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NÖRLUND SPACE OF DOUBLE ENTIRE SEQUENCES

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Abstract. Let Γ^2 denote the spaces of all double entire sequences. Let Λ^2 denote the spaces of all double analytic sequences. This paper is devoted to a study of the general properties of Nörlund space of double entire sequences $\eta\left(\Gamma^2_\pi\right)$, Γ^2 and also study some of the properties of $\eta\left(\Gamma^2_\pi\right)$ and $\eta\left(\Lambda^2_\pi\right)$

1. Introduction

Let (x_{mn}) be a double sequence of real or complex numbers. Then the series $\sum_{m,n=1}^{\infty} x_{mn}$is called a double series. The double series $\sum_{m,n=1}^{\infty} x_{mn}$is said to be convergent if and only if the double sequence $(S_{mn})$is convergent, where

$$S_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n = 1, 2, 3, ...) \text{ (see[1])}.$$

We denote w^2 as the class of all complex double sequences (x_{mn}). A sequence $x = (x_{mn})$is said to be double analytic if

$$sup_{mn} |x_{mn}|^{1/m+n} < \infty.$$

The vector space of all prime sense double analytic sequences are usually denoted by Λ^2. A sequence $x = (x_{mn})$ is called double entire sequence if

$$|x_{mn}|^{1/m+n} \to 0 \text{ as } m, n \to \infty.$$

The vector space of all prime sense double entire sequences are usually denoted by Γ^2. The space Λ^2 as well as Γ^2 is a metric space with the metric

$$d(x, y) = sup_{mn} \left\{ |x_{mn} - y_{mn}|^{1/m+n} : m, n : 1, 2, 3, ... \right\}, \tag{1.1}$$

forall $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ in Γ^2.

A sequence $\pi = (\pi_{mn})$ is said to be double analytic rate if

$$sup_{mn} \left| \frac{x_{mn}}{\pi_{mn}} \right|^{1/m+n} < \infty.$$

The vector space of all prime sense double analytic rate sequences are usually denoted by Λ^2_π.
A sequence $\pi = (\pi_{mn})$ is called double entire sequence rate if

$$\left| \frac{x_{mn}}{\pi_{mn}} \right|^{1/m+n} \to 0 \text{ as } m, n \to \infty.$$

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The vector space of all prime sense double entire rate sequences are usually denoted by Γ^2_π. The space Λ^2_π as well as Γ^2_π is a metric space with the metric

$$d(x,y) = sup_{mn}\left\{\left|\frac{x_{mn}-y_{mn}}{\pi_{mn}}\right|^{1/m+n} : m,n : 1,2,3,...\right\}, \tag{1.2}$$

forall $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ in Γ^2.

Let $(P_{m,n})^\infty_{m,n=0}$ be a sequence of non-negative real numbers with $p_{00} > 0$. Consider the transformation

$$y_{mn} = \frac{1}{\sum_{i=0}^m \sum_{j=0}^n p_{ij}} \sum_{i=0}^m \sum_{j=0}^n p_{ij} x_{m-i,n-j}$$

for $m,n = 0,1,2,\cdots$. The set of all (x_{mn}) for which $(y_{mn}) \in \Gamma^2$ is called the Nörlund space of double entire sequence. The Nörlund space of double entire sequence is denoted by $\eta\left(\Gamma^2\right)$. Similarly the set of all (x_{mn}) for which $(y_{mn}) \in \Lambda^2$ is called the Nörlund space of double analytic sequence is denoted by $\eta\left(\Lambda^2\right)$. We write $P_{mn} = p_{00} + \cdots + p_{mn}$, for $m,n = 0,1,2,\cdots$.

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for $m,n = 0,1,2,\cdots$. The set of all (x_{mn}) for which $(y_{mn}) \in \Gamma^2$ is called the Nörlund space of double entire rate sequence. The Nörlund space of double entire rate sequence is denoted by $\eta\left(\Gamma^2_\pi\right)$. Similarly the set of all (x_{mn}) for which $(y_{mn}) \in \Lambda^2$ is called the Nörlund space of double analytic rate sequence is denoted by $\eta\left(\Lambda^2_\pi\right)$. We write $P_{mn} = p_{00} + \cdots + p_{mn}$, for $m,n = 0,1,2,\cdots$.

Absorbent is a neighbourhood of zero and $\sigma\left(X, X^{'}\right)-$ is a subsequence of schauder basis converges to weakly.

All absolutely convex absorbent closed subset of locally convex Topological Vector Space X is called barrel. X is called barreled space if each barrel is a neighbourhood of zero.

A locally convex Topological Vector Space X is said to be semi reflexive if each bounded closed set in X is $\sigma\left(X, X^{'}\right)-$compact.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\delta_{ij}$ for all $m, n \in \mathbb{N}$, where

$$\delta_{mn} = \begin{pmatrix} 0 & 0 & ...0 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \\ . \\ . \\ . \\ 0 & 0 & ...1/\pi & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \end{pmatrix}$$

with $1/\pi$ in the $(m,n)^{th}$ position and zero other wise. An FK-space(or a metric space)X is said to have AK property if (δ_{mn}) is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$. Consider the constant sequence $\pi = (\pi_{mn})$ and it is defined by

$$\pi_{mn} = \begin{pmatrix} \pi_{11} & \pi_{12} & ..., & \pi_{1m} & ... \\ \pi_{21} & \pi_{22} & ..., & \pi_{2m} & ... \\ . \\ . \\ . \\ \pi_{m1} & \pi_{m2} & ..., & \pi_{mn} & ... \\ 0 & 0 & ...0 & 0 & ... \end{pmatrix}$$

We need the following inequality in the sequel of the paper:

Lemma 1: For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p$$

2. Preliminaries

Let us define the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{ (x_{mn}) \in w^2 : sup_{m,n \in N} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_p(t) := \left\{ (x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn} - L|^{t_{mn}} = 1 \, for\, some\, L \in \mathbb{C} \right\},$$

$$\mathcal{C}_{0p}(t) := \left\{ (x_{mn}) \in w^2 : P - lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 0 \right\},$$

$$\mathcal{L}_u(t) := \left\{ (x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);$$

where $t = (t_{mn})$ is the sequence of positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}; \mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [10,11] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [12] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [13] and Tripathy [8] have recently introduced the statistical convergence and Cauchy for double sequences independently and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [14] and Mursaleen and Edely [15] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the $M-$core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the $M-$core of x. More recently, Altay and Basar [16] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Quite recently Basar and Sever [17] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [18,19] have studied the space $\chi^2_M(p,q,u)$ of double sequences and proved some inclusion relations and also studied characterization and general properties of gai sequences via double Orlicz space of χ^2_M of χ^2 establishing some inclusion relations.

Some initial works on double sequence spaces is found in Bromwich[3]. Later on it was investigated by Hardy[5], Moricz[6], Moricz and Rhoades[7], Basarir and Solankan[2], Tripathy[8], Tripathy and Dutta ([26],[27]), Tripathy and Sarma ([28],[29],[30]), Colak and Turkmenoglu[4], Turkmenoglu[9], and many others.

3. Main Results

3.1. **Proposition.** $\eta\left(\Gamma^2_\pi\right) = \Gamma^2_\pi$

Proof: Let $x = (x_{mn}) \in \eta\left(\Gamma^2_\pi\right)$. Then $y \in \Gamma^2_\pi$ so that for every $\epsilon > 0$, we have a positive integer n_0 such that

$$\left|\frac{p_{00}(x_{mn}/\pi_{mn}) + \cdots + p_{mn}(x_{00}/\pi_{00})}{P_{mn}}\right| < \epsilon^{m+n} \text{ for all } m,n \geq n_0$$

Take $p_{00} = 1; p_{11} = \cdots = p_{mn} = 0$. We then have $\left|\frac{x_{mn}}{\pi_{mn}}\right| < \epsilon^{m+n}, \forall m,n \geq n_0$. Therefore $x = (x_{mn}) \in \Gamma^2_\pi$. Hence

$$\eta\left(\Gamma^2_\pi\right) \subset \Gamma^2_\pi \tag{3.1}$$

On the other hand, let $x = (x_{mn}) \in \Gamma^2_\pi$. But for any given $\epsilon > 0$, there exists a positive integer n_0 such that $\left|\frac{x_{mn}}{\pi_{mn}}\right| < \epsilon^{m+n}, \forall m,n \geq n_0$. We have

$$\begin{aligned}\left|\frac{y_{mn}}{\pi_{mn}}\right| &\leq \left|\frac{p_{00}(x_{mn}/\pi_{mn}) + \cdots + p_{mn}(x_{00}/\pi_{00})}{P_{mn}}\right| \\ &\leq \frac{1}{P_{mn}}\left[p_{00}\left|\frac{x_{mn}}{\pi_{mn}}\right| + \cdots + p_{mn}\left(|x_{00}/\pi_{00}|\right)\right] \\ &\leq \frac{1}{P_{mn}}\left[p_{00}\epsilon^{m+n} + \cdots + p_{mn}\epsilon^{0+0}\right] \\ &\leq \frac{\epsilon^{m+n}}{P_{mn}}\left[p_{00} + \cdots + p_{mn}\right] \\ &\leq \frac{\epsilon^{m+n}}{P_{mn}}P_{mn} = \epsilon^{m+n} \forall m,n \geq n_0.\end{aligned}$$

Therefore $(y_{mn}) \in \Gamma^2_\pi$. Consequently $x \in \eta\left(\Gamma^2_\pi\right)$. Hence

$$\Gamma^2_\pi \subset \eta\left(\Gamma^2_\pi\right) \tag{3.2}$$

From (3.1) and (3.2) we obtain $\eta\left(\Gamma_\pi^2\right)=\Gamma_\pi^2$. This completes the proof.

3.2. **Proposition.** $\eta\left(\Lambda_\pi^2\right)=\Lambda_\pi^2$
Proof: Let $(x_{mn})\in\Lambda_\pi^2$. Then there exists a positive constant T such that

$\left|\frac{x_{mn}}{\pi_{mn}}\right|\leq T^{m+n}$ for $m,n=0,1,2,\cdots$.

$$\begin{aligned}\left|\frac{y_{mn}}{\pi_{mn}}\right| &\leq \frac{p_{00}T^{m+n}+\cdots+p_{mn}T^{0+0}}{P_{mn}}\\ &\leq \frac{T^{m+n}}{P_{mn}}\left[p_{00}+\cdots+\frac{p_{mn}}{T^{m+n}}\right]\\ &\leq \frac{T^{m+n}}{P_{mn}}\left[p_{00}+\cdots+p_{mn}\right]\\ &\leq \frac{T^{m+n}}{P_{mn}}P_{mn}=T^{m+n},\text{ for } m,n=0,1,2,\cdots.\end{aligned}$$

Hence $(y_{mn})\in\Lambda_\pi^2$. But then $x=(x_{mn})\in\eta\left(\Gamma_\pi^2\right)$. Consequently

$$\Lambda_\pi^2\subset\eta\left(\Lambda_\pi^2\right) \tag{3.3}$$

On the other hand let $(x_{mn}/\pi_{mn})\in\eta\left(\Lambda_\pi^2\right)$. Then $(y_{mn}/\pi_{mn})\in\Lambda_\pi^2$. Hence there exists a positive constant T such that $\left|\frac{y_{mn}}{\pi_{mn}}\right|<T^{m+n}$ for $m,n=0,1,2,\cdots$. This in turn implies that

$$\left|\frac{p_{00}(x_{mn}/\pi_{mn})+\cdots+P_{mn}(x_{00}/\pi_{00})}{P_{mn}}\right|<T^{m+n}$$

Hence

$$\frac{1}{P_{mn}}\left(\left|p_{00}\left(x_{mn}/\pi_{mn}\right)+\cdots+p_{mn}\left(x_{00}/\pi_{00}\right)\right|\right)<T^{m+n}$$

and thus

$$\left|p_{00}\left(x_{mn/\pi_{mn}}\right)+\cdots+p_{mn}\left(x_{00}/\pi_{00}\right)\right|<P_{mn}T^{m+n}.$$

Take $p_{00}=1; p_{11}=\cdots=p_{mn}=0$. Then it follows that $P_{mn}=1$ and so $\left|\frac{x_{mn}}{\pi_{mn}}\right|<T^{m+n}$ for all m,n. Consequently $x=(x_{mn})\in\Lambda_\pi^2$. Hence

$$\eta\left(\Lambda_\pi^2\right)\subset\Lambda_\pi^2 \tag{3.4}$$

From (3.3) and (3.4) we get $\eta\left(\Lambda_\pi^2\right)=\Lambda_\pi^2$. This completes the proof.

3.3. **Proposition.** Γ_π^2 is not a barreled space
Proof: Let

$$A=\left\{x\in\Gamma_\pi^2:\left|\frac{x_{mn}}{\pi_{mn}}\right|^{\frac{1}{m+n}}\leq\frac{1}{m+n},\forall m,n\right\}.$$

Then A is an absolutely convex, closed absorbent in Γ_π^2. But A is not a neighbour hood of zero. Hence Γ_π^2 is not barreled.

3.4. **Proposition.** Γ_π^2 is not semi reflexive
Proof: Let $\left\{\delta^{(mn)}\right\}\in U$ be the unit closed ball in Γ_π^2. Clearly no subsequence of $\left\{\delta^{(mn)}\right\}$ can converge weakly to any $y\in\Gamma_\pi^2$. Hence Γ_π^2 is not semi reflexive.

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