CERTAIN INEQUALITIES RELATED TO THE CHEBYSHEV'S FUNCTIONAL INVOLVING A RIEMANN-LIOUVILLE OPERATOR

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ABSTRACT. In this paper, the Riemann-Liouville fractional integral is used to establish some integral results related to Chebyshev's functional in the case of differentiable functions whose derivatives belong to the space $L_p([0, \infty[)$.

1. INTRODUCTION

The fractional calculus has attracted great attention during last few decades. One of the important reasons for such deep interest in the subject is its ability to model many natural phenomena, see, for instance, the papers [7, 11]. In this work, we consider the celebrated Chebyshev functional [1]

$$T(f, g, p) := \int_a^b p(x) \int_a^b p(x) f(x) g(x) \, dx - \int_a^b p(x) f(x) \, dx \int_a^b p(x) g(x) \, dx \tag{1.1}$$

where f and g are two integrable functions on $[a, b]$ and p is a positive and integrable function on $[a, b]$.

The functional (1.1) has wide applicability in numerical quadrature, transform theory, probability and statistical problems, and the bounding of special functions. Its basic appeal stems from a desire to approximate, for example, information in the form of a particular measure of the product of functions in terms of the products of the individual function measures. It is, also, of great interest in differential and difference equations [4, 10].

In [5], S.S. Dragmir proved that

$$2|T(f, g, p)| \leq ||f'||_r ||g'||_s \left[\int_a^b \int_a^b |x - y| p(x) p(y) dx dy \right], \tag{1.2}$$

where f, g are two differentiable functions and $f' \in L_r(a, b), g' \in L_s(a, b), r > 1, r^{-1} + s^{-1} = 1$.

Many researchers have given considerable attention to (1.1) and several inequalities related to this functional have appeared in the literature, to mention a few, see [2, 3, 8, 9, 12, 13] and the references cited therein.

The main aim of this paper is to establish some new inequalities for (1.1) by using the Riemann-Liouville fractional integrals. We give our results in the case of differentiable functions whose derivatives belong to $L_p([0, \infty[$. Other class of "no weighted" inequalities are also obtained. Our results have some relationships with some inequalities obtained in [5, 10].

2. Basic Definitions of the Fractional Integrals

Definition 2.1: A real valued function $f(t), t \geq 0$ is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $p > \mu$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C([0, \infty[)$.

Definition 2.2: A function $f(t), t \geq 0$ is said to be in the space $C_\mu^n, \mu \in \mathbb{R}$, if $f^{(n)} \in C_\mu$.

Definition 2.3: The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a function $f \in C_\mu, (\mu \geq -1)$ is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau; \quad \alpha > 0, t > 0,$$
$$J^0 f(t) = f(t), \tag{2.1}$$

where $\Gamma(\alpha) := \int_0^\infty e^{-u} u^{\alpha-1} du$.

For the convenience of establishing the results, we give the semigroup property:

$$J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t), \alpha \geq 0, \beta \geq 0, \tag{2.2}$$

which implies the commutative property

$$J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t). \tag{2.3}$$

For more details, one may consult [6].

3. Results

Theorem 3.1. *Let p be a positive function on $[0, \infty[$ and let f and g be two differentiable functions on $[0, \infty[$. If $f' \in L_r([0, \infty[), g' \in L_s([0, \infty[), r > 1, r^{-1} + s^{-1} = 1$, then for all $t > 0, \alpha > 0$, we have:*

$$2\left| J^\alpha p(t) J^\alpha pfg(t) - J^\alpha pf(t) J^\alpha pg(t) \right|$$

$$\leq \frac{||f'||_r ||g'||_s}{\Gamma^2(\alpha)} \int_0^t \int_0^t (t - \tau)^{\alpha-1} (t - \rho)^{\alpha-1} |\tau - \rho| p(\tau) p(\rho) d\tau d\rho \tag{3.1}$$

$$\leq ||f'||_r ||g'||_s t (J^\alpha p(t))^2.$$

Proof. Let f and g be two functions satisfying the conditions of Theorem 3.1 and let p be a positive function on $[0, \infty[$.
Define

$$H(\tau, \rho) := (f(\tau) - f(\rho))(g(\tau) - g(\rho)); \ \tau, \rho \in (0, t), t > 0. \tag{3.2}$$

Multiplying (3.2) by $\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} p(\tau); \ \tau \in (0, t)$ and and integrating the resulting identity with respect to τ from 0 to t, we can state that

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} p(\tau) H(\tau, \rho) d\tau$$
$$= J^\alpha pfg(t) - f(\rho) J^\alpha pg(t) - g(\rho) J^\alpha pf(t) + f(\rho)g(\rho) J^\alpha p(t). \tag{3.3}$$

Now, multiplying (3.3) by $\frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} p(\rho); \ \rho \in (0, t)$ and integrating the resulting identity with respect to ρ over $(0, t)$, we can write

$$\frac{1}{\Gamma^2(\alpha)} \int_0^t \int_0^t (t-\tau)^{\alpha-1} (t-\rho)^{\alpha-1} p(\tau) p(\rho) H(\tau, \rho) d\tau d\rho$$
$$= 2 \Big(J^\alpha p(t) J^\alpha pfg(t) - J^\alpha pf(t) J^\alpha pg(t) \Big). \tag{3.4}$$

On the other hand, we have

$$H(\tau, \rho) = \int_\tau^\rho \int_\tau^\rho f'(y) g'(z) dy dz. \tag{3.5}$$

Using Holder inequality for double integral, we can write

$$|H(\tau, \rho)| \le \Big| \int_\tau^\rho \int_\tau^\rho |f'(y)|^r dy dz \Big|^{r-1} \Big| \int_\tau^\rho \int_\tau^\rho |g'(z)|^s dy dz \Big|^{s-1}. \tag{3.6}$$

Since

$$\Big| \int_\tau^\rho \int_\tau^\rho |f'(y)|^r dy dz \Big|^{r-1} = |\tau - \rho|^{r-1} \Big| \int_\tau^\rho |f'(y)|^r dy \Big|^{r-1} \tag{3.7}$$

and

$$\Big| \int_\tau^\rho \int_\tau^\rho |g'(y)|^r dy dz \Big|^{s-1} = |\tau - \rho|^{s-1} \Big| \int_\tau^\rho |g'(z)|^s dz \Big|^{s-1}, \tag{3.8}$$

then, we can estimate H as follows:

$$|H(\tau, \rho)| \le |\tau - \rho| \Big| \int_\tau^\rho |f'(y)|^r dy \Big|^{r-1} \Big| \int_\tau^\rho |g'(z)|^s dz \Big|^{s-1}. \tag{3.9}$$

On the other hand, we have

$$\frac{1}{\Gamma^2(\alpha)} \int_0^t \int_0^t (t-\tau)^{\alpha-1} (t-\rho)^{\alpha-1} p(\tau) p(\rho) |H(\tau, \rho)| d\tau d\rho$$

$$\le \frac{1}{\Gamma^2(\alpha)} \int_0^t \int_0^t (t-\tau)^{\alpha-1} (t-\rho)^{\alpha-1} |\tau - \rho| p(\tau) p(\rho) \Big| \int_\tau^\rho |f'(y)|^r dy \Big|^{r-1} \Big| \int_\tau^\rho |g'(z)|^s dz \Big|^{s-1} d\tau d\rho. \tag{3.10}$$

Applying again Holder inequality to the right-hand side of (3.10), we can state that

$$\frac{1}{\Gamma^2(\alpha)} \int_0^t \int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}p(\tau)p(\rho)|H(\tau,\rho)|d\tau d\rho$$

$$\leq \left[\frac{1}{\Gamma^r(\alpha)} \int_0^t \int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}|\tau-\rho|p(\tau)p(\rho) \left| \int_\tau^\rho |f'(y)|^r dy \right| d\tau d\rho \right]^{r^{-1}}$$

$$\times \left[\frac{1}{\Gamma^s(\alpha)} \int_0^t \int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}|\tau-\rho|p(\tau)p(\rho) \left| \int_\tau^\rho |g'(z)|^s dz \right| d\tau d\rho \right]^{s^{-1}}.$$

(3.11)

Now, using the fact that

$$\left| \int_\tau^\rho |f'(y)|^r dy \right| \leq ||f'||_r^r, \left| \int_\tau^\rho |g'(z)|^s dz \right| \leq ||g'||_s^s,$$

(3.12)

we obtain

$$\frac{1}{\Gamma^2(\alpha)} \int_0^t \int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}p(\tau)p(\rho)|H(\tau,\rho)|d\tau d\rho$$

$$\leq \left[\frac{||f'||_r^r}{\Gamma^r(\alpha)} \int_0^t \int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}|\tau-\rho|p(\tau)p(\rho)d\tau d\rho \right]^{r^{-1}}$$

(3.13)

$$\times \left[\frac{||g'||_s^s}{\Gamma^s(\alpha)} \int_0^t \int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}|\tau-\rho|p(\tau)p(\rho)d\tau d\rho \right]^{s^{-1}}.$$

From (3.13), we get

$$\frac{1}{\Gamma^2(\alpha)} \int_0^t \int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}p(\tau)p(\rho)|H(\tau,\rho)|d\tau d\rho$$

$$\leq \frac{||f'||_r||g'||_s}{\Gamma^2(\alpha)} \left[\int_0^t \int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}|\tau-\rho|p(\tau)p(\rho)d\tau d\rho \right]^{r^{-1}}$$

(3.14)

$$\times \left[\int_0^t \int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}|\tau-\rho|p(\tau)p(\rho)d\tau d\rho \right]^{s^{-1}}.$$

Since $r^{-1} + s^{-1}$, then we have

$$\frac{1}{\Gamma^2(\alpha)} \int_0^t \int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}p(\tau)p(\rho)|H(\tau,\rho)|d\tau d\rho$$

$$\leq \frac{||f'||_r||g'||_s}{\Gamma^2(\alpha)} \left[\int_0^t \int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}|\tau-\rho|p(\tau)p(\rho)d\tau d\rho \right].$$

(3.15)

By the relations (3.4) and (3.15) and using the properties of the modulus, we get the first inequality in (3.1).

Now we shall prove the second inequality of (3.1). We have

$$0 \leq \tau \leq t, 0 \leq \rho \leq t.$$

Hence,

$$0 \leq |\tau - \rho| \leq t.$$

Therefore, we have

$$\frac{1}{\Gamma^2(\alpha)} \int_0^t \int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}p(\tau)p(\rho)|H(\tau,\rho)|d\tau d\rho$$

$$\leq \frac{||f'||_r||g'||_s t}{\Gamma^2(\alpha)} \left[\int_0^t \int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}p(\tau)p(\rho)d\tau d\rho \right] \qquad (3.16)$$

$$= ||f'||_r||g'||_s t(J^\alpha p(t))^2.$$

Theorem 3.1 is thus proved.   □

Remark. *Taking $\alpha = 1$ in the first inequality in Theorem 3.1, we obtain the inequality (1.2) on $[0,t]$.*

We shall further generalize Theorem 3.1 by considering two fractional positive parameters.

Theorem 3.2. *Let p be a positive function on $[0,\infty[$ and let f and g be two differentiable functions on $[0,\infty[$. If $f' \in L_r([0,\infty[), g' \in L_s([0,\infty[), r > 1, r^{-1} + s^{-1} = 1$, then for all $t > 0, \alpha > 0, \beta > 0$, we have*

$$\left| J^\alpha p(t)J^\beta pfg(t) + J^\beta p(t)J^\alpha pfg(t) - J^\alpha pf(t)J^\beta pg(t) - J^\beta pf(t)J^\alpha pg(t) \right|$$

$$\leq \frac{||f'||_r||g'||_s}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}|\tau-\rho)|p(\tau)p(\rho)d\tau d\rho$$

$$\leq ||f'||_r||g'||_s t J^\alpha p(t)J^\beta p(t). \qquad (3.17)$$

Proof. Using the identity (3.3), we can state that

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}p(\tau)p(\rho)H(\tau,\rho)d\tau d\rho$$

$$= J^\alpha p(t)J^\beta pfg(t) + J^\beta p(t)J^\alpha pfg(t) - J^\alpha pf(t)J^\beta pg(t) - J^\beta pf(t)J^\alpha pg(t). \qquad (3.18)$$

From the relation (3.9), we can obtain the following estimation

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}p(\tau)|H(\tau,\rho)|d\tau \qquad (3.19)$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}|\tau-\rho|p(\tau)\left| \int_\tau^\rho |f'(y)|^r dy \right|^{r^{-1}} \left| \int_\tau^\rho |g'(z)|^s dz \right|^{s^{-1}} d\tau.$$

Therefore, we have

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}p(\tau)p(\rho)|H(\tau,\rho)|d\tau d\rho$$

$$\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}|\tau-\rho|p(\tau)p(\rho)\left| \int_\tau^\rho |f'(y)|^r dy \right|^{r^{-1}} \left| \int_\tau^\rho |g'(z)|^s dz \right|^{s^{-1}} d\tau d\rho. \qquad (3.20)$$

Applying Holder inequality for double integral to the right-hand side of (3.20), yields

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}p(\tau)p(\rho)|H(\tau,\rho)|d\tau d\rho$$

$$\leq \left[\frac{1}{\Gamma^r(\alpha)} \int_0^t \int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}|\tau-\rho|p(\tau)p(\rho)\left|\int_\tau^\rho |f'(y)|^r dy\right| d\tau d\rho\right]^{r^{-1}}$$

$$\times \left[\frac{1}{\Gamma^s(\beta)} \int_0^t \int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}|\tau-\rho|p(\tau)p(\rho)\left|\int_\tau^\rho |g'(z)|^s dz\right| d\tau d\rho\right]^{s^{-1}}. \tag{3.21}$$

By (3.12) and (3.21), we get

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}p(\tau)p(\rho)|H(\tau,\rho)|d\tau d\rho$$

$$\leq \frac{||f'||_r ||g'||_s}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}|\tau-\rho|p(\tau)p(\rho)d\tau d\rho. \tag{3.22}$$

Using (3.18) and (3.22) and the properties of modulus, we get the first inequality in (3.17). $\qquad\square$

Remark. (*i*) *Applying Theorem 3.2 for $\alpha = \beta$, we obtain Theorem 3.1*
(*ii*) *Taking $\alpha = \beta = 1$ in the first inequality in Theorem 3.1, we obtain the inequality (1.2) on $[0,t]$.*

In Theorem 3.1, if we set $p(x) = 1$, we arrive at the following corollary :

Corollary 3.3. *Let f and g be two differentiable functions on $[0,\infty[$. If $f' \in L_r([0,\infty[), g' \in L_s([0,\infty[), r > 1, r^{-1} + s^{-1} = 1$, then for all $t > 0, \alpha > 0$, we have:*

$$\left|\frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha fg(t) - J^\alpha f(t)J^\alpha g(t)\right|$$

$$\leq ||f'||_r ||g'||_s \frac{t^{2\alpha+1}}{2\Gamma^2(\alpha+1)}. \tag{3.23}$$

In Theorem 3.2, if we set $p(x) = 1$, we obtain the following corollary:

Corollary 3.4. *Let f and g be two differentiable functions on $[0,\infty[$. If $f' \in L_r([0,\infty[), g' \in L_s([0,\infty[), r > 1, r^{-1} + s^{-1} = 1$, then for all $t > 0, \alpha > 0$, we have:*

$$\left|\frac{t^\alpha}{\Gamma(\alpha+1)}J^\beta fg(t) + \frac{t^\beta}{\Gamma(\beta+1)}J^\alpha fg(t) - J^\alpha f(t)J^\beta g(t) - J^\beta f(t)J^\alpha g(t)\right|$$

$$\leq ||f'||_r ||g'||_s \frac{t^{\alpha+\beta+1}}{\Gamma(\alpha+1)\Gamma(\beta+1)}. \tag{3.24}$$

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