

**AN IMPLICIT ITERATION SCHEME WITH ERRORS FOR
COMMON FIXED POINTS OF GENERALIZED
ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS**

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ABSTRACT. In this paper, we study an implicit iteration scheme with errors for a finite family of generalized asymptotically quasi-nonexpansive mappings and give the necessary and sufficient condition to converge to common fixed points for the said scheme in Banach spaces. The results presented in this paper generalize, improve and unify the corresponding results in [3, 4, 7, 9, 11, 12, 13, 15, 16, 17, 18].

1. INTRODUCTION

It is well known that the concept of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [8] who proved that every asymptotically nonexpansive self-mapping of nonempty closed bounded and convex subset of a uniformly convex Banach space has fixed point. In 1973, Petryshyn and Williamson [16] gave necessary and sufficient conditions for Mann iterative sequence [14] to converge to fixed points of quasi-nonexpansive mappings. In 1997, Ghosh and Debnath [7] extended the results of Petryshyn and Williamson [16] and gave necessary and sufficient conditions for Ishikawa iterative sequence to converge to fixed points for quasi-nonexpansive mappings.

Liu [13] extended results of [7, 16] and gave necessary and sufficient conditions for Ishikawa iterative sequence with errors to converge to fixed point of asymptotically quasi-nonexpansive mappings.

In 2003, Zhou et al. [26] introduced a new class of generalized asymptotically nonexpansive mapping and gave a necessary and sufficient condition for the modified Ishikawa and Mann iterative sequences to converge to fixed points for the class of mappings. Atsushiba [1] studied the necessary and sufficient condition for the convergence of iterative sequences to a common fixed point of the finite family of

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asymptotically nonexpansive mappings in Banach spaces. Suzuki [21], Zeng and Yao [25] discussed a necessary and sufficient condition for common fixed points of two nonexpansive mappings and a finite family of nonexpansive mappings, and proved some convergence theorems for approximating a common fixed point, respectively (see, also [2, 5, 6, 19, 27]).

Recently, Lan [11] introduced a new class of generalized asymptotically quasi-nonexpansive mappings and gave necessary and sufficient condition for the 2-step modified Ishikawa iterative sequences to converge to fixed points for the class of mappings.

More recently, Nantadilok [15] (Thai J. Math. 6(2) (2008), 297-308) extended and improved the result of Lan [11] and gave the necessary and sufficient condition for convergence of common fixed point for three-step iteration scheme with errors for generalized asymptotically quasi-nonexpansive mappings.

In 2001, Xu and Ori [24] have introduced an implicit iteration process for a finite family of nonexpansive mappings in a Hilbert space H . Let C be a nonempty subset of H . Let T_1, T_2, \dots, T_N be self-mappings of C and suppose that $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \neq \emptyset$, the set of common fixed points of $T_i, i = 1, 2, \dots, N$. An implicit iteration process for a finite family of nonexpansive mappings is defined as follows, with $\{t_n\}$ a real sequence in $(0, 1)$, $x_0 \in C$:

$$\begin{aligned} x_1 &= t_1 x_0 + (1 - t_1) T_1 x_1, \\ x_2 &= t_2 x_1 + (1 - t_2) T_2 x_2, \\ &\vdots \\ x_N &= t_N x_{N-1} + (1 - t_N) T_N x_N, \\ x_{N+1} &= t_{N+1} x_N + (1 - t_{N+1}) T_1 x_{N+1}, \\ &\vdots \end{aligned}$$

which can be written in the following compact form:

$$x_n = t_n x_{n-1} + (1 - t_n) T_n x_n, \quad n \geq 1 \tag{1.1}$$

where $T_k = T_{k \bmod N}$. (Here the mod N function takes values in \mathcal{N}). And they proved the weak convergence of the process (1.1).

In 2003, Sun [20] extended the process (1.1) to a process for a finite family of asymptotically quasi-nonexpansive mappings, with $\{\alpha_n\}$ a real sequence in $(0, 1)$ and an initial point $x_0 \in C$, which is defined as follows:

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + (1 - \alpha_1) T_1 x_1, \\
&\vdots \\
x_N &= \alpha_N x_{N-1} + (1 - \alpha_N) T_N x_N, \\
x_{N+1} &= \alpha_{N+1} x_N + (1 - \alpha_{N+1}) T_1^2 x_{N+1}, \\
&\vdots \\
x_{2N} &= \alpha_{2N} x_{2N-1} + (1 - \alpha_{2N}) T_N^2 x_{2N}, \\
x_{2N+1} &= \alpha_{2N+1} x_{2N} + (1 - \alpha_{2N+1}) T_1^3 x_{2N+1}, \\
&\vdots
\end{aligned}$$

which can be written in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_i^k x_n, \quad n \geq 1 \quad (1.2)$$

where $n = (k - 1)N + i$, $i \in \mathcal{N}$.

Sun [20] proved the strong convergence of the process (1.2) to a common fixed point, requiring only one member T in the family $\{T_i : i \in \mathcal{N}\}$ to be semi-compact. The result of Sun [20] generalized and extended the corresponding main results of Wittmann [23] and Xu and Ori [24].

The purpose of this paper is to study an implicit iteration process with errors which converges strongly to a common fixed point of a finite family of generalized asymptotically quasi-nonexpansive mappings in Banach spaces. Also prove some strong convergence theorems for said mappings.

Let X be a normed space, C be a nonempty closed convex subset of X , and $T_i : C \rightarrow C$, $i = \{1, 2, \dots, N\} = \mathcal{N}$ be N generalized asymptotically quasi-nonexpansive mappings with respect to $\{r_{n_i}\}$ and $\{s_{n_i}\}$ with $\sum_{n=1}^{\infty} r_{n_i} < \infty$ and $\sum_{n=1}^{\infty} s_{n_i} < \infty$. Define a sequence $\{x_n\}$ in C as follows:

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + \beta_1 T_1 x_1 + \gamma_1 u_1, \\
x_2 &= \alpha_2 x_1 + \beta_2 T_2 x_2 + \gamma_2 u_2, \\
&\vdots \\
x_N &= \alpha_N x_{N-1} + \beta_N T_N x_N + \gamma_N u_N, \\
x_{N+1} &= \alpha_{N+1} x_N + \beta_{N+1} T_1^2 x_{N+1} + \gamma_{N+1} u_{N+1}, \\
&\vdots \\
x_{2N} &= \alpha_{2N} x_{2N-1} + \beta_{2N} T_N^2 x_{2N} + \gamma_{2N} u_{2N}, \\
x_{2N+1} &= \alpha_{2N+1} x_{2N} + \beta_{2N+1} T_1^3 x_{2N+1} + \gamma_{2N+1} u_{2N+1}, \\
&\vdots
\end{aligned}$$

which can be written in the following compact form:

$$x_n = \alpha_n x_{n-1} + \beta_n T_i^k x_n + \gamma_n u_n, \quad n \geq 1 \quad (1.3)$$

where $n = (k-1)N + i$, $i \in \mathcal{N}$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$ and $\{u_n\}$ is a bounded sequence in C .

2. Preliminaries

In the sequel, we need the following definitions and lemmas for our main results in this paper.

Definition 2.1. (see [15]) Let X be a real Banach space, C be a nonempty subset of X and $F(T)$ denotes the set of fixed points of T . A mapping $T: C \rightarrow C$ is said to be

(1) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad (2.1)$$

for all $x, y \in C$,

(2) quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\| \quad (2.2)$$

for all $x \in C$ and $p \in F(T)$,

(3) asymptotically nonexpansive if there exists a sequence $\{r_n\} \subset [0, \infty)$ with $r_n \rightarrow 0$ as $n \rightarrow \infty$ such that

$$\|T^n x - T^n y\| \leq (1 + r_n)\|x - y\| \quad (2.3)$$

for all $x, y \in C$,

(4) asymptotically quasi-nonexpansive if (3) holds for all $x \in C$ and $y \in F(T)$;

(5) generalized quasi-nonexpansive with respect to $\{s_n\}$, if there exists a sequence $\{s_n\} \subset [0, 1)$ with $s_n \rightarrow 0$ as $n \rightarrow \infty$ such that

$$\|T^n x - p\| \leq \|x - p\| + s_n\|x - T^n x\| \quad (2.4)$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$,

(6) generalized asymptotically quasi-nonexpansive with respect to $\{r_n\}$ and $\{s_n\} \subset [0, 1)$ with $r_n \rightarrow 0$ and $s_n \rightarrow 0$ as $n \rightarrow \infty$ such that

$$\|T^n x - p\| \leq (1 + r_n)\|x - p\| + s_n\|x - T^n x\| \quad (2.5)$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$.

Remark 2.1. From the above definitions, it is clear that:

(i) a nonexpansive mapping is a generalized asymptotically quasi-nonexpansive mapping;

(ii) a quasi-nonexpansive mapping is a generalized asymptotically quasi-nonexpansive mapping;

(iii) an asymptotically nonexpansive mapping is a generalized asymptotically quasi-nonexpansive mapping;

(iv) a generalized quasi-nonexpansive mapping is a generalized asymptotically quasi-nonexpansive mapping.

However, the converse of the above statements are not true.

Lemma 2.1. (see [22]) *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad n \geq 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n \rightarrow \infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence converging to zero, then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.2. (see [11]) *Let C be nonempty closed subset of a Banach space X and $T: C \rightarrow C$ be a generalized asymptotically quasi-nonexpansive mapping with the fixed point set $F(T) \neq \emptyset$. Then $F(T)$ is closed subset in C .*

3. Main Results

In this section, we prove strong convergence theorems of an implicit iteration scheme with bounded errors for a finite family of generalized asymptotically quasi-nonexpansive mappings in a real Banach space.

Theorem 3.1. *Let X be a real arbitrary Banach space, C be a nonempty closed convex subset of X . Let $T_i: C \rightarrow C$, $i = \{1, 2, \dots, N\} = \mathcal{N}$ be N generalized asymptotically quasi-nonexpansive mappings with respect to $\{r_{n_i}\}$ and $\{s_{n_i}\}$ for all $i \in \mathcal{N}$ such that $\sum_{n=1}^{\infty} \frac{r_{n_i} + 2s_{n_i}}{1 - s_{n_i}} < \infty$. Let $\{x_n\}$ be the sequence defined by (1.3) with $\sum_{n=1}^{\infty} \gamma_n < \infty$. If $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ and $\{\beta_n\} \subset (s, 1 - s)$ for some $s \in (0, \frac{1}{2})$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point of $\{T_i : i \in \mathcal{N}\}$ if and only if $\liminf_{n \rightarrow \infty} d(x_n, \mathcal{F}) = 0$, where $d(x, \mathcal{F})$ denotes the distance between x and the set \mathcal{F} .*

Proof. The necessity is obvious and it is omitted. Now we prove the sufficiency. Let $p \in \mathcal{F}$, then it follows from (2.5), we have

$$\begin{aligned} \|x_n - T_i^k x_n\| &\leq \|x_n - p\| + \|T_i^k x_n - p\| \\ &\leq \|x_n - p\| + (1 + r_{k_i})\|x_n - p\| + s_{k_i}\|x_n - T_i^k x_n\| \\ &\leq (2 + r_{k_i})\|x_n - p\| + s_{k_i}\|x_n - T_i^k x_n\| \\ &\leq (2 + r_{k_i})\|x_n - p\| + s_{k_i}\|x_n - T_i^k x_n\| \end{aligned}$$

which implies that

$$\|x_n - T_i^k x_n\| \leq \frac{2 + r_{k_i}}{1 - s_{k_i}} \|x_n - p\|. \quad (3.1)$$

Again for any $p \in \mathcal{F}$, where $n = (k - 1)N + i$, $T_n = T_{n(\text{mod } N)} = T_i$, $i \in \mathcal{N}$, it follows that from (1.3), we note that

$$\begin{aligned}
\|x_n - p\| &= \|\alpha_n x_{n-1} + \beta_n T_i^k x_n + \gamma_n u_n - p\| \\
&= \|\alpha_n (x_{n-1} - p) + \beta_n (T_i^k x_n - p) + \gamma_n (u_n - p)\| \\
&\leq \alpha_n \|x_{n-1} - p\| + \beta_n \|T_i^k x_n - p\| + \gamma_n \|u_n - p\| \\
&\leq \alpha_n \|x_{n-1} - p\| + \beta_n \left[(1 + r_{k_i}) \|x_n - p\| + s_{k_i} \|x_n - T_i^k x_n\| \right] \\
&\quad + \gamma_n \|u_n - p\| \\
&\leq \alpha_n \|x_{n-1} - p\| + \beta_n \left[(1 + r_{k_i}) \|x_n - p\| + s_{k_i} \cdot \left(\frac{2 + r_{k_i}}{1 - s_{k_i}} \right) \|x_n - p\| \right] \\
&\quad + \gamma_n \|u_n - p\| \\
&\leq \alpha_n \|x_{n-1} - p\| + \beta_n \left(\frac{1 + r_{k_i} + s_{k_i}}{1 - s_{k_i}} \right) \|x_n - p\| + \gamma_n \|u_n - p\| \\
&\leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n) \left(1 + \frac{r_{k_i} + 2s_{k_i}}{1 - s_{k_i}} \right) \|x_n - p\| + \gamma_n \|u_n - p\| \\
&\leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n)(1 + w_{k_i}) \|x_n - p\| + \gamma_n \|u_n - p\| \\
&\leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n + w_{k_i}) \|x_n - p\| + \gamma_n \|u_n - p\| \tag{3.2}
\end{aligned}$$

where $w_{k_i} = \frac{r_{k_i} + 2s_{k_i}}{1 - s_{k_i}}$. Since $\lim_{n \rightarrow \infty} \gamma_n = 0$, there exists a natural number n_1 such that for $n > n_1$, $\gamma_n \leq \frac{s}{2}$. Hence

$$\alpha_n = 1 - \beta_n - \gamma_n \geq 1 - (1 - s) - \frac{s}{2} = \frac{s}{2}$$

for $n > n_1$. Thus, we have from (3.2) that

$$\alpha_n \|x_n - p\| \leq \alpha_n \|x_{n-1} - p\| + w_{k_i} \|x_n - p\| + \gamma_n \|u_n - p\|$$

and

$$\begin{aligned}
\|x_n - p\| &\leq \|x_{n-1} - p\| + \frac{w_{k_i}}{\alpha_n} \|x_n - p\| + \frac{\gamma_n}{\alpha_n} \|u_n - p\| \\
&\leq \|x_{n-1} - p\| + \frac{2}{s} w_{k_i} \|x_n - p\| + \frac{2}{s} \gamma_n \|u_n - p\|. \tag{3.3}
\end{aligned}$$

Since $\sum_{k=1}^{\infty} \frac{r_{k_i} + 2s_{k_i}}{1 - s_{k_i}} < \infty$, it follows that $\sum_{k=1}^{\infty} w_{k_i} < \infty$ for all $i \in \mathcal{N}$ and $\lim_{n \rightarrow \infty} w_{n_i} = 0$ for each $i \in \mathcal{N}$. Hence there exists a natural number n_2 , as $n > \frac{n_2}{N} + 1$, that is, $n > n_2$ such that

$$w_{n_i} \leq \frac{s}{4}, \quad \forall i \in \mathcal{N}. \tag{3.4}$$

Then (3.3) becomes

$$\|x_n - p\| \leq \frac{s}{s - 2w_{k_i}} \|x_{n-1} - p\| + \frac{2\gamma_n}{s - 2w_{k_i}} \|u_n - p\|. \tag{3.5}$$

Let

$$1 + t_{k_i} = \frac{s}{s - 2w_{k_i}} = 1 + \frac{2w_{k_i}}{s - 2w_{k_i}}.$$

Then

$$t_{k_i} = \frac{2w_{k_i}}{s - 2w_{k_i}} < \frac{4}{s} w_{k_i}.$$

Therefore

$$\sum_{k=1}^{\infty} t_{k_i} < \frac{4}{s} \sum_{k=1}^{\infty} w_{k_i} < \infty, \quad \forall i \in \mathcal{N} \quad (3.6)$$

and (3.5) becomes

$$\begin{aligned} \|x_n - p\| &\leq (1 + t_{k_i})\|x_{n-1} - p\| + \frac{2\gamma_n}{s - 2w_{k_i}}\|u_n - p\| \\ &\leq (1 + t_{k_i})\|x_{n-1} - p\| + \frac{4}{s}\gamma_n K, \end{aligned} \quad (3.7)$$

where $K = \sup_{n \geq 1} \{\|u_n - p\|\}$, since $\{u_n\}$ is a bounded sequence in C . This implies that

$$d(x_n, \mathcal{F}) \leq (1 + t_{k_i})d(x_{n-1}, \mathcal{F}) + \frac{4}{s}\gamma_n K. \quad (3.8)$$

Since $\sum_{k=1}^{\infty} t_{k_i} < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$, it follows from Lemma 2.1, we know that $\lim_{n \rightarrow \infty} d(x_n, \mathcal{F}) = 0$.

Next, we will prove that $\{x_n\}$ is a Cauchy sequence. Notice that when $x > 0$, $1 + x \leq e^x$, from (3.7) we have

$$\begin{aligned} \|x_{n+m} - p\| &\leq (1 + t_{k_i})\|x_{n+m-1} - p\| + \frac{4K}{s}\gamma_{n+m} \\ &\leq (1 + t_{k_i})\left[(1 + t_{k_i})\|x_{n+m-2} - p\| + \frac{4K}{s}\gamma_{n+m-1}\right] + \frac{4K}{s}\gamma_{n+m} \\ &\leq (1 + t_{k_i})^2\|x_{n+m-2} - p\| + \frac{4K}{s}(1 + t_{k_i})\left[\gamma_{n+m-1} + \gamma_{n+m}\right] \\ &\leq (1 + t_{k_i})^2\left[(1 + t_{k_i})\|x_{n+m-3} - p\| + \frac{4K}{s}\gamma_{n+m-2}\right] \\ &\quad + \frac{4K}{s}(1 + t_{k_i})\left[\gamma_{n+m-1} + \gamma_{n+m}\right] \\ &\leq (1 + t_{k_i})^3\|x_{n+m-3} - p\| \\ &\quad + \frac{4K}{s}(1 + t_{k_i})\left[\gamma_{n+m-2} + \gamma_{n+m-1} + \gamma_{n+m}\right] \\ &\leq \\ &\quad \vdots \\ &\leq \\ &\leq \exp\left\{\sum_{i=1}^N \sum_{k=1}^{\infty} t_{k_i}\right\} \|x_n - p\| + \frac{4K}{s} \exp\left\{\sum_{i=1}^N \sum_{k=1}^{\infty} t_{k_i}\right\} \sum_{j=n+1}^{n+m} \gamma_j \\ &\leq K' \|x_n - p\| + \frac{4KK'}{s} \sum_{j=n+1}^{n+m} \gamma_j, \end{aligned} \quad (3.9)$$

for all $p \in \mathcal{F}$ and $m, n \in \mathbb{N}$, where $K' = \exp\left\{\sum_{i=1}^N \sum_{k=1}^{\infty} t_{k_i}\right\} < \infty$. Since $\lim_{n \rightarrow \infty} d(x_n, \mathcal{F}) = 0$ and $\sum_{k=1}^{\infty} w_{k_i} < \infty$ ($i \in \mathcal{N}$), there exists a natural number n_1 such that for $n \geq n_1$,

$$d(x_n, \mathcal{F}) < \frac{\varepsilon}{4K'} \quad \text{and} \quad \sum_{j=n_1+1}^{n+m} \gamma_j \leq \frac{s \cdot \varepsilon}{16KK'}. \quad (3.10)$$

Thus there exists a point $q \in \mathcal{F}$ such that $d(x_{n_1}, q) < \frac{\varepsilon}{4K'}$. It follows from (3.9) that for all $n \geq n_1$ and $m \geq 1$, we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - q\| + \|x_n - q\| \\ &\leq K' \|x_{n_1} - q\| + \frac{4KK'}{s} \sum_{j=n_1+1}^{n+m} \gamma_j \\ &\quad + K' \|x_{n_1} - q\| + \frac{4KK'}{s} \sum_{j=n_1+1}^{n+m} \gamma_j \\ &< K' \cdot \frac{\varepsilon}{4K'} + \frac{4KK'}{s} \cdot \frac{s \cdot \varepsilon}{16KK'} \\ &\quad + K' \cdot \frac{\varepsilon}{4K'} + \frac{4KK'}{s} \cdot \frac{s \cdot \varepsilon}{16KK'} \\ &= \varepsilon. \end{aligned} \quad (3.11)$$

This implies that $\{x_n\}$ is a Cauchy sequence X . Thus the completeness of X implies that $\{x_n\}$ must be convergent. Let $\lim_{n \rightarrow \infty} x_n = p$, that is, $\{x_n\}$ converges to p . Then $p \in C$, because C is a closed subset of X . By Lemma 2.2 we know that the set \mathcal{F} is closed. From the continuity of $d(x_n, \mathcal{F})$ with

$$d(x_n, \mathcal{F}) \rightarrow 0 \quad \text{and} \quad x_n \rightarrow p \quad \text{as} \quad n \rightarrow \infty,$$

we get

$$d(p, \mathcal{F}) = 0$$

and so $p \in \mathcal{F} = \bigcap_{i=1}^N F(T_i)$, that is, p is a common fixed point of $\{T_i : i \in \mathcal{N}\}$. This completes the proof. \square

Theorem 3.2. *Let X be a real arbitrary Banach space, C be a nonempty closed convex subset of X . Let $T_i: C \rightarrow C$, $i = \{1, 2, \dots, N\} = \mathcal{N}$ be N generalized asymptotically quasi-nonexpansive mappings with respect to $\{r_{n_i}\}$ and $\{s_{n_i}\}$ for all $i \in \mathcal{N}$ such that $\sum_{n=1}^{\infty} \frac{r_{n_i} + 2s_{n_i}}{1 - s_{n_i}} < \infty$. Let $\{x_n\}$ be the sequence defined by (1.3) with $\sum_{n=1}^{\infty} \gamma_n < \infty$. If $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ and $\{\beta_n\} \subset (s, 1 - s)$ for some $s \in (0, \frac{1}{2})$. Then $\{x_n\}$ converges strongly to a common fixed point p of the mappings $\{T_i : i \in \mathcal{N}\}$ if and only if there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to p .*

Proof. The proof of Theorem 3.2 follows from Lemma 2.1 and Theorem 3.1. \square

Theorem 3.3. *Let X be a real arbitrary Banach space, C be a nonempty closed convex subset of X . Let $T_i: C \rightarrow C$, $i = \{1, 2, \dots, N\} = \mathcal{N}$ be N generalized asymptotically quasi-nonexpansive mappings with respect to $\{r_{n_i}\}$ and $\{s_{n_i}\}$ for all $i \in \mathcal{N}$ such that $\sum_{n=1}^{\infty} \frac{r_{n_i} + 2s_{n_i}}{1 - s_{n_i}} < \infty$. Let $\{x_n\}$ be the sequence defined by (1.3) with*

$\sum_{n=1}^{\infty} \gamma_n < \infty$. If $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ and $\{\beta_n\} \subset (s, 1-s)$ for some $s \in (0, \frac{1}{2})$. Suppose that the mappings $\{T_i : i \in \mathcal{N}\}$ satisfy the following conditions:

(C₁) $\lim_{n \rightarrow \infty} \|x_n - T_i x_n\| = 0$ for all $i \in \{1, 2, \dots, N\} = \mathcal{N}$;

(C₂) there exists a constant $A > 0$ such that $\{\|x_n - T_i x_n\|\} \geq Ad(x_n, \mathcal{F})$ for all $i \in \{1, 2, \dots, N\} = \mathcal{N}$ and for all $n \geq 1$.

Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in \mathcal{N}\}$.

Proof. From condition (C₁) and (C₂), we have $\lim_{n \rightarrow \infty} d(x_n, \mathcal{F}) = 0$, it follows as in the proof of Theorem 3.1, that $\{x_n\}$ must converges strongly to a common fixed point of the mappings $\{T_i : i \in \mathcal{N}\}$. \square

Remark 3.1. Theorem 3.1 extends, improves and unifies the corresponding results of [3, 7, 11, 12, 13, 15, 16, 18]. Especially Theorem 3.1 extends, improves and unifies Theorem 1 and 2 in [13], Theorem 1 in [12] and Theorem 3.2 in [18] in the following ways:

(1) The asymptotically quasi-nonexpansive mapping in [12], [13] and [18] is replaced by finite family of generalized asymptotically quasi-nonexpansive mappings.

(2) The usual Ishikawa [10] iteration scheme in [12], the usual modified Ishikawa iteration scheme with errors in [13] and the usual modified Ishikawa iteration scheme with errors for two mappings are extended to the implicit iteration scheme with errors for a finite family of mappings.

Remark 3.2. Theorem 3.2 extends, improves and unifies Theorem 3 in [13] and Theorem 3.3 extends, improves and unifies Theorem 3 in [12] in the following aspects:

(1) The asymptotically quasi-nonexpansive mapping in [12] and [13] is replaced by finite family of generalized asymptotically quasi-nonexpansive mappings.

(2) The usual Ishikawa [10] iteration scheme in [12] and the usual modified Ishikawa iteration scheme with errors in [13] are extended to the implicit iteration scheme with errors for a finite family of mappings.

Remark 3.3. Theorem 3.1 also extends the corresponding results of [11] and [15] to the case of implicit iteration scheme with errors for a finite family of mappings considered in this paper.

Remark 3.4. Our results also extend the corresponding results of [9, 17] and [4] to the case of more general class of asymptotically nonexpansive mappings and implicit iteration scheme with errors respectively, considered in this paper.

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