by the Second Fundamental Theorem, we get

$$\begin{aligned} T\left(r, f^{(k)}\right) &\leq \overline{N}\left(r, f^{(k)}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}}\right) + S(r,f) \\ &\leq \overline{N}\left(r, f^{(k)}\right) + \overline{N}\left(r, g^{(k)}\right) + N_2\left(r, \frac{1}{f^{(k)}}\right) + S(r,f) \end{aligned} \tag{2.16}$$

Similarly

$$T\left(r, g^{(k)}\right) \leq \overline{N}\left(r, g^{(k)}\right) + \overline{N}\left(r, f^{(k)}\right) + N_2\left(r, \frac{1}{g^{(k)}}\right) + S(r,g). \tag{2.17}$$

From (2.16) and (2.17), we get a contradiction as in case 1.
Let $f^{(k)}(z_0) = 1$ for some $z_0 \in \mathbb{C}$. Since $E(1, f^{(k)}) = E(1, g^{(k)})$, we get $g^{(k)}(z_0) = 1$ and so from (2.15) it follows that $a/d = 1$. Therefore $f^{(k)} \equiv g^{(k)}$. This completes the proof of Lemma 2.3.

3. **Proof of Theorem 1.1**

Let

$$F = -\frac{1}{b}(f)^{n-1}(f+a) \quad \textit{and} \quad G = -\frac{1}{b}(g)^{n-1}(g+a). \tag{3.1}$$

Therefore

$$\overline{N}(r,F) = \overline{N}(r,f) \quad \textit{and} \quad \overline{N}(r,G) = \overline{N}(r,g). \tag{3.2}$$

We have

$$N_2\left(r, \frac{1}{F}\right) = \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}_{(2}\left(r, \frac{1}{F}\right), \tag{3.3}$$

where

$$\overline{N}\left(r, \frac{1}{F}\right) = \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f+a}\right) \tag{3.4}$$

$$\overline{N}_{(2}\left(r, \frac{1}{F}\right) = \overline{N}_{1)}\left(r, \frac{1}{f}\right) + \overline{N}_{(2}\left(r, \frac{1}{f}\right) + \overline{N}_{(2}\left(r, \frac{1}{f+a}\right). \tag{3.5}$$

By our hypothesis, $N_{1)}\left(r, \frac{1}{f}\right) = S(r,f)$ and from (3.3), (3.4) and (3.5), we get

$$N_2\left(r, \frac{1}{F}\right) \leq N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f+a}\right) + S(r,f). \tag{3.6}$$

Similarly

$$N_2\left(r, \frac{1}{G}\right) \leq N\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g+a}\right) + S(r,g). \tag{3.7}$$

Adding (3.6) and (3.7), we get

$$\begin{aligned} N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) &\leq N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f+a}\right) + N\left(r, \frac{1}{g}\right) \\ &\quad + N\left(r, \frac{1}{g+a}\right) + S(r,f) + S(r,g) \\ &\leq 2\left(T(r,f) + T(r,g)\right) + S(r,f) + S(r,g). \end{aligned} \tag{3.8}$$

From (3.2) and (3.8), we get

$$\begin{aligned} &\overline{N}(r,F) + N_2\left(r, \frac{1}{F}\right) + \overline{N}(r,G) + N_2\left(r, \frac{1}{G}\right) \\ &\quad \leq \overline{N}(r,f) + \overline{N}(r,g) + 2\left(T(r,f) + T(r,g)\right) + S(r,f) + S(r,g) \end{aligned} \tag{3.9}$$

Since $\Theta(\infty,f)>\frac{2}{n-1}$ and $\Theta(\infty,g)>\frac{2}{n-1}$, (hypothesis of the theorem) and from (3.9), we get

$$\begin{aligned}
&\lim_{\substack{r\to\infty\\ r\notin E}}\frac{\overline{N}\left(r,F\right)+N_2\left(r,\frac{1}{F}\right)+\overline{N}\left(r,G\right)+N_2\left(r,\frac{1}{G}\right)}{T\left(r,F\right)+T\left(r,G\right)}\\
<\;&\lim_{\substack{r\to\infty\\ r\notin E}}\frac{\left(\frac{n-3}{n-1}\right)T(r,f)+\left(\frac{n-3}{n-1}\right)T(r,g)+2\left(T(r,f)+T(r,g)\right)}{n\left[T(r,f)+T(r,g)\right]}\\
\le\;&\lim_{\substack{r\to\infty\\ r\notin E}}\frac{\left(\frac{n-3}{n-1}+2\right)\left[T(r,f)+T(r,g)\right]}{n\left[T(r,f)+T(r,g)\right]}=\frac{3n-5}{n(n-1)}\le\frac{1}{2},\qquad for\;\; n\ge 5
\end{aligned}$$

Therefore,

$$\lim_{\substack{r\to\infty\\ r\notin E}}\frac{\overline{N}\left(r,F\right)+N_2\left(r,\frac{1}{F}\right)+\overline{N}\left(r,G\right)+N_2\left(r,\frac{1}{G}\right)}{T\left(r,F\right)+T\left(r,G\right)}<\frac{1}{2},\qquad for\;\; n\ge 5$$

and also $E[1,F]=E[1,G]$, since $E[S,f]=E[S,g]$ and $E[\infty,F]=E[\infty,G]$, since $E[\infty,f]=E[\infty,g]$. Therefore by Lemma 2.3 for $k=0$, we get either $F\equiv G$ or $FG\equiv 1$.

Consider $FG\equiv 1$, that is,

$$\left[-\frac{1}{b}(f)^{n-1}(f+a)\right]\;\left[-\frac{1}{b}(g)^{n-1}(g+a)\right]\equiv 1,$$

$$(f)^{n-1}(f+a)\;(g)^{n-1}(g+a)\equiv b^2. \tag{3.10}$$

If F has no poles, then f has no poles. Then from (3.10) it follows that g has neither zero nor $-a$ point. So by the deficiency relation we get $\Theta(\infty,f)=0$, which contradicts the given condition $\Theta(\infty,f)>\frac{2}{n-1}$.

If z_0 is a pole of F, then z_0 is a pole of f, it follows that z_0 is either a zero or $-a$ point of g and this contradicts $E(\{\infty\},f)=E(\{\infty\},g)$.

Thus, $FG\equiv 1$ is not possible. Therefore $F\equiv G$, that is

$$-\frac{1}{b}(f)^{n-1}(f+a)\equiv-\frac{1}{b}(g)^{n-1}(g+a).$$

Suppose $f\not\equiv g$

(i) Let $h=g/f$. Then $f=\frac{1-h^{n-1}}{1-h^n}$ and $g=\frac{\left(1-h^{n-1}\right)h}{1-h^n}$.

$$T(r,f)=(n-1)T[r,h]$$

$$\overline{N}(r,f)=\sum_{j=1}^{n-1}\overline{N}\left(r,\frac{1}{h-\alpha_j}\right)\ge(n-3)T(r,g),$$

where $\alpha_j\ne 1$ ($j=1,2,\cdots,n-1$ are roots of the algebraic equation $h^n=1$. Therefore

$$1-\overline{\lim_{r\to\infty}}\frac{\overline{N}(r,f)}{T(r,f)}\le 1-\overline{\lim_{r\to\infty}}\frac{(n-3)T(r,h)}{(n-1)T(r,h)}\le 1-\frac{n-3}{n-1}=\frac{2}{n-1},$$

that is

$$\Theta(\infty,f)\le\frac{2}{n-1},$$

which is a contradiction to our hypothesis $\Theta(\infty,f)>\frac{2}{n-1}$.

Thus $f\equiv g$.

This completes the proof of Theorem 1.1.

Acknowledgments. The authors would like to thank the referee for his/her comments and careful observations that helped us to improve this article.

References

[1] G. Frank, M. Reinders, *A unique range set for meromorphic function with 11 elements*, Complex Variables, Theory Appl. **37(1-4)**(1998) 185-193.
[2] F. Gross, *Factorization of meromorphic functions and some open problems*, In Lecture Notes in Math., Springer, Berlin **599**(1977) 51-69.
[3] W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, (1964).
[4] I. Lahiri, *On a question of Hong Xun Yi*, Arch. Math (Brno), **38**, (2002), 119-128.
[5] Yan. Xu, *Meromorphic functions sharing one finite set*, Comput.Math.Appl, **45**, (2003), 1489-1495.
[6] C. C. Yang , *On deficiencies of differential polynomials*, Math Z., **125**, (1972), 107-112.
[7] L. Yang , *Value Distribution Theory*, Springer Verlag, Berli, (1993).
[8] H. X. Yi, C. C Yang,*Uniqueness Theory of Meromorphic Functions*, Science Press, Beijing, (1995).
[9] H. X. Yi ,*Unicity theorems for meromorphic or entire functions II*, Bull. Austral. Math. Soc., **52**, (1995), 215-224.

Subhas S Bhoosnurmath
Deparment of Mathematics, Karnatak University, Dharwad, Karnataka state, India
E-mail address: ssbmath@gmail.com

Renukadevi S Dyavanal
Deparment of Mathematics, Karnatak University, Dharwad, Karnataka state, India
E-mail address: renukadyavanal@gmail.com