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EXISTENCE OF POSITIVE RADIAL SOLUTIONS FOR SOME NONLINEAR ELLIPTIC SYSTEMS

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G.A. AFROUZI, T.A. ROUSHAN

Abstract. In this paper we study a class of nonvariational elliptic systems, by using the Gidas-Spruck Blow-up method. first, we obtain a priori estimates, and then using Leray-Schauder topological degree theory, we establish the existence of positive radial solutions vanishing at infinity.

1. Introduction

The aim of this paper is to prove the existence of radial positive solutions, vanishing at infinity, the so-called fundamental states, for the system

$$\begin{cases} -\Delta_p u = \lambda\, f(x,u,v), & \text{in } \mathbb{R}^N, \\ -\Delta_q v = \mu\, g(x,u,v), & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

under superlinear assumptions on the nonlinearities. Here $1 < p, q < N$; f and g are real-valued functions; λ and μ are positive parameters.

Djellit and Tas in [2] investigated the system (1.1) by using fixed point theorems. In this work, following the same ideas in [3] (A. Djellit, M. Moussaoui, S. Tas, Existence of radial positive solutions vanishing at infinity for asymptotically homogeneous systems, Electronic J. Diff. Eqns., Vol. 2010(2010), No. 54, 1-10.), we establish existence of nontrivial positive radial solution vanishing at infinity, namely "ground states", under the following hypotheses, for system (1.1). For other related works in the literature, we refer the reader to [1, 5, 6].

This paper is divided into three sections, organized as follows: in Section 2, we give some notation and hypotheses; Section 3 is devoted to prove existence of ground states.

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2. Notation and hypotheses

let us denote

$$X = \left\{ (u,v) \in C^0([0,+\infty[) \times C^0([0,+\infty[),\ \lim_{r\to+\infty} u(r) = \lim_{r\to+\infty} v(r) = 0 \right\}$$

the Banach space endowed with the norm

$$X = \|(u,v)\| = \|u\|_\infty + \|v\|_\infty,\ \|u\|_\infty = \sup_{r\in[0,+\infty[} |u(r)|.$$

In addition, let $\lambda = \mu = 1$ and we specify assumptions on f and g. So, we define the last functions by

$$f(|x|,u,v) = a_1(|x|)|u|^{\alpha-1}u + a_2(|x|)|v|^{\beta-1}v,$$
$$g(|x|,u,v) = a_3(|x|)|u|^{\gamma-1}u + a_4(|x|)|v|^{\delta-1}v,$$

and we assume that

(H1) the functions $a_i : [0,+\infty[\to [0,+\infty[$ is continuous for each $i = 1,\dots,4$. Also there exist $\theta_1,\theta_2 > p$; $\theta_3,\theta_4 > q$ and $R > 0$ such that $a_i(\xi) = O(\xi^{-\theta_i}), \forall \xi > R$. And $\tilde{a_i} = \min a_i(r) > 0$. $i = 2,3$.
(H2) $\min(\alpha,\beta) > p$; $\min(\gamma,\delta) > q$, i.e. system (1.1) is superlinear.

Let $K = \{(u,v) \in X,\ u \geq 0,\ v \geq 0\}$ a positive cone of X, we will show that system (1.1) has a solution in K.

A function $\phi : \mathbb{R} \to \mathbb{R}$ defined in a neighborhood at the infinity (respect. at the origin) is said asymptotically homogeneous at the infinity (respect. at the origin) of order $\rho > 0$ for all $\sigma > 0$, we have $\lim\limits_{s\to+\infty} \frac{\phi(\sigma s)}{\phi(s)} = \sigma^\rho$ (respect. $\lim\limits_{s\to 0} \frac{\phi(\sigma s)}{\phi(s)} = \sigma^\rho$).

Hence, if ϕ be one of the form $|u|^{\alpha-1}u,\ |v|^{\beta-1}v,\ |u|^{\gamma-1}u$ or $|v|^{\delta-1}v$ then it is asymptotically homogeneous of order α,β,γ and δ respectively.

let $\alpha_1 = \frac{p(q-1)+\beta q}{\beta\gamma-(p-1)(q-1)}$, $\alpha_2 = \frac{q(p-1)+\gamma p}{\beta\gamma-(p-1)(q-1)}$, $\beta_1 = \alpha_1 - \frac{N-p}{p-1}$ and $\beta_2 = \alpha_2 - \frac{N-q}{q-1}$. So, $\max(\beta_1,\ \beta_2) \geq 0$, $\alpha_1\alpha - \alpha_1(p-1) - p < 0$ and $\alpha_2\delta - \alpha_2(q-1) - q < 0$.

Proposition 2.1 ([4]). *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous, odd, asymptotically homogeneous at infinity (respect. at the origin) of order ρ such that $t\phi(t) > 0$ for all $t \neq 0$ and $\phi(t) \to \infty$ as $t \to \infty$, then*
(i) For all $\varepsilon \in]0,\rho[$, there exists $t_0 > 0$ such that $\forall t \geq t_0$ (respect. $0 \leq t \leq t_0$), $c_1 t^{\rho-\varepsilon} \leq \phi(t) \leq c_2 t^{\rho+\varepsilon}$; c_1, c_2 are positive constants. Moreover
$\forall s \in [t_0,t] :\ (\rho+1-\varepsilon)\phi(s) \leq (\rho+1+\varepsilon)\phi(t)$.
(ii) If $\{w_n\},\{t_n\}$ are real sequences such that $w_n \to w$ and $t_n \to +\infty$ (respect $t_n \to 0$) then $\lim\limits_{n\to+\infty} \frac{\phi(t_n w_n)}{\phi(t_n)} = w^\rho$.

For $h \geq 0$ and $\lambda \in [0,1]$, we define two families of operators T_h and S_λ from X to itself by $T_h(u,v) = (w,z)$ such that (w,z) satisfies the system

$$\left\{\begin{array}{l} -(r^{N-1}|w'(r)|^{p-2}w'(r))' = r^{N-1}a_1(r)(u(r))^\alpha + r^{N-1}a_2(r)[(v(r))^\beta + h] \\ \qquad\qquad\text{in } [0,+\infty[, \\ -(r^{N-1}|z'(r)|^{q-2}z'(r))' = r^{N-1}a_3(r)(u(r))^\gamma + r^{N-1}a_4(r)(v(r))^\delta \\ \qquad\qquad\text{in } [0,+\infty[, \\ w'(0) = z'(0) = 0,\ \lim\limits_{r\to+\infty} w(r) = \lim\limits_{r\to+\infty} z(r) = 0, \end{array}\right. \tag{2.1}$$

and $S_\lambda(u,v)=(w,z)$ such that (w,z) satisfies the system

$$\begin{cases} -(r^{N-1}|w'(r)|^{p-2}w'(r))' = \lambda r^{N-1}a_1(r)(u(r))^\alpha + \lambda r^{N-1}a_2(r)(v(r))^\beta \\ \qquad \text{in } [0,+\infty[, \\ -(r^{N-1}|z'(r)|^{q-2}z'(r))' = \lambda r^{N-1}a_3(r)(u(r))^\gamma + r^{N-1}a_4(r)(v(r))^\delta \\ \qquad \text{in } [0,+\infty[, \\ w'(0)=z'(0)=0, \ \lim_{r\to+\infty} w(r) = \lim_{r\to+\infty} z(r) = 0. \end{cases} \tag{2.2}$$

3. Existence of solutions

To show the existence result, it is necessary to state some lemmas.

Lemma 3.1. *Let $u \in C^1([0,+\infty[) \cap C^2([0,+\infty[)$ be a nontrivial positive radial solution of the problem*

$$-(r^{N-1}|u'(r)|^{p-2}u'(r))' \geq 0, \qquad \text{in } [0,+\infty[,$$

then we have
(i) $u'(r) \leq 0$ for $r \geq 0$,
(ii) The function $H_p(r) = ru'(r) + \frac{N-p}{p-1}u(r), r \geq 0$, is nonnegative and non-increasing. In particular, the function $r \mapsto r^{\frac{N-p}{p-1}}u(r)$ is non-decreasing in $[0,+\infty[$.

Proof. (i) Let u be a nontrivial positive radial solution of the problem

$$-(r^{N-1}|u'(r)|^{p-2}u'(r))' \geq 0, \qquad \text{in } [0,+\infty[.$$

Suppose that $0<s<r$. Integrating from s to r, we obtain $r^{N-1}|u^{'}(r)|^{p-2}u^{'}(r) \leq s^{N-1}|u^{'}(s)|^{p-2}u^{'}(s)$. Letting $s\to 0$, we get $u^{'}(r)\leq 0$.

Obviously, if $u^{'}(r)=0$ then $u^{'}(s)=0$ for $0\leq s\leq r$. This means that u is either constant on $[0,+\infty[$ or there exists $r_0 \geq 0$ such that $u^{'}(r)<0$ for $r>r_0$ and $u^{'}(r)=0, u(r)=u(0)$ for $0\leq r\leq r_0$. Consequently u is non-increasing and $u(0)>0$.

(ii) Since u is a positive solution of the problem

$$-(r^{N-1}|u^{'}(r)|^{p-2}u^{'}(r))^{'} \geq 0, \qquad \text{in } [0,+\infty[,$$

we have $-r^{N-1}(p-1)|u^{'}(r)|^{p-2}u''(r) - (N-1)r^{N-2}(p-1)|u^{'}(r)|^{p-2}u^{'}(r) \geq 0$. In other words $ru''(r) + \frac{N-p}{p-1}u^{'}(r) \leq 0$. This yields that H_p is non-increasing.

To show that $H_p(r)\geq 0$ for all $r\geq 0$, we use a contradiction argument. Indeed, assume that there exists $r_1>0$ such that $H_p(r_1)<0$. Since H_p is non-increasing, for all $r>r_1, H_p(r)\leq H_p(r_1)$ or $u^{'}(r)+\frac{N-p}{p-1}\frac{u(r)}{r} \leq \frac{H_p(r_1)}{r}$. On the other hand $u(r)>0$, $\frac{N-p}{p-1}>0$, hence $u^{'}(r)\leq \frac{H_p(r_1)}{r}$. Consequently $u(r)-u(r_1)\leq H_p(r_1)\ln\frac{r}{r_1}, r>r_1$. It follows immediately that $\lim_{r\to+\infty} u(r) = -\infty$. This contradicts u begin positive. In particular, $\frac{H_p(r)}{ru(r)} \geq 0, \ \forall r>0$.

Finally, we obtain $\frac{u^{'}(r)}{u(r)} + \frac{N-p}{p-1}\frac{1}{r} \geq 0$. In other words, $(\ln r^{\frac{N-p}{p-1}}u(r))^{'} \geq 0$. This implies that the function $r\mapsto r^{\frac{N-p}{p-1}}u(r)$ is non-decreasing. □

Lemma 3.2. *Under hypothesis (H_1), the operators T_h and S_λ are compact.*

Proof. By following the same argument in [[2], Lemma 6], the lemma will be proved. □

We remark that the ground states of (1.1) are precisely the fixed points of the operator T_0.

Theorem 3.3. *Suppose that* (H_1) *and* (H_2) *hold, the system*

$$\begin{cases} -\Delta_p u = a_2(|x|)|v|^{\beta-1}v, & \text{in } \mathbb{R}^N \\ -\Delta_q v = a_3(|x|)|u|^{\gamma-1}u, & \text{in } \mathbb{R}^N, \end{cases} \tag{3.1}$$

has no nontrivial radial positive solutions; in particular (3.1) has no ground state.

Proof. Let (u,v) be a radial positive solution of system (3.1). Then (u,v) satisfies the differential system

$$\begin{cases} -(r^{N-1}|u^{'}(r)|^{p-2}u^{'}(r))^{'} = r^{N-1}a_2(r)(v(r))^{\beta} & \text{in } [0,+\infty[, \\ -(r^{N-1}|v^{'}(r)|^{q-2}v^{'}(r))^{'} = r^{N-1}a_3(r)(u(r))^{\gamma} & \text{in } [0,+\infty[, \\ u^{'}(0) = v^{'}(0) = 0. \end{cases} \tag{3.2}$$

Hence,

$$-(r^{N-1}|u^{'}(r)|^{p-2}u^{'}(r))^{'} \geq r^{N-1}\tilde{a_2}v^{\beta}, \tag{3.3}$$

$$-(r^{N-1}|v^{'}(r)|^{q-2}v^{'}(r))^{'} \geq r^{N-1}\tilde{a_3}u^{\gamma}. \tag{3.4}$$

First, consider the case $\beta_1 > 0$ or $\beta_2 > 0$. Integrating both (3.3) and (3.4) from 0 to r and taking into account that $u^{'}(r) < 0, v^{'}(r) < 0$ for all $r > 0$, we obtain

$$-u^{'}(r) \geq (\frac{\tilde{a_2}}{N})^{\frac{1}{p-1}} r^{\frac{1}{p-1}} v^{\frac{\beta}{p-1}},$$

$$-v^{'}(r) \geq (\frac{\tilde{a_3}}{N})^{\frac{1}{q-1}} r^{\frac{1}{q-1}} u^{\frac{\gamma}{q-1}}.$$

By lemma (3.1), we have $H_p \geq 0, H_q \geq 0$, thus

$$0 \geq -ru^{'}(r) - \frac{N-p}{p-1}u(r) \geq (\frac{\tilde{a_2}}{N})^{\frac{1}{p-1}} r^{\frac{p}{p-1}} v^{\frac{\beta}{p-1}} - \frac{N-p}{p-1}u(r),$$

$$0 \geq -rv^{'}(r) - \frac{N-q}{q-1}v(r) \geq (\frac{\tilde{a_3}}{N})^{\frac{1}{q-1}} r^{\frac{q}{q-1}} u^{\frac{\gamma}{q-1}} - \frac{N-q}{q-1}v(r).$$

This yields

$$u(r) \geq Cr^{\frac{p}{p-1}} v^{\frac{\beta}{p-1}}, \tag{3.5}$$

$$v(r) \geq Cr^{\frac{q}{q-1}} u^{\frac{\gamma}{q-1}}. \tag{3.6}$$

Combining these two inequalities, we have

$$u(r) \leq Cr^{-\alpha_1}, \tag{3.7}$$

$$v(r) \leq Cr^{-\alpha_2}. \tag{3.8}$$

Since $r^{\frac{N-p}{p-1}}u(r)$ and $r^{\frac{N-q}{q-1}}v(r)$ are non-decreasing, for all $r > r_0 > 0$,

$$u(r) \geq Cr^{\frac{-N-p}{p-1}} r_0^{\frac{N-p}{p-1}} u(r_0) = Cr^{\frac{-N-p}{p-1}}, \tag{3.9}$$

$$v(r) \geq Cr^{\frac{-N-q}{q-1}} r_0^{\frac{N-q}{q-1}} u(r_0) = Cr^{\frac{-N-q}{-1}}. \tag{3.10}$$

Inequalities (3.7) – (3.10) imply either $r^{\beta_1} \leq C$ or $r^{\beta_2} \leq C$. This is a contradiction.

Suppose now that $\beta_1 = 0$ (we may prove in a similar manner for $\beta_2 = 0$). Integrating with respect to r the first equation of (3.2) from $r_0 > 0$ to r and by using (3.3), we obtain

$$r^{N-1}|u'(r)|^{p-1} - r_0^{N-1}|u'(r_0)|^{p-1} \geq \tilde{a_2}\int_{r_0}^{r} s^{N-1}v^{\beta}(s)ds.$$

Then (3.6) yields

$$v^{\beta}(s) \geq Cs^{\frac{\beta q}{q-1}}u^{\frac{\beta\gamma}{q-1}}(s)$$

consequently

$$r^{N-1}|u'(r)|^{p-1} \geq C\int_{r_0}^{r} s^{N-1+\frac{\beta q}{q-1}}u^{\frac{\beta\gamma}{q-1}}(s)ds.$$

Taking into account inequality (3.9) and the fact that $\beta_1 = 0$, we have

$$r^{N-1}|u'(r)|^{p-1} \geq C\int_{r_0}^{r} s^{N-1+\frac{\beta q}{q-1}-\frac{N-p}{p-1}\frac{\beta\gamma}{q-1}(s)}ds = C\int_{r_0}^{r} s^{-1}ds = C\ln\frac{r}{r_0}.$$

On the other hand, for $r > 0$, $H_p(r) \geq 0$ implies $(\frac{N-p}{p-1})^{p-1}u^{p-1}(r) \geq r^{p-1}|u'(r)|^{p-1}$. Hence

$$u^{p-1}(r) \geq Cr^{p-1}|u'(r)|^{p-1} \geq Cr^{p-N}\ln\frac{r}{r_0}.$$

Then we write

$$r^{\frac{N-p}{p-1}}u(r) \geq C(\ln\frac{r}{r_0})^{\frac{1}{p-1}}.$$

This together with (3.7) yields a contradiction.

□

We now show that the radial positive solutions of system (2.1) are bounded.

Theorem 3.4. *Assume (H_1) and (H_2). If (u, v) is a ground state of (2.1), then there exists a constant $C > 0$ (independent of u and v) such that $\|(u, v)\|_X \leq C$.*

Proof. Let (u, v) be a ground state of (2.1) for $h = 0$, then (u, v) satisfies the system

$$\begin{cases} -(r^{N-1}|u'(r)|^{p-2}u'(r))' = r^{N-1}a_1(r)|u(r)|^{\alpha-1}u(r) + r^{N-1}a_2(r)|v(r)|^{\beta-1}v(r) \\ \qquad \text{in } [0, +\infty[, \\ -(r^{N-1}|v'(r)|^{q-2}v'(r))' = r^{N-1}a_3(r)|u(r)|^{\gamma-1}u(r) + r^{N-1}a_4(r)|v(r)|^{\delta-1}v(r) \\ \qquad \text{in } [0, +\infty[, \\ u'(0) = v'(0) = 0, \ \lim_{r\to+\infty} u(r) = \lim_{r\to+\infty} v(r) = 0, \end{cases} \tag{3.11}$$

Assume now that there exists a sequence of positive solutions of (3.11) such that $\|u_n\|_\infty \to \infty$ as $n \to +\infty$ or $\|v_n\|_\infty \to \infty$ as $n \to +\infty$. Taking $\gamma_n = \|u_n\|_\infty^{\frac{1}{\alpha_1}} + \|v_n\|_\infty^{\frac{1}{\alpha_2}}$, since $\alpha_1 > 0$ and $\alpha_2 > 0$, we get $\gamma_n \to +\infty$ as $n \to +\infty$. Now we introduce the transformations

$$y = \gamma_n r, \ w_n(y) = \frac{u_n(r)}{\gamma_n^{\alpha_1}}, \ z_n(y) = \frac{v_n(r)}{\gamma_n^{\alpha_2}}.$$

It is clear that for all $y \in [0,+\infty[, 0 \leq w_n(y) \leq 1, 0 \leq z_n(y) \leq 1$. Furthermore it is easy to see that for any n the pair (w_n, z_n) is a solution of the system

$$\begin{cases} -(y^{N-1}|w_n^{'}(y)|^{p-2}w_n^{'}(y))^{'} = y^{N-1}a_1(\frac{y}{\gamma_n})\frac{|\gamma_n^{\alpha_1}w_n(y)|^{\alpha-1}\gamma_n^{\alpha_1}w_n(y)}{\gamma_n^{\alpha_1(p-1)+p}} \\ \qquad +y^{N-1}a_2(\frac{y}{\gamma_n})\frac{|\gamma_n^{\alpha_2}z_n(y)|^{\beta-1}\gamma_n^{\alpha_2}z_n(y)}{\gamma_n^{\alpha_1(p-1)+p}}, \qquad \text{in } [0,+\infty[, \\ -(y^{N-1}|z_n^{'}(y)|^{q-2}z_n^{'}(y))^{'} = y^{N-1}a_3(\frac{y}{\gamma_n})\frac{|\gamma_n^{\alpha_1}w_n(y)|^{\gamma-1}\gamma_n^{\alpha_1}w_n(y)}{\gamma_n^{\alpha_2(q-1)+q}} \\ \qquad +y^{N-1}a_4(\frac{y}{\gamma_n})\frac{|\gamma_n^{\alpha_2}z_n(y)|^{\delta-1}\gamma_n^{\alpha_2}z_n(y)}{\gamma_n^{\alpha_2(q-1)+q}}, \qquad \text{in } [0,+\infty[, \\ w_n^{'}(0) = z_n^{'}(0) = 0, \lim\limits_{r\to+\infty} w_n(r) = \lim\limits_{r\to+\infty} z_n(r) = 0. \end{cases} \tag{3.12}$$

Let $R > 0$ be fixed. We claim that $\{w_n^{'}\}$ and $\{z_n^{'}\}$ are bounded in $C([0,R])$. Indeed passing to a subsequence of $\{w_n^{'}\}$ (denoted again $\{w_n^{'}\}$)assume that $\|w_n^{'}\|_{C([0,R])} \to +\infty$ as $n \to +\infty$. Hence there exists a sequence $\{y_n\}$ in $[0,R]$ such that for all $A > 0$, there exists $n_0 \in N$ such that for all $n \geq n_0, |w_n^{'}(y_n)| > A$. Integrating with respect to y the first equation of system (3.12), we obtain

$$\begin{aligned} &|w_n^{'}(y_n)|^{p-1} \\ &= \frac{1}{y_n^{N-1}}\int_0^{y_n}\Big(y^{N-1}a_1(\frac{y}{\gamma_n})\frac{(\gamma_n^{\alpha_1}w_n(y))^{\alpha}}{\gamma_n^{\alpha_1(p-1)+p}} + y^{N-1}a_2(\frac{y}{\gamma_n})\frac{(\gamma_n^{\alpha_2}z_n(y))^{\beta}}{\gamma_n^{\alpha_1(p-1)+p}}\Big)dy. \end{aligned}$$

From the part (i) of Proposition (2.1), and the fact that $\{w_n\}$ and $\{z_n\}$ are bounded, we obtain

$$c_{11}^1\gamma_n^{\alpha_1(\alpha-\varepsilon)-\alpha_1(p-1)-p} \leq \frac{(\gamma_n^{\alpha_1}w_n(y))^{\alpha}}{\gamma_n^{\alpha_1(p-1)+p}} \leq c_{11}^2\gamma_n^{\alpha_1(\alpha+\varepsilon)-\alpha_1(p-1)-p},$$

$$c_{12}^1\gamma_n^{\alpha_1(\beta-\varepsilon)-\alpha_1(p-1)-p} \leq \frac{(\gamma_n^{\alpha_2}z_n(y))^{\beta}}{\gamma_n^{\alpha_1(p-1)+p}} \leq c_{12}^2\gamma_n^{\alpha_2(\beta+\varepsilon)-\alpha_1(p-1)-p}.$$

By choosing ε sufficiently small, since $\alpha_1\alpha - \alpha_1(p-1) - p < 0$ we get

$$\frac{(\gamma_n^{\alpha_1}w_n(y))^{\alpha}}{\gamma_n^{\alpha_1(p-1)+p}} \to 0, \ \frac{(\gamma_n^{\alpha_2}z_n(y))^{\beta}}{\gamma_n^{\alpha_1(p-1)+p}} \to c_1 \ as \ n \to +\infty$$

where c_1 is positive constant. So there exists $n_1 \in N$ such that for any $n \geq n_1$, we have

$$|w_n^{'}(y_n)|^{p-1} \leq \frac{a_2(0)}{y_n^{N-1}}c_1\int_0^{y_n} y^{N-1}dy = \frac{c_1}{N}a_2(0)y_n \leq \frac{Rc_1}{N}a_2(0) \equiv c.$$

Setting $n \geq \max(n_0, n_1)$, we have $A < |w_n^{'}(y_n)| \leq c$. This contradicts the fact that A may be infinitely large. Similarly we prove that $\{z_n^{'}\}$ is bounded in $C([0,R])$. Consequently $\{w_n\}$ and $\{z_n\}$ are equicontinuous in $C([0,R])$. By Arzela-Ascoli theorem, there exists a subsequence of $\{w_n\}$ denoted again $\{w_n\}$ (respect.$\{z_n\}$) such that $w_n \to w$ (respect.$z_n \to z$) in $C([0,R])$.

On the other hand,

$$\|w_n\|_{\infty}^{\frac{1}{\alpha_1}} + \|z_n\|_{\infty}^{\frac{1}{\alpha_2}} = 1,$$

this implies that the real-valued sequences $\{\|w_n\|_{\infty}\}$ and $\{\|z_n\|_{\infty}\}$ are bounded. Hence there exist subsequences denoted again $\{\|w_n\|_{\infty}\}$ and $\{\|z_n\|_{\infty}\}$ such that $\|w_n\|_{\infty} \to w_0, \|z_n\|_{\infty} \to z_0$ and $w_0^{\frac{1}{\alpha_1}} + z_0^{\frac{1}{\alpha_2}} = 1$. In view of the uniqueness of the limit in $C([0,R])$, we get $\|w\|_{\infty}^{\frac{1}{\alpha_1}} + \|z\|_{\infty}^{\frac{1}{\alpha_2}} = 1$. This implies that (w,z) is not identically null. Integrating from 0 to $y \in [0, R]$, the first and the second equation of system (3.12), we obtain

$$w_n(0) - w_n(y) = \int_0^y (g_n(s))^{\frac{1}{p-1}} ds, \tag{3.13}$$

$$z_n(0) - z_n(y) = \int_0^y (h_n(s))^{\frac{1}{q-1}} ds. \tag{3.14}$$

Clearly $g_n(y)$ and $h_n(y)$ are defined by

$$g_n(y) = \frac{1}{y^{N-1}} \int_0^y \left(s^{N-1} a_1(\frac{s}{\gamma_n}) \frac{(\gamma_n^{\alpha_1} w_n(s))^{\alpha}}{\gamma_n^{\alpha_1(p-1)+p}} + s^{N-1} a_2(\frac{s}{\gamma_n}) \frac{(\gamma_n^{\alpha_2} z_n(s))^{\beta}}{\gamma_n^{\alpha_1(p-1)+p}} \right) ds,$$

$$h_n(y) = \frac{1}{y^{N-1}} \int_0^y \left(s^{N-1} a_3(\frac{s}{\gamma_n}) \frac{(\gamma_n^{\alpha_1} w_n(s))^{\gamma}}{\gamma_n^{\alpha_2(q-1)+q}} + s^{N-1} a_4(\frac{s}{\gamma_n}) \frac{(\gamma_n^{\alpha_2} z_n(s))^{\delta}}{\gamma_n^{\alpha_2(q-1)+q}} \right) ds.$$

By Proposition (2.1), we obtain

$$\frac{(\gamma_n^{\alpha_1} w_n(s))^{\alpha}}{\gamma_n^{\alpha_1(p-1)+p}} \to 0, \ \frac{(\gamma_n^{\alpha_2} z_n(s))^{\delta}}{\gamma_n^{\alpha_2(q-1)+q}} \to 0,$$
$$\frac{(\gamma_n^{\alpha_2} z_n(s))^{\beta}}{\gamma_n^{\alpha_1(p-1)+p}} = \frac{(\gamma_n^{\alpha_2})^{\beta}}{\gamma_n^{\alpha_1(p-1)+p}} \frac{(\gamma_n^{\alpha_2} z_n(s))^{\beta}}{\gamma_n^{\alpha_2\beta}} \to cz^{\beta}(s)$$
$$\frac{(\gamma_n^{\alpha_1} w_n(s))^{\gamma}}{\gamma_n^{\alpha_2(q-1)+q}} = \frac{(\gamma_n^{\alpha_1})^{\gamma}}{\gamma_n^{\alpha_2(q-1)+q}} \frac{(\gamma_n^{\alpha_1} w_n(s))^{\gamma}}{\gamma_n^{\alpha_1\gamma}} \to cw^{\gamma}(s),$$

as $n \to +\infty$. By the Lebesgue theorem on dominated convergence, it follows that

$$g_n(y) \to \frac{c}{y^{N-1}} \int_0^y s^{N-1} a_2(0) z^{\beta}(s) ds,$$
$$h_n(y) \to \frac{c}{y^{N-1}} \int_0^y s^{N-1} a_3(0) w^{\gamma}(s) ds,$$

as $n \to +\infty$. Passing to the limit in (3.13) and (3.14), we obtain

$$w(0) - w(y) = c \int_0^y \frac{1}{\tau^{N-1}} \left(\int_0^{\tau} s^{N-1} a_2(0) z^{\beta}(s) ds \right)^{\frac{1}{p-1}} d\tau,$$
$$z(0) - z(y) = c \int_0^y \frac{1}{\tau^{N-1}} \left(\int_0^{\tau} s^{N-1} a_3(0) w^{\gamma}(s) ds \right)^{\frac{1}{q-1}} d\tau.$$

In this way $w \geq 0, z \geq 0, w, z \in C^1([0, R]) \cap C^2(]0, R])$ and satisfy the system

$$\begin{cases} -(y^{N-1}|w'(y)|^{p-2} w'(y))' = ca_2(0) y^{N-1} (z(y))^{\beta}, & \text{in } [0, R], \\ -(y^{N-1}|z'(y)|^{q-2} z'(y))' = ca_3(0) y^{N-1} (w(y))^{\gamma}, & \text{in } [0, R], \\ w'(0) = z'(0) = 0 \end{cases} \tag{3.15}$$

If we use the same argument on $[0, R^*]$ where $R^* > R$, we obtain a solution (w^*, z^*) of system (3.15) with R^* instead of R, which coincide with (w, z) in $[0, R]$. To this end, we indefinitely extend (w, z) to $[0, +\infty[$. By Lemma (3.1) $w(y) > 0, z(y) > 0$, for all $y \geq 0$. The pair (w, z) also satisfies system (3.15). In other words (w, z) is a radial positive solution of (3.2). This contradicts Theorem (3.3).

□

Lemma 3.5. *Under hypothesis* (H_1) *and* (H_2)*, there exists* $h_0 > 0$ *such that the problem* $(u, v) = T_h(u, v)$ *has no solution for* $h \geq h_0$

Proof. Let $(u,v) \in X$ of the above problem. Then (u,v) satisfies system

$$\begin{cases} -(r^{N-1}|u'(r)|^{p-2}u'(r))' = r^{N-1}a_1(r)u^{\alpha}(r) + r^{N-1}a_2(r)[(v(r))^{\beta} + h] \\ \qquad\qquad\qquad\qquad \text{in } [0,+\infty[, \\ -(r^{N-1}|v'(r)|^{q-2}v'(r))' = r^{N-1}a_3(r)u^{\gamma}(r) + r^{N-1}a_4(r)(v(r))^{\delta} \\ \qquad\qquad\qquad\qquad \text{in } [0,+\infty[, \\ u'(0) = v'(0) = 0, \ \lim_{r\to+\infty} u(r) = \lim_{r\to+\infty} v(r) = 0 \end{cases} \tag{3.16}$$

Assume that there exists a sequence $\{h_n\}$, $h_n \to +\infty$ as $n \to +\infty$, such that (3.16) admits a pair of solutions $\{(u_n,v_n)\}$. In view of Lemma (3.1), we have $u_n(r) > 0, v_n(r) > 0, u_n^{'}(r) \leq 0$ and $v_n^{'}(r) \leq 0$, for all $n \in N$. By replacing (u_n,v_n) to (u,v) in (3.16) and integrating the first equation of system (3.16), from R to $2R$, $R > 0$, we obtain

$$u_n(R) \geq \int_R^{2R} \left(\eta^{1-N} \int_0^{\eta} \xi^{N-1} a_2(\xi) h_n d\xi\right)^{\frac{1}{p-1}} d\eta \geq cRh_n{}^{\frac{1}{p-1}}$$

Here

$$c = \left(\frac{1}{(2R)^{N-1}} \int_0^R \xi^{N-1} a_2(\xi) d\xi\right)^{\frac{1}{p-1}}.$$

Consequently $u_n(R) \geq cRh_n{}^{\frac{1}{p-1}}$. Passing to the limit we get $u_n(R) \to +\infty$. On the other hand, integrating the second equation of (3.16), from R to $2R$, we get

$$v_n(R) \geq \int_R^{2R} \left(\eta^{1-N} \int_0^{\eta} \xi^{N-1} a_3(\xi)(u_n(\xi))^{\gamma} d\xi\right)^{\frac{1}{q-1}} d\eta \geq cRu_n(R)^{\frac{\gamma}{q-1}}$$

By Proposition (2.1), we have $v_n(R) \geq c(u_n(R))^{\frac{\gamma-\varepsilon}{q-1}}$, in a same way, we obtain $u_n(R) \geq c(v_n(R))^{\frac{\beta-\varepsilon}{p-1}}$. It follows from the last two inequalities, that

$$(u_n(R))^{\frac{(\beta-\varepsilon)(\gamma-\varepsilon)-(p-1)(q-1)}{(p-1)(q-1)}} \leq \frac{1}{c}.$$

This is a contradiction, since $u_n(R)$ increases to infinitely.

□

Lemma 3.6. *There exists $\bar{\rho} > 0$ such that for all $\rho \in]0,\bar{\rho}[$ and all $(u,v) \in X$ satisfying $\|(u,v)\| = \rho$, the equation $(u,v) = S_\lambda(u,v)$ has no solution.*

Proof. Assume that there exists $\{\rho_n\} \in [0,+\infty[, \rho_n \to 0; \{\lambda_n\} \subset [0,1]$ and $(u_n,v_n) \in X$ such that $(u_n,v_n) = S_\lambda(u_n,v_n)$ with $\|(u_n,v_n)\| = \rho_n$. So, we get

$$\|u_n\|_\infty \leq c\lambda_n^{\frac{1}{p-1}} \left(\|u_n\|_\infty^{\frac{\alpha-\varepsilon}{p-1}} + \|v_n\|_\infty^{\frac{\beta-\varepsilon}{p-1}}\right),$$

$$\|v_n\|_\infty \leq c\lambda_n^{\frac{1}{q-1}} \left(\|u_n\|_\infty^{\frac{\gamma-\varepsilon}{q-1}} + \|v_n\|_\infty^{\frac{\delta-\varepsilon}{q-1}}\right).$$

Adding the last two inequalities, we obtain

$$\|(u_n,v_n)\| \leq C\left(\|(u_n,v_n)\|^{\frac{\alpha-\varepsilon}{p-1}} + \|(u_n,v_n)\|^{\frac{\beta-\varepsilon}{p-1}} + \|(u_n,v_n)\|^{\frac{\gamma-\varepsilon}{q-1}} + \|(u_n,v_n)\|^{\frac{\delta-\varepsilon}{q-1}}\right).$$

This yields

$$1 \leq C\left(\|(u_n,v_n)\|^{\frac{\alpha-\varepsilon}{p-1}-1} + \|(u_n,v_n)\|^{\frac{\beta-\varepsilon}{p-1}-1} + \|(u_n,v_n)\|^{\frac{\gamma-\varepsilon}{q-1}-1} + \|(u_n,v_n)\|^{\frac{\delta-\varepsilon}{q-1}-1}\right).$$

The above inequality contradicts the fact that $\|(u_n, v_n)\| = \rho_n \to 0$ as $n \to +\infty$. □

Theorem 3.7. *Under hypothesis (H_1) and (H_2), system (1.1) has positive radial solution.*

Proof. To show the existence of ground states for (1.1) (or (2.1) with h=0), it is sufficient to prove that the compact operator T_0 admits a fixed point. By virtue of Theorem (3.4), the eventual fixed point (u, v) of T_0 are bounded; in fact there exists $C > 0$ such that $\|(u, v)\|_X \leq C$. Let us choose $R_1 > C$ and let us designate by B_{R_1} the ball of X, centered at the origin with radius R_1. To this end, the Leray-Schauder degree $deg_{LS}(I - T_h, B_{R_1}, 0)$ is well defined. we recall that I denote the identical operator in X. Moreover, by Lemma (3.5), we have $deg_{LS}(I - T_h, B_{R_1}, 0) = 0$ for all $h \geq h_0$. It follows from the homotopy invariance of the Leray-Schauder degree that

$$deg_{LS}(I - T_0, B_{R_1}, 0) = deg_{LS}(I - T_h, B_{R_1}, 0) = 0.$$

on the other hand, by Lemma (3.6), there exists $0 < \rho < \bar{\rho} < R_1$ such that $deg_{LS}(I - S_\lambda, B_\rho, 0)$ is well defined. Once again, the homotopy invariance of the Leray-Schauder degree yields

$$\begin{aligned} 1 &= deg_{LS}(I, B_\rho, 0) = deg_{LS}(I - S_\lambda, B_\rho, 0) \\ &= deg_{LS}(I - S_1, B_\rho, 0) = deg_{LS}(I - T_0, B_\rho, 0). \end{aligned}$$

Using the additivity of the the Leray-Schauder degree,

$$deg_{LS}(I - T_0, B_{R_1} \setminus B_\rho, 0) = deg_{LS}(I - T_0, B_{R_1}, 0) - deg_{LS}(I - T_0, B_\rho, 0) = -1.$$

This implies that T_0 has fixed point in $B_{R_1} \setminus B_\rho$. Consequently, there exists a nontrivial ground state.

□

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Ghasem Alizadeh Afrouzi

Department of Mathematics, Faculty of Mathematical Sciences,, University of Mazandaran, Babolsar, Iran

E-mail address: afrouzi@umz.ac.ir

Tahere Azimi Roushan

Department of Mathematics, Faculty of Mathematical Sciences,, University of Mazandaran, Babolsar, Iran

E-mail address: t.roushan@umz.ac.ir