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FINITE BLASCHKE PRODUCTS AND CIRCLES THAT PASS THROUGH THE ORIGIN

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Abstract. Let $B(z)$ be a finite Blaschke product of degree n and $\mathbf{C}$ be the unit circle. It is well-known that for any given Blaschke product $B(z)$ of degree n and any specified point λ of $\mathbf{C}$, there exist n distinct points of $\mathbf{C}$ that $B(z)$ maps to λ. In this paper, we discuss the determination of these points using circles passing through the origin.

1. Introduction

A Blaschke product of degree n is a function defined by

$$B(z) = \beta \prod_{i=1}^{n} \frac{z - a_i}{1 - \overline{a_i} z} \tag{1.1}$$

where $|\beta| = 1$ and the a_i are complex numbers of modulus less than one for $1 \leq i \leq n$. The degree is simply the number of zeros of B counted according to their multiplicity: the Blaschke product has zeros precisely at the points a_1, a_2, ..., a_n and a zero that appears exactly m times in this list is said to be of multiplicity m. This is the general form for a rational function which takes the closed unit disc $\overline{\mathbf{D}} = \{z : |z| \leq 1\}$ to itself, and it is usually referred as a finite Blaschke product. $B(z)$ is an n-to-one map of $\overline{\mathbf{D}}$ onto itself and has modulus one on the unit circle $\mathbf{C} = \{z : |z| = 1\}$, (see [8]).

In this paper, we look at some geometric properties of finite Blaschke products. It is well-known that for any given Blaschke product B of degree n and any specified point λ of $\mathbf{C}$, there exist n distinct points of $\mathbf{C}$ that B maps to λ. We shall try to determine the points of $\mathbf{C}$ that B maps to λ for Blaschke products of degree n using circles passing through the origin. We start with the Blaschke products of degree two and three and then, we give an open problem for Blaschke products of degree $n \geq 4$.

In Section 2 and Section 3, we obtain the complete solutions of our problem for the Blaschke products of degrees 2 and 3, respectively. In these cases, our results

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are based on the two theorems obtained in [3]. In Section 4, we deal with the cases $n \geq 4$ and we solve the open problem for some special cases. We use the notion of decomposition of finite Blaschke products (see [2], [6] and [11]) and following uniqueness theorem for monic Blaschke products (see [7]):

Theorem 1.1. [7] *Let*

$$A(z) = \prod_{j=1}^{n}((z-a_j)/(1-\overline{a_j}z)) \text{ and } B(z) = \prod_{j=1}^{n}((z-b_j)/(1-\overline{b_j}z)),$$

with a_j and $b_j \in \mathbf{D} = \{z : |z| < 1\}$ for $j = 1, ..., n$. Suppose that $A(\lambda_j) = B(\lambda_j)$ for n distinct points $\lambda_1, ..., \lambda_n$ in $\mathbf{D}$. Then $A \equiv B$.

2. Blaschke Products of Degree Two

In this section, we consider the Blaschke products of the form

$$B(z) = \frac{z(z-a)}{1-\overline{a}z},$$

where $a \neq 0$, $|a| < 1$. In [3], the following theorem was proved:

Theorem 2.1. (*see* [3]*, Theorem* 2). *Let $B(z) = \frac{z(z-a)}{1-\overline{a}z}$ be a Blaschke product with $a \neq 0$. For λ in $\mathbf{C}$, let z_1 and z_2 be the two distinct points satisfying $B(z_1) = B(z_2) = \lambda$. Then the line joining z_1 and z_2 passes through the point a. Conversely, if we consider any line L through the point a, then for the points z_1 and z_2 at which L intersects $\mathbf{C}$ it is the case that $B(z_1) = B(z_2)$.*

Here we look at the circles that pass through the origin. By means of the circles through the points 0 and $\frac{1}{\overline{a}}$, we determine the two distinct points of $\mathbf{C}$ that B maps to λ.

We prove the following theorem:

Theorem 2.2. *Let $a \neq 0$ be any complex number with $|a| < 1$ and $B(z) = \frac{z(z-a)}{1-\overline{a}z}$ be a Blaschke product of degree 2. The unit circle $\mathbf{C}$ and any circle through the points 0 and $\frac{1}{\overline{a}}$ have exactly two distinct intersection points z_1 and z_2. Then we have $B(z_1) = B(z_2)$ for these intersection points.*

Conversely, for λ in $\mathbf{C}$, let z_1 and z_2 be the two distinct points satisfying $B(z_1) = B(z_2) = \lambda$. Then the circle through the points 0, z_1 and z_2 passes through the point $\frac{1}{\overline{a}}$.

Proof. For the first part of the proof we use the inversion map $z \to \frac{1}{\overline{z}}$ in the unit circle $\mathbf{C}$. Since $|a| < 1$, we have $\left|\frac{1}{\overline{a}}\right| > 1$ and therefore, any circle passing through the points 0 and $\frac{1}{\overline{a}}$ must intersect the unit circle $\mathbf{C}$ at two distinct points z_1 and z_2. The image of this circle under the inversion map $z \to \frac{1}{\overline{z}}$ is a line passing through the points z_1, z_2 and a. Then by Theorem 2.1, we have $B(z_1) = B(z_2)$.

Conversely, let λ be a fixed point on the unit circle $\mathbf{C}$ and let z_1 and z_2 be the two distinct points satisfying $B(z_1) = B(z_2) = \lambda$. Let L be the line joining z_1 and z_2. By Theorem 2.1, if L does not pass through 0, the image of L under the inversion map $w = \frac{1}{\overline{z}}$ is the circle passes through the points z_1, z_2, 0 and $\frac{1}{\overline{a}}$ or itself if L passes through 0. This completes the proof of the theorem. □

In the proof of Theorem 2.2, we use the basic properties of the inversion map $z \to \frac{1}{\overline{z}}$ in the unit circle $\mathbf{C}$. For more details about inversion maps on the complex plane one can see [1] and [10].

3. Blaschke Products of Degree Three

In this section, we consider a Blaschke product with three distinct zeros. Composing with a Möbius transformation, we may assume that one zero is at the origin and that our Blaschke product has the form

$$B(z) = z\frac{z-a_1}{1-\overline{a_1}z}\frac{z-a_2}{1-\overline{a_2}z}.$$

We have the following theorem:

Theorem 3.1. (*see* [3], *Theorem* 1). *Let B be a Blaschke product of degree three with distinct zeros at the points* 0, a_1 *and* a_2. *For λ on the unit circle, let z_1, z_2 and z_3 denote the points mapped to λ under B. Then the lines joining z_j and z_k for $j \neq k$ are tangent to the ellipse E with equation*

$$|w-a_1| + |w-a_2| = |1-\overline{a_1}a_2|. \tag{3.1}$$

Conversely, every point on E is the point of tangency of a line segment joining two distinct points z_1 and z_2 on the unit circle for which $B(z_1) = B(z_2)$.

Let F be the image of the ellipse E with equation (3.1) under the inversion map $w = \frac{1}{\overline{z}}$. It can be easily seen that the curve F has the equation

$$|a_1|\left|z-\frac{1}{\overline{a_1}}\right| + |a_2|\left|z-\frac{1}{\overline{a_2}}\right| = |1-\overline{a_1}a_2|\,|z|.$$

Now we prove the following theorem.

Theorem 3.2. *Let $a_1 \neq 0$ and $a_2 \neq 0$ be any distinct complex numbers with $|a_1| < 1$, $|a_2| < 1$ and $B(z) = z\frac{z-a_1}{1-\overline{a_1}z}\frac{z-a_2}{1-\overline{a_2}z}$ be a Blaschke product of degree* 3. *The unit circle* **C** *and any circle through the point* 0 *and tangent to the curve F with equation*

$$|a_1|\left|z-\frac{1}{\overline{a_1}}\right| + |a_2|\left|z-\frac{1}{\overline{a_2}}\right| = |1-\overline{a_1}a_2|\,|z| \tag{3.2}$$

have exactly two distinct intersection points z_1 and z_2. Then we have $B(z_1) = B(z_2)$ for these intersection points.

Conversely, for λ on the unit circle **C**, *let z_1, z_2 and z_3 be the three distinct points satisfying $B(z_1) = B(z_2) = B(z_3) = \lambda$. Then the circle through the points z_j, z_k, and* 0 *($j \neq k$ and $1 \leq j, k \leq 3$) is tangent to the curve F with equation (3.2).*

Proof. Since the ellipse E with equation (3.1) is contained in the unit disc (see [5], p. 785), the curve F (being the image of the ellipse E under the inversion map $w = \frac{1}{\overline{z}}$) lies outside of the unit circle. Therefore, any circle passing through the point 0 and tangent to the curve F must intersect the unit circle **C** at two distinct points z_1 and z_2. The image of any such circle under the inversion map $z \to \frac{1}{\overline{z}}$ is a line passing through the points z_1 and z_2 and tangent to the ellipse E with equation (3.1). Then by Theorem 3.1, we have $B(z_1) = B(z_2)$.

Conversely, let λ be a fixed point on the unit circle **C** and let z_1, z_2 and z_3 be the three distinct points satisfying $B(z_1) = B(z_2) = B(z_3) = \lambda$. Let γ be any circle that passes through the points 0, z_j and z_k for $j \neq k$ and $1 \leq j, k \leq 3$. Let L be any line joining z_j and z_k. Then by Theorem 3.1, the image of L under the inversion map $w = \frac{1}{\overline{z}}$ is a circle through the points z_j, z_k, and 0 and tangent to the curve F with equation (3.2) since the inversion map $w = \frac{1}{\overline{z}}$ is an anti-conformal map. This completes the proof of the theorem. □

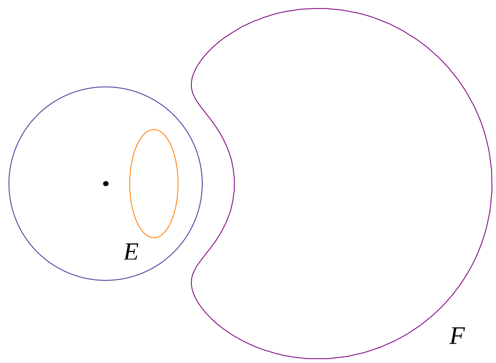

FIGURE 1. Blaschke product of degree 3 with $a_1 = \frac{1}{2} - \frac{1}{2}i$ and $a_2 = \frac{1}{2} + \frac{1}{2}i$; the ellipse E and its Blaschke mate F.

If any two circles passing through the point 0 and tangent to the curve F with equation (3.2) have only one common intersection point with the unit circle $\mathbf{C}$, then we have found the three points z_1, z_2 and z_3 of $\mathbf{C}$ satisfying $B(z_1) = B(z_2) = B(z_3) = \lambda$.

We call the curve F with equation (3.2) as the *Blaschke mate* of the ellipse E with equation (3.1) (see Figure 1).

4. Blaschke Products of Higher Degree

Let $B(z) = z \prod_{i=1}^{n-1} \frac{z-a_i}{1-\overline{a_i}z}$ be a Blaschke product with n distinct zeros. In the cases $n = 2$ or $n = 3$, we have seen that the circles or lines, pass through 0 and have a common property, are enough to determine the points z_i and z_j on the unit circle for which $B(z_i) = B(z_j)$. We hope to find the corresponding common property for Blaschke products of degree $n \geq 4$. It seems that it is not easy to find a complete solution of this problem for $n \geq 4$. Now we formulate our problem as an open problem.

Open Problem 1. *Let* $B(z) = z \prod_{i=1}^{n-1} \frac{z-a_i}{1-\overline{a_i}z}$ *be a Blaschke product with* n *distinct zeros and of degree* $n \geq 4$. *Let* z_i *and* z_j *be any intersection points of the unit circle with a circle or a line through* 0.

For which circles or lines through 0 *do we have* $B(z_i) = B(z_j)$*?*

In this section, we solve this open problem for two special cases. At first, we give the answer of the open problem for some Blaschke products of degree $2n$.

Now we consider a Blaschke product with four distinct zeros and we give the following theorem:

Theorem 4.1. *Let a_1, a_2, a_3 be three distinct nonzero complex numbers with $|a_i| < 1$ for $1 \leq i \leq 3$ and $B(z) = z\prod\limits_{i=1}^{3} \frac{z-a_i}{1-\overline{a_i}z}$ be a Blaschke product of degree 4 with the condition that one of its zeros, say a_1, satisfies the following equation:*

$$a_1 + \overline{a_1}a_2a_3 = a_2 + a_3. \tag{4.1}$$

i) If L is any line through the point a_1, then for the points z_1 and z_2 at which L intersects $\mathbf{C}$, we have $B(z_1) = B(z_2)$.

ii) The unit circle $\mathbf{C}$ and any circle through the points 0 and $\frac{1}{\overline{a_1}}$ have exactly two distinct intersection points z_1 and z_2. Then we have $B(z_1) = B(z_2)$ for these intersection points.

Proof. Let $B(z)$ be any Blaschke product of degree 4 with the condition that one of its zeros, say a_1, satisfies the equation (4.1). At first we show that $B(z)$ can be written as a composition of two Blaschke products of degrees 2 as

$$B(z) = B_2 \circ B_1(z),$$

where

$$B_1(z) = \frac{z(z-a_1)}{1-\overline{a_1}z} \text{ and } B_2(z) = \frac{z(z+a_2a_3)}{1+\overline{a_2a_3}z}.$$

Indeed, it is clear that

$$B_2 \circ B_1(0) = B_2 \circ B_1(a_1) = 0.$$

Using the equation (4.1), after some straightforward computations, we have also

$$B_2 \circ B_1(a_2) = B_2 \circ B_1(a_3) = 0.$$

Then by Theorem 1.1, we obtain $B \equiv B_2 \circ B_1$.

(i) Let L be any line through the point a_1 and z_1, z_2 be the points at which L intersects $\mathbf{C}$. By Theorem 2.1, we have $B_1(z_1) = B_1(z_2)$. Hence we obtain

$$B(z_1) = B_2(B_1(z_1)) = B_2(B_1(z_2)) = B(z_2).$$

(ii) Let us consider any circle through the points 0 and $\frac{1}{\overline{a_1}}$. Since $|a_1| < 1$, we have $\left|\frac{1}{\overline{a_1}}\right| > 1$. Therefore, any circle passing through the points 0 and $\frac{1}{\overline{a_1}}$ must intersect the unit circle $\mathbf{C}$ at two distinct points z_1 and z_2. The image of this circle under the inversion map $z \to \frac{1}{\overline{z}}$ is a line passing through the points z_1, z_2 and a_1. Then by the Case (i) we have $B(z_1) = B(z_2)$. □

For the converse of Theorem 4.1 (i), let $B(z) = B_2(B_1(z)) = \lambda$ for a fixed $\lambda \in \mathbf{C}$ and let $u = B_1(z)$. Then there are 2 distinct points u_1 and u_2 of the unit circle $\mathbf{C}$ such that $B_2(u_k) = \lambda$ where $1 \leq k \leq 2$ (notice that the line joining u_1 and u_2 passes through the point $-a_2a_3$ by Theorem 2.1). For every u_k, there are two points z_{k_1} and z_{k_2} of the unit circle $\mathbf{C}$ such that $B_1(z_{k_1}) = B_1(z_{k_2}) = u_k$. By Theorem 2.1, the line joining z_{k_1} and z_{k_2} passes through the point a_1. Therefore we have 2 lines pass through the point a_1, (see Figure 2). Now we can extend the arguments used in the Theorem 4.1 for Blaschke products of degree $2n$.

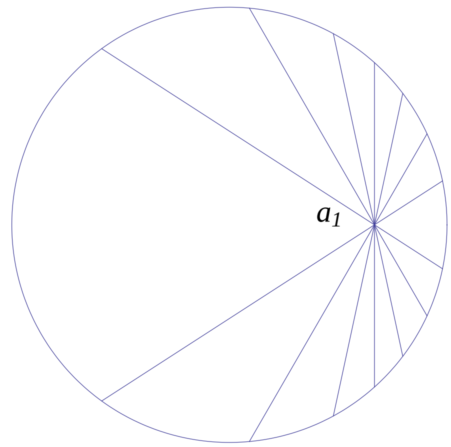

FIGURE 2. Blaschke product of degree 4 with $a_1 = \frac{2}{3}$, $a_2 = \frac{1}{2} - \frac{1}{2}i$ and $a_3 = \frac{1}{2} + \frac{1}{2}i$.

Theorem 4.2. *Let a_1, a_2, ..., a_{2n-1} be $2n-1$ distinct nonzero complex numbers with $|a_i| < 1$ for $1 \leq i \leq 2n-1$ and $B(z) = z \prod_{i=1}^{2n-1} \frac{z-a_i}{1-\overline{a_i}z}$ be a Blaschke product of degree $2n$ with the condition that one of its zeros, say a_1, satisfies the following equations:*

$$a_1 + \overline{a_1}a_2a_3 = a_2 + a_3,$$
$$a_1 + \overline{a_1}a_4a_5 = a_4 + a_5,$$
$$\ldots$$
$$a_1 + \overline{a_1}a_{2n-2}a_{2n-1} = a_{2n-2} + a_{2n-1}.$$

i) If L is any line through the point a_1, then for the points z_1 and z_2 at which L intersects $\mathbf{C}$, we have $B(z_1) = B(z_2)$.

ii) The unit circle $\mathbf{C}$ and any circle through the points 0 and $\frac{1}{\overline{a_1}}$ have exactly two distinct intersection points z_1 and z_2. Then we have $B(z_1) = B(z_2)$ for these intersection points.

Proof. By the same arguments used in the proof of Theorem 4.1, we can show that $B(z)$ can be written as a composition of two Blaschke products of degrees 2 and n as

$$B(z) = B_2 \circ B_1(z),$$

where

$$B_1(z) = \frac{z(z-a_1)}{1-\overline{a_1}z}$$

and

$$B_2(z) = \frac{z(z+a_2a_3)(z+a_4a_5)\ldots(z+a_{2n-2}a_{2n-1})}{(1+\overline{a_2a_3}z)\,(1+\overline{a_4a_5}z)\ldots(1+\overline{a_{2n-2}a_{2n-1}}z)}.$$

Then the proof follows similarly. □

Theorem 4.1 is the special case of Theorem 4.2 for $n = 2$.

For the converse of Theorem 4.2 (i), let $B(z) = B_2(B_1(z)) = \lambda$ for a fixed $\lambda \in \mathbf{C}$ and let $u = B_1(z)$. Then there are n distinct points u_1, u_2, ..., u_n of the unit circle $\mathbf{C}$ such that $B_2(u_k) = \lambda$ where $1 \leq k \leq n$. For every u_k, there are two points z_{k_1}

and z_{k_2} of the unit circle $\mathbf{C}$ such that $B_1(z_{k_1}) = B_1(z_{k_2}) = u_k$. By Theorem 2.1, the line joining z_{k_1} and z_{k_2} passes through the point a_1. Therefore we have n lines pass through the point a_1.

Now we consider the Blaschke products of degree 6 with six distinct zeros. We can give the following theorem:

Theorem 4.3. *Let a_1, a_2, a_3, a_4, a_5 be five distinct nonzero complex numbers with $|a_i| < 1$ for $1 \leq i \leq 5$ and $B(z) = z\prod_{i=1}^{5}\frac{z-a_i}{1-\overline{a_i}z}$ be a Blaschke product of degree 6 with the condition that two of its zeros, say a_1 and a_2, satisfy the following equations:*

$$a_1 + a_2 + a_3a_4a_5\overline{a_1a_2} = a_3 + a_4 + a_5 \tag{4.2}$$

and

$$a_1a_2 + a_3a_4a_5(\overline{a_1} + \overline{a_2}) = a_3a_4 + a_3a_5 + a_4a_5. \tag{4.3}$$

i)Let E be the ellipse with equation

$$|z - a_1| + |z - a_2| = |1 - \overline{a_1}a_2| \tag{4.4}$$

and L be any line tangent to the ellipse E. For the points z_1 and z_2 on the unit circle $\mathbf{C}$ at which L intersects $\mathbf{C}$, we have $B(z_1) = B(z_2)$.

ii) Let F be the image of the ellipse E under the inversion map $w = \frac{1}{z}$. The curve F has the equation

$$|a_1|\left|z - \frac{1}{\overline{a_1}}\right| + |a_2|\left|z - \frac{1}{\overline{a_2}}\right| = |1 - \overline{a_1}a_2|\,|z|\,.$$

Then the unit circle $\mathbf{C}$ and any circle through the point 0 and tangent to the curve F have exactly two distinct intersection points z_1 and z_2. For these intersection points we have $B(z_1) = B(z_2)$.

Proof. Let $B(z)$ be any Blaschke product of degree 6 with the condition that two of its zeros, say a_1 and a_2, satisfy the equations (4.2) and (4.3). Using Theorem 1.1, we show that $B(z)$ can be written as a composition of two Blaschke products of degrees 3 and 2 as $B(z) = B_2 \circ B_1(z)$ where

$$B_1(z) = \frac{z(z-a_1)(z-a_2)}{(1-\overline{a_1}z)(1-\overline{a_2}z)} \text{ and } B_2(z) = \frac{z(z-a_3a_4a_5)}{1-\overline{a_3a_4a_5}z}.$$

Indeed, it is clear that

$$B_2 \circ B_1(0) = B_2 \circ B_1(a_1) = B_2 \circ B_1(a_2) = 0.$$

Using the equations (4.2) and (4.3), it can be easily checked that

$$B_2 \circ B_1(a_3) = B_2 \circ B_1(a_4) = B_2 \circ B_1(a_5) = 0.$$

Then by Theorem 1.1, we obtain $B \equiv B_2 \circ B_1$.

Now the rest of the proof can be easily seen by the same arguments used in the proofs of Theorem 3.2 and Theorem 4.1. □

For the converse of Theorem 4.3 (i), let $B(z) = B_2(B_1(z)) = \lambda$ for a fixed $\lambda \in \mathbf{C}$. Then there are 2 distinct points u_1 and u_2 of the unit circle $\mathbf{C}$ such that $B_2(u_k) = \lambda$ where $1 \leq k \leq 2$ (notice that the line joining u_1 and u_2 passes through the point $a_3a_4a_5$ by Theorem 2.1). For every u_k, there are three points z_{k_1}, z_{k_2} and z_{k_3} of the unit circle $\mathbf{C}$ such that $B_1(z_{k_1}) = B_1(z_{k_2}) = B_1(z_{k_3}) = u_k$. By Theorem 3.2, the line joining z_{k_i} and z_{k_j} for $i \neq j$ is tangent to the ellipse E with equation (4.4). Therefore we have 6 lines tangent to the ellipse E.

We can extend the arguments used in the Theorem 4.3 for Blaschke products of degree $3n$.

Theorem 4.4. *Let a_1, a_2, ..., a_{3n-1} be $3n-1$ distinct nonzero complex numbers with $|a_i| < 1$ for $1 \leq i \leq 3n-1$ and $B(z) = z \prod_{i=1}^{3n-1} \frac{z-a_i}{1-\overline{a_i}z}$ be a Blaschke product of degree $3n$ with the condition that two of its zeros, say a_1 and a_2, satisfy the following equations:*

$$\begin{aligned} a_1 + a_2 + a_3a_4a_5\overline{a_1a_2} &= a_3 + a_4 + a_5, \\ a_1a_2 + a_3a_4a_5(\overline{a_1} + \overline{a_2}) &= a_3a_4 + a_3a_5 + a_4a_5, \\ a_1 + a_2 + a_6a_7a_8\overline{a_1a_2} &= a_6 + a_7 + a_8, \\ a_1a_2 + a_6a_7a_8(\overline{a_1} + \overline{a_2}) &= a_6a_7 + a_6a_8 + a_7a_8, \\ &\ldots \\ a_1 + a_2 + a_{3n-3}a_{3n-2}a_{3n-1}\overline{a_1a_2} &= a_{3n-3} + a_{3n-2} + a_{3n-1}, \\ a_1a_2 + a_{3n-3}a_{3n-2}a_{3n-1}(\overline{a_1} + \overline{a_2}) &= a_{3n-3}a_{3n-2} + a_{3n-3}a_{3n-1} + a_{3n-2}a_{3n-1}. \end{aligned} \tag{4.5}$$

i) Let E be the ellipse with equation

$$|z - a_1| + |z - a_2| = |1 - \overline{a_1}a_2|$$

and L be any line tangent to the ellipse E. For the points z_1 and z_2 on the unit circle $\mathbf{C}$ at which L intersects $\mathbf{C}$, we have $B(z_1) = B(z_2)$.

ii) Let F be the image of the ellipse E under the inversion map $w = \frac{1}{\overline{z}}$. The curve F has the equation

$$|a_1| \left| z - \frac{1}{\overline{a_1}} \right| + |a_2| \left| z - \frac{1}{\overline{a_2}} \right| = |1 - \overline{a_1}a_2| \, |z| \, .$$

The unit circle $\mathbf{C}$ and any circle through the point 0 and tangent to the curve F have exactly two distinct intersection points z_1 and z_2. Then we have $B(z_1) = B(z_2)$ for these intersection points.

Proof. By the same arguments used in the proof of Theorem 4.3, we can show that $B(z)$ can be written as a composition of two Blaschke products of degrees 3 and n as $B(z) = B_2 \circ B_1(z)$ where

$$B_1(z) = \frac{z(z - a_1)(z - a_2)}{(1 - \overline{a_1}z)(1 - \overline{a_2}z)}$$

and

$$B_2(z) = \frac{z(z - a_3a_4a_5)(z - a_6a_7a_8)...(z - a_{3n-3}a_{3n-2}a_{3n-1})}{(1 - \overline{a_3a_4a_5}z)(1 - \overline{a_6a_7a_8}z)...(1 - \overline{a_{3n-3}a_{3n-2}a_{3n-1}}z)}.$$

Then the proof follows similarly. □

Theorem 4.3 is the special case of Theorem 4.4 for $n = 2$.

For the converse of Theorem 4.4 (i), let $B(z) = B_2(B_1(z)) = \lambda$ for a fixed $\lambda \in \mathbf{C}$. Then there are n distinct points u_1, u_2, ..., u_n of the unit circle $\mathbf{C}$ such that $B_2(u_k) = \lambda$ where $1 \leq k \leq n$. For every u_k, there are three points z_{k_1}, z_{k_2} and z_{k_3} of the unit circle $\mathbf{C}$ such that $B_1(z_{k_1}) = B_1(z_{k_2}) = B_1(z_{k_3}) = u_k$. By Theorem 3.2, the line joining z_{k_i} and z_{k_j} for $i \neq j$ is tangent to the ellipse E with equation (4.4). Therefore we have $3n$ lines tangent to the ellipse E.

Finally, we note that for a degree n Blaschke product the role of the ellipse plays for degree 3 is replaced by algebraic curves of higher class. These curves are referred to by some authors as Poncelet curves (see [4] and [9]).

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