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INTEGRAL TYPE MODIFICATION FOR $q-$LAGUERRE POLYNOMIALS

(COMMUNICATED BY PROFESSOR F. MARCELLAN)

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Abstract. Özarslan gave the approximation properties of linear positive operators including the q-Laguerre polynomials in [13]. In this paper, we will give Kantorovich type generalization for this operator with the help of Riemann type q-integral. We also get approximation properties for the generalized operator with modulus.

1. Introduction

In 1960, The Meyer-König and Zeller operators

$$M_n(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{k+n+1}\right)\binom{n+k}{k} x^k (1-x)^{n+1}$$

$(0 \leq x < 1)$ were introduced by Meyer-König and Zeller in [11].

In order to give the monotonicity properties, Cheney and Sharma [1] modified these operators as:

$$M_n^*(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{k+n}\right)\binom{n+k}{k} x^k (1-x)^{n+1}$$

$(0 \leq x < 1)$.

In [1], they also introduced the operators

$$P_n(f;x) = \exp\left(\frac{tx}{1-x}\right)\sum_{k=0}^{\infty} f\left(\frac{k}{k+n}\right) L_k^{(n)}(t)\, x^k (1-x)^{n+1}$$

$(0 \leq x < 1$ and $-\infty < t \leq 0)$ where $L_k^{(n)}(t)$ denotes the Laguerre polynomials. Since $L_k^{(n)}(0) = \binom{n+k}{k}$, then $M_n^*(f;x)$ is the special case of the operators $P_n(f;x)$.

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The q-type generalization of the linear positive operators was initiated by Phillips in [14]. He introduced the q-type generalization of the classical Bernstein operator and obtained the rate of convergence and the Voronovskaja type asymptotic formula for these operators.

q-Laguerre polynomials were defined by (Hahn [7, p. 29], Jackson [8, p. 57] and Moak [12, p. 21, eq. 23])

$$L_n^{(\alpha)}(x;q)=\frac{\left(q^{\alpha+1};q\right)_n}{(q;q)_n}\sum_{k=0}^{n}\frac{(q^{-n};q)_k\,q^{\binom{k}{2}}\,(1-q)^k\left(q^{n+\alpha+1}x\right)^k}{(q^{\alpha+1};q)_k\,(q;q)_k}$$

where

$$(\alpha;q)_n=\begin{cases}1 & ;n=0\\ (1-\alpha)(1-\alpha q)\ldots\left(1-\alpha q^{n-1}\right) & ;n\in\mathbb{N},\ \alpha\in\mathbb{C}\end{cases}.$$

Moak gave the following recurrence relation [12, p. 29, eq. 4.14] and generating function [12, p. 29, eq. 4.17] for the q-Laguerre polynomials:

$$tL_{k-1}^{(\alpha+1)}(t;q)=[k+\alpha]\,q^{-\alpha-k}L_{k-1}^{(\alpha)}(t;q)-[k]\,q^{-\alpha-k}L_k^{(\alpha)}(t;q)$$

$$(\operatorname{Re}\alpha>-1,\ k=1,2,\ldots),$$

$$\begin{aligned}F_\alpha(x,t)&=\frac{\left(xq^{\alpha+1};q\right)_\infty}{(x;q)_\infty}\sum_{m=0}^{\infty}\frac{q^{m^2+\alpha m}\left[-(1-q)xt\right]^m}{(q;q)_m\,(xq^{\alpha+1};q)_m}\\&=\sum_{k=0}^{\infty}L_k^{(\alpha)}(t;q)\,x^k\qquad(\operatorname{Re}\alpha>1)\end{aligned}\tag{1.1}$$

where

$$(a;q)_\infty=\prod_{j=0}^{\infty}\left(1-aq^j\right),\quad(a\in\mathbb{C}).$$

Trif [16] defined the Meyer-König and Zeller operators based on q-integer as follows:

$$M_{n,q}(f;x)=\prod_{j=0}^{n}\left(1-q^jx\right)\sum_{k=0}^{\infty}f\left(\frac{[k]}{[k+n]}\right)\begin{bmatrix}n+k\\k\end{bmatrix}x^k$$

$(0\leq x<1)$ where

$$[k]=\begin{cases}\left(1-q^k\right)/(1-q) & ;q\neq1\\ 1 & ;q=1\end{cases},$$

$$[k]!=\begin{cases}[1]\,[2]\ldots[k] & ;k\geq1\\ 1 & ;k=0\end{cases}$$

and

$$\begin{bmatrix}n+k\\k\end{bmatrix}=\frac{[n+k]!}{[n]!\,[k]!}$$

$(k,\ n\in\mathbb{N})$ for $q\in(0,1]$.

In [13], Özarslan defined the q-analogue for $P_n(f;x)$ operators as follow:

$$P_{n,q}(f;x)=\frac{1}{F_n(x,t)}\sum_{k=0}^{\infty}f\left(\frac{[k]}{[k+n]}\right)L_k^{(n)}(t;q)\,x^k\tag{1.2}$$

where $x\in[0,1]$, $t\in(-\infty,0]$, $q\in(0,1]$ and $\{F_n(x,t)\}_{n\in\mathbb{N}}$ is the generating functions for the q-Laguerre polynomials. Since $L_k^{(n)}(0;q)=\begin{bmatrix} n+k \\ k \end{bmatrix}$ and $F_n(x,0)=\prod_{j=0}^{n}\left(1-q^jx\right)$, then $M_{n,q}(f;x)$ is the special case of the operators $P_{n,q}(f;x)$.

Let us recall the concepts of q-differential, q-derivative and q-integral respectively.

For an arbitrary function $f(x)$, the q-differential is given by

$$d_qf(x)=f(qx)-f(x).$$

For an arbitrary function $f(x)$, the q-derivative is defined as

$$D_qf(x)=\frac{d_qf(x)}{d_qx}=\frac{f(qx)-f(x)}{(q-1)x}.$$

Now suppose that $0<a<b$, $0<q<1$ and f is a real-valued function. The q-Jackson integral of f over the interval $[0,b]$ and a general interval $[a,b]$ are defined by (see [9])

$$\int_0^a f(t)\,d_qt=(1-q)\,a\sum_{j=0}^{\infty}f\left(q^ja\right)q^j$$

and

$$\int_a^b f(t)\,d_qt=\int_0^b f(t)\,d_qt-\int_0^a f(t)\,d_qt$$

respectively.

It is clear that q-Jackson integral of f over an interval $[a,b]$ contains two infinite sums, so some problems are encountered in deriving the q-analogues of some well-known integral inequalities which are used to compute order of approximation of linear positive operators containing q-Jackson integral. In order to overcome these problems Gauchman [5] and Marinković et al. [10] introduced a new type of q-integral. This new q-integral is called as Riemann type q-integral and defined by

$$\int_a^b f(t)\,d_q^Rt=(1-q)(b-a)\sum_{j=0}^{\infty}f\left(a+(b-a)q^j\right)q^j$$

where a,b and q are some real numbers such that $0<a<b$ and $0<q<1$. Contrary to the classical definition of q-integral, this definition includes only points within the interval of integration.

Now, we give a Kantorovich type generalization of operators P_n, M_n^*, $M_{n,q}$ and $P_{n,q}$. This Kantorovich type generalization was studied by Dalmanoğlu [3], Radu [15] and etc. We consider the sequence of Kantorovich type linear positive operators as follow:

$$(K_{n,q}f)(x,t)=\frac{1}{F_{n,q}(x,t)}\sum_{k=0}^{\infty}\left(\int_{[k]/[n+k]}^{[k+1]/[n+k]}f(t)\,d_q^Rt\right)q^{-k}[n+k]\,L_k^{(n)}(t;q)\,x^k, \tag{1.3}$$

where $x\in[0,1]$, $t\in(-\infty,0]$, $q\in(0,1]$, $n>1$ and $\{F_n(x,t)\}_{n\in\mathbb{N}}$ is the generating functions for the q-Laguerre polynomials which was given in (1.1).

2. Approximation Properties of the $(K_{n,q}f)(x,t)$ Operators

We have the following theorem for the convergence of $(K_{n,q}f)(x,t)$ operators.

Theorem 1. *Let $q := q_n$ be a sequence satisfying $\lim_n q_n = 1$ and $0 < q_n < 1$. If $f \in C[0,1]$ and $\frac{|t|}{[n]} \to 0$ $(n \to \infty)$ then $(K_{n,q}f)$ converges to f uniformly on $[0,b]$ $(0 < b < 1)$.*

Proof. By Korovkin's theorem, it is sufficient for us to prove that $(K_{n,q}f)$ is a positive linear operator and that the desired convergence occurs whenever f is a quadratic function. It is obvious that $(K_{n,q}f)$ is linear and positive operators.

$$(K_{n,q}e_0)(x,t) = \frac{1}{F_{n,q}(x,t)} \sum_{k=0}^{\infty} \left(\int_{[k]/[n+k]}^{[k+1]/[n+k]} d_q^R t \right) q^{-k} [n+k] L_k^{(n)}(t;q) x^k.$$

Since

$$\int_{[k]/[n+k]}^{[k+1]/[n+k]} d_q^R t = \frac{q^k}{[n+k]}$$

and from (1.1), we get

$$(K_{n,q}e_0)(x,t) = 1. \tag{2.1}$$

By considering the function $f(s) = e_1(s) = s$, we obtain

$$(K_{n,q}e_1)(x,t) = \frac{1}{F_n(x,t)} \sum_{k=0}^{\infty} \left(\int_{[k]/[n+k]}^{[k+1]/[n+k]} t d_q^R t \right) q^{-k} [n+k] L_k^{(n)}(t;q) x^k.$$

One can easily compute that

$$\int_{[k]/[n+k]}^{[k+1]/[n+k]} t d_q^R t = \frac{q^k}{[n+k]^2} \left([k] + \frac{q^k}{[2]} \right),$$

then we have

$$(K_{n,q}e_1)(x,t) = \frac{1}{F_n(x,t)} \sum_{k=0}^{\infty} \frac{1}{[n+k]} \left([k] + \frac{q^k}{[2]} \right) L_k^{(n)}(t;q) x^k.$$

Since $q^k < 1$ for $0 < q < 1$ and $[k+n] \geq [n]$, we can write

$$(K_{n,q}e_1)(x,t) \leq P_{n,q}(e_1;x) + \frac{1}{[2][n]} P_{n,q}(e_0;x).$$

If we use $P_{n,q}(e_0;x) = 1$ and $P_{n,q}(e_1;x) \leq x - \dfrac{tx}{[n](1-bq^{n+1})}$ from [13], then we get

$$(K_{n,q}e_1)(x,t) - x \leq -\frac{tx}{[n](1-bq^{n+1})} + \frac{1}{[2][n]}. \tag{2.2}$$

On the other hand, we see

$$(K_{n,q}e_1)(x,t) \geq P_{n,q}(e_1;x).$$

If we use $P_{n,q}(e_1;x) \geq x$ from [13], we obtain

$$(K_{n,q}e_1)(x,t) \geq x. \tag{2.3}$$

From (2.2) and (2.3), we have

$$0 \leq \left(K_{n,q}e_1\right)(x,t) - x \leq -\frac{tx}{[n]\left(1-bq^{n+1}\right)} + \frac{1}{[2]\,[n]}. \tag{2.4}$$

From (2.4), it is obvious that

$$\left\|\left(K_{n,q}e_1\right)(x,t) - x\right\|_{C[0,b]} \leq \frac{|t|\,b}{[n]\left(1-bq^{n+1}\right)} + \frac{1}{[2]\,[n]}. \tag{2.5}$$

We proceed with the consideration of the function $f(s) = e_2(s) = s^2$.

$$\left(K_{n,q}e_2\right)(x,t) = \frac{1}{F_n(x,t)} \sum_{k=0}^{\infty} \left(\int_{[k]/[n+k]}^{[k+1]/[n+k]} t^2 d_q^R t \right) q^{-k}\,[n+k]\,L_k^{(n)}(t;q)\,x^k.$$

One can easily see that

$$\int_{[k]/[n+k]}^{[k+1]/[n+k]} t^2 d_q^R t = \frac{q^k}{[n+k]^3} \left([k]^2 + \frac{2q^k}{[2]}\,[k] + \frac{q^{2k}}{[3]} \right).$$

So, we acquire

$$\left(K_{n,q}e_2\right)(x,t) = \frac{1}{F_n(x,t)} \sum_{k=0}^{\infty} \frac{1}{[n+k]^2} \left([k]^2 + \frac{2q^k}{[2]}\,[k] + \frac{q^{2k}}{[3]} \right) L_k^{(n)}(t;q)\,x^k.$$

Since $q^k < 1$ for $0 < q < 1$ and $[k+n] \geq [n]$, we can write

$$\left(K_{n,q}e_2\right)(x,t) \leq P_{n,q}(e_2;x) + \frac{2}{[2]\,[n]} P_{n,q}(e_1;x) + \frac{1}{[3]\,[n]^2} P_{n,q}(e_0;x).$$

If we use $P_{n,q}(e_0;x) = 1$, $P_{n,q}(e_1;x) \leq x - \dfrac{tx}{[n]\left(1-bq^{n+1}\right)}$ and

$$P_{n,q}(e_2;x) \leq x^2 - \frac{t\left(x^2+x\right)}{[n]\left(1-bq^{n+1}\right)} + \frac{x}{[n]},$$

from [13], then we get

$$\begin{aligned} \left(K_{n,q}e_2\right)(x,t) - x^2 \quad &\leq \quad -\frac{t\left(x^2+x\right)}{[n]\left(1-bq^{n+1}\right)} + \frac{x}{[n]} + \frac{1}{[3]\,[n]^2} \\ &\quad + \frac{2}{[2]\,[n]} \left(x - \frac{tx}{[n]\left(1-bq^{n+1}\right)} \right). \end{aligned} \tag{2.6}$$

On the other hand, using the equality

$$s^2 = (s-x)^2 + 2xs - x^2$$

we may write

$$\left(K_{n,q}e_2\right)(x,t) - x^2 = \left(K_{n,q}\left(e_1-x\right)^2\right)(x,t) + 2x\left(K_{n,q}\left(e_1-x\right)\right)(x,t).$$

By (2.4) and positivity of $K_{n,q}$, it follows that

$$\left(K_{n,q}e_2\right)(x,t) - x^2 \geq 0. \tag{2.7}$$

Thus from (2.6) and (2.7), we have

$$0 \leq (K_{n,q}e_2)(x,t) - x^2 \leq -\frac{t\left(x^2+x\right)}{[n]\left(1-bq^{n+1}\right)} + \frac{x}{[n]} + \frac{1}{[3]\,[n]^2} + \frac{2}{[2]\,[n]}\left(x - \frac{tx}{[n]\left(1-bq^{n+1}\right)}\right). \quad (2.8)$$

From (2.8), it is clear that

$$\left\|(K_{n,q}e_2)(x,t) - x^2\right\|_{C[0,b]} \leq \frac{|t|\left(b^2+b\right)}{[n]\left(1-bq^{n+1}\right)} + \frac{2\,|t|\,b}{[2]\,[n]^2\left(1-bq^{n+1}\right)} + \left(1+\frac{2}{[2]}\right)\frac{b}{[n]} + \frac{1}{[3]\,[n]^2}. \quad (2.9)$$

After replacing q by a sequence q_n such that $\lim_n q_n = 1$, we have from (2.1), (2.5) and (2.9) $(K_{n,q}e_i)(x,t_0) \rightrightarrows e_i(x) = x^i$ $(i = 0,1,2.)$ on $[0,b]$. ∎

3. Rates of Convergence

In this section, we compute the rates of convergence by means of modulus of continuity, elements of Lipschitz class and second order modulus of smoothness.

Let $f \in C[0,b]$. The modulus of continuity of f denotes by $\omega(f,\delta)$, is defined to be

$$\omega(f,\delta) = \sup_{\substack{s,x\in[0,b]\\ |s-x|<\delta}} |f(s) - f(x)|.$$

It is well known that a necessary and sufficient condition for a function $f \in C[0,b]$ is

$$\lim_{\delta\to 0}\omega(f,\delta) = 0.$$

It is also well known that for any $\delta > 0$ and each $s \in [0,b]$

$$|f(s) - f(x)| \leq \omega(f,\delta)\left(1 + \frac{|s-x|}{\delta}\right). \quad (3.1)$$

Before giving the theorem on the rate of convergence of the operator $K_{n,q}f$, let us first examine its second moment:

$$\left(K_{n,q}(e_1-x)^2\right)(x,t) = (K_{n,q}e_2)(x,t) - x^2 - 2x\left[(K_{n,q}e_1)(x,t) - x\right],$$

$$\left\|\left(K_{n,q}(e_1-x)^2\right)(x,t)\right\|_{C[0,b]} \leq \left\|(K_{n,q}e_2)(x,t) - x^2\right\|_{C[0,b]} + 2\,\|x\|_{C[0,b]}\,\|(K_{n,q}e_1)(x,t) - x\|_{C[0,b]}.$$

Using (2.9) and (2.5), we can write

$$\left\|\left(K_{n,q}(e_1-x)^2\right)(x,t)\right\|_{C[0,b]} \leq \frac{|t|\left(3b^2+b\right)}{[n]\left(1-bq^{n+1}\right)} + \frac{2\,|t|\,b}{[2]\,[n]^2\left(1-bq^{n+1}\right)} + \left(1+\frac{4}{[2]}\right)\frac{b}{[n]} + \frac{1}{[3]\,[n]^2}. \quad (3.2)$$

The following theorem gives the rate of convergence of the operator $K_{n,q}f$ to the function f by means of modulus of continuity.

Theorem 2. *Let* $q := q_n$ *be a sequence satisfying* $\lim_n q_n = 1$ *and* $0 < q_n < 1$. *For all* $f \in C[0,b]$ *and* $\frac{|t|}{[n]} \to 0$ $(n \to \infty)$

$$\left\| (K_{n,q}f)(x,t) - f(x) \right\|_{C[0,b]} \leq 2\omega(f,\delta_n) \tag{3.3}$$

where

$$\delta_n = \left[\frac{|t|\left(3b^2+b\right)}{[n]\left(1-bq^{n+1}\right)} + \frac{2|t|b}{[2][n]^2\left(1-bq^{n+1}\right)} + \left(1+\frac{4}{[2]}\right)\frac{b}{[n]} + \frac{1}{[3][n]^2} \right]^{1/2}.$$

Proof. Let $f \in C[0,b]$. By using (3.1), linearity and monotonicity of $K_{n,q}f$, we obtain

$$\begin{aligned} |(K_{n,q}f)(x,t) - f(x)| &\leq (K_{n,q}|f(s)-f(x)|)(x,t) \\ &\leq \omega(f,\delta)\left(K_{n,q}\left(1+\frac{|s-x|}{\delta}\right)\right)(x,t) \\ &= \omega(f,\delta)\left[1+\frac{1}{\delta}(K_{n,q}|s-x|)(x,t)\right]. \end{aligned} \tag{3.4}$$

In [2], Dalmanoğlu and Doğru show that the Riemann type q-integral is a positive operator and it satisfies the following Hölder's inequality:

Let $0 < a < b$, $0 < q < 1$ and $\frac{1}{m} + \frac{1}{n} = 1$. Then

$$R_q(|fg|;a;b) \leq \left(R_q(|f|^m;a;b)\right)^{1/m}\left(R_q(|g|^n;a;b)\right)^{1/n}. \tag{3.5}$$

Therefore, by using the Cauchy-Schwarz inequality for the Riemann type q-integral with $m=2$ and $n=2$ in (3.5), we have

$$\int_{[k]/[n+k]}^{[k+1]/[n+k]} |t-x|\, d_q^R t \leq \left(\int_{[k]/[n+k]}^{[k+1]/[n+k]} (t-x)^2\, d_q^R t\right)^{1/2}\left(\int_{[k]/[n+k]}^{[k+1]/[n+k]} d_q^R t\right)^{1/2}.$$

Now applying the Cauchy-Schwarz inequality for the sum with $p = \frac{1}{2}$ and $q = \frac{1}{2}$ and taking into consideration (3.2), one can write

$$\begin{aligned} &(K_{n,q}|s-x|)(x,t) \leq \\ &\left\{\sum_{k=0}^{\infty}\frac{1}{F_{n,q}(x,t)}\left(\int_{[k]/[n+k]}^{[k+1]/[n+k]}(t-x)^2\, d_q^R t\right)q^{-k}[n+k]\,L_k^{(n)}(t;q)\,x^k\right\}^{1/2} \\ &\times\left\{\sum_{k=0}^{\infty}\frac{1}{F_{n,q}(x,t)}\left(\int_{[k]/[n+k]}^{[k+1]/[n+k]} d_q^R t\right)q^{-k}[n+k]\,L_k^{(n)}(t;q)\,x^k\right\}^{1/2} \\ &= \left(\left(K_{n,q}(e_1-x)^2\right)(x,t)\right)^{1/2}\left((K_{n,q}e_0)(x,t)\right)^{1/2} \\ &\leq \left[-\frac{t\left(3b^2+b\right)}{[n]\left(1-bq^{n+1}\right)} - \frac{2tb}{[2][n]^2\left(1-bq^{n+1}\right)} + \left(1+\frac{4}{[2]}\right)\frac{b}{[n]} + \frac{1}{[3][n]^2}\right]^{1/2} \end{aligned} \tag{3.6}$$

If we write (3.6) in (3.4) and choose $\delta = \delta_n$, then we arrive at the desired result. ∎

Next, we compute the approximation order of operator $K_{n,q}f$ in term of the elements of the usual Lipschitz class.

Let $f \in C[0,b]$ and $0 < \alpha \leq 1$. We recall that f belongs to $Lip_M(\alpha)$ if the inequality

$$|f(\zeta) - f(\eta)| \leq M|\zeta - \eta|^{\alpha}; \ \zeta, \eta \in [0,b] \tag{3.7}$$

holds.

Theorem 3. *Let $q := q_n$ be a sequence satisfying $\lim_n q_n = 1$ and $0 < q_n < 1$. For all $f \in Lip_M(\alpha)$ and $\frac{|t|}{[n]} \to 0$ $(n \to \infty)$*

$$\|(K_{n,q}f)(x,t) - f(x)\|_{C[0,b]} \leq M\delta_n^{\alpha} \tag{3.8}$$

where δ_n is the same as in Theorem 2.

Proof. Let $f \in C[0,b]$. By (3.7), linearity and monotonicity of $K_{n,q}f$, we have

$$\begin{aligned} |(K_{n,q}f)(x,t) - f(x)| &\leq (K_{n,q}|f(s) - f(x)|)(x,t) \\ &\leq \frac{M}{F_{n,q}(x,t)} \\ &\quad \times \sum_{k=0}^{\infty} \left(\int_{[k]/[n+k]}^{[k+1]/[n+k]} |t-x|^{\alpha} d_q^R t \right) q^{-k}[n+k] L_k^{(n)}(t;q) x^k. \end{aligned} \tag{3.9}$$

On the other hand, by using the Hölder inequality for the Riemann type q-integral with $m = \frac{2}{\alpha}$ and $n = \frac{2}{2-\alpha}$, we have

$$\int_{[k]/[n+k]}^{[k+1]/[n+k]} |t-x|^{\alpha} d_q^R t \leq \left(\int_{[k]/[n+k]}^{[k+1]/[n+k]} (t-x)^2 d_q^R t \right)^{\alpha/2} \left(\int_{[k]/[n+k]}^{[k+1]/[n+k]} d_q^R t \right)^{(2-\alpha)/2}.$$

If we write above inequality in (3.9) and then apply the Hölder inequality for the sum with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, we get

$|(K_{n,q}f)(x,t) - f(x)| \leq$

$$M\left[\sum_{k=0}^{\infty} \frac{1}{F_{n,q}(x,t)} \left(\int_{[k]/[n+k]}^{[k+1]/[n+k]} (t-x)^2 d_q^R t \right) q^{-k}[n+k] L_k^{(n)}(t;q) x^k \right]^{\alpha/2}$$

$$\times \left[\sum_{k=0}^{\infty} \frac{1}{F_{n,q}(x,t)} \left(\int_{[k]/[n+k]}^{[k+1]/[n+k]} d_q^R t \right) q^{-k}[n+k] L_k^{(n)}(t;q) x^k \right]^{(2-\alpha)/2}$$

and so we have

$$|(K_{n,q}f)(x,t) - f(x)| \leq M\left(\left(K_{n,q}(e_1 - x)^2 \right)(x,t) \right)^{\alpha/2}.$$

If we use (3.2) and choose $\delta = \delta_n$, then the proof is completed. ∎

Finally, we establish a local approximation theorem for the operator $K_{n,q}f$.

Let $\Omega^2 := \{g \in C[0,b] : g', g'' \in C[0,b]\}$. For any $\delta > 0$, the Peetre's K-functional is defined by

$$K_2(\varphi;\delta) = \inf_{g\in\Omega^2} \{\|\varphi - g\| + \delta \|g''\|\}$$

where $\|.\|$ is the uniform norm on $C[0,b]$ (see [6]). From [4](p.177, Theorem 2.4), there exists an absolute constant $C > 0$ such that

$$K_2(f;\delta) \le C\omega_2\left(f;\sqrt{\delta}\right) \tag{3.10}$$

where the second order modulus of smoothness of $f \in C[0,b]$ is denoted by

$$\omega_2\left(f;\sqrt{\delta}\right) = \sup_{0<h\le\sqrt{\delta}} \sup_{x,x+2h\in[0,b]} |f(x+2h) - 2f(x+h) + f(x)|.$$

We recall the usual modulus of continuity of $f \in C[0,b]$ by

$$\omega\left(f;\sqrt{\delta}\right) = \sup_{0<h\le\sqrt{\delta}} \sup_{x,x+h\in[0,b]} |f(x+h) - f(x)|.$$

Now consider the following operator

$$(L_{n,q}f)(x,t) = (K_{n,q}f)(x,t) - f\left(x - \frac{tx}{[n](1-bq^{n+1})} + \frac{1}{[2][n]}\right) + f(x) \tag{3.11}$$

for $x \in [0,1]$.

Lemma 4. *Let $g \in \Omega^2$. Then we have*

$$\begin{aligned} |(L_{n,q}g)(x,t) - g(x)| \le & \left\{ \frac{-t(3x^2+x)}{[n](1-bq^{n+1})} - \frac{2tx}{[2][n]^2(1-bq^{n+1})} + \left(1+\frac{4}{[2]}\right)\frac{x}{[n]} \right. \\ & \left. + \frac{1}{[3][n]^2} + \left(\frac{-tx}{[n](1-bq^{n+1})} + \frac{1}{[2][n]}\right)^2 \right\} \|g''\|. \end{aligned} \tag{3.12}$$

Proof. By definition of the operator $L_{n,q}f$, (2.1) and (2.4), it is seen that

$$\begin{aligned} (L_{n,q}(s-x))(x,t) &= (K_{n,q}(s-x))(x,t) + \frac{tx}{[n](1-bq^{n+1})} - \frac{1}{[2][n]} \\ &= 0. \end{aligned} \tag{3.13}$$

Let $x \in [0,1]$ and $g \in \Omega^2$. Then by using the Taylor formula

$$g(s) - g(x) = (s-x)g'(x) + \int_x^s (s-u)g''(u)\,du$$

and (3.13), we have

$$
\begin{aligned}
&(L_{n,q}g)(x,t)-g(x)= \\
&\qquad g'(x)\left(L_{n,q}(s-x)\right)(x,t)+(L_{n,q}(\int_x^s (s-u)\,g''(u)\,du))(x,t) \\
&\leq \quad (L_{n,q}(\int_x^s (s-u)\,g''(u)\,du))(x,t) \\
&= \quad (K_{n,q}(\int_x^s (s-u)\,g''(u)\,du))(x,t) \\
&\quad -\int_x^{x-\frac{tx}{[n](1-bq^{n+1})}+\frac{1}{[2][n]}} (x-\frac{tx}{[n](1-bq^{n+1})}+\frac{1}{[2][n]}-u)g''(u)\,du.
\end{aligned}
$$

The monotonicity of $K_{n,q}f$ gives

$$
\begin{aligned}
&\left|(L_{n,q}g)(x,t)-g(x)\right|\leq \\
&\qquad \left|\int_x^{x-\frac{tx}{[n](1-bq^{n+1})}+\frac{1}{[2][n]}} (x-\frac{tx}{[n](1-bq^{n+1})}+\frac{1}{[2][n]}-u)g''(u)\,du\right| \\
&\qquad +(K_{n,q}\left|\int_x^s (s-u)\,g''(u)\,du\right|)(x,t). \qquad (3.14)
\end{aligned}
$$

On the other hand, it is clear that

$$
\left|\int_x^s (s-u)\,g''(u)\,du\right|\leq (s-x)^2\left\|g''\right\|. \qquad (3.15)
$$

Now let

$$
I:=\int_x^{x-\frac{tx}{[n](1-bq^{n+1})}+\frac{1}{[2][n]}} \left(x-\frac{tx}{[n](1-bq^{n+1})}+\frac{1}{[2][n]}-u\right)g''(u)\,du.
$$

Then we may write

$$
I\leq\left(-\frac{tx}{[n](1-bq^{n+1})}+\frac{1}{[2][n]}\right)^2\left\|g''\right\|. \qquad (3.16)
$$

Substituting (3.15) and (3.16) into (3.14), we have

$$
\begin{aligned}
\left|(L_{n,q}g)(x,t)-g(x)\right| \quad &\leq \quad \Big\{\left(K_{n,q}(s-x)^2\right)(x,t) \\
&\qquad +\left(-\frac{tx}{[n](1-bq^{n+1})}+\frac{1}{[2][n]}\right)^2\Big\}\left\|g''\right\|. \qquad (3.17)
\end{aligned}
$$

Using (3.2) in (3.17), it follows that

$$\begin{aligned} \left|\left(L_{n,q}g\right)(x,t)-g(x)\right| &\leq \left\{\frac{-t\left(3x^2+x\right)}{[n]\left(1-bq^{n+1}\right)}-\frac{2tx}{[2][n]^2\left(1-bq^{n+1}\right)}\right. \\ &+\frac{1}{[3][n]^2}+\left(1+\frac{4}{[2]}\right)\frac{x}{[n]} \\ &\left.+\left(\frac{-tx}{[n]\left(1-bq^{n+1}\right)}+\frac{1}{[2][n]}\right)^2\right\}\left\|g''\right\|. \end{aligned}$$

This completes the proof. ∎

Theorem 5. *Let $q:=q_n$ be a sequence satisfying $\lim_n q_n=1$ and $0<q_n<1$. For each $f\in C[0,1]$ and $x\in[0,1]$, we have*

$$\left|\left(K_{n,q}f\right)(x,t)-f(x)\right|\leq C\omega_2\left(f;\sqrt{\delta_n(x)}\right)+\omega\left(f;\left|\frac{-tx}{[n]\left(1-bq^{n+1}\right)}+\frac{1}{[2][n]}\right|\right)$$

where

$$\begin{aligned} \delta_n(x) &= \frac{-t\left(3x^2+x\right)}{[n]\left(1-bq^{n+1}\right)}-\frac{2tx}{[2][n]^2\left(1-bq^{n+1}\right)}+\left(1+\frac{4}{[2]}\right)\frac{x}{[n]} \\ &+\frac{1}{[3][n]^2}+\left(\frac{-tx}{[n]\left(1-bq^{n+1}\right)}+\frac{1}{[2][n]}\right)^2 \end{aligned}$$

and C is a positive constant.

Proof. From (3.11), we have

$$\left|\left(L_{n,q}f\right)(x,t)\right|\leq 3\left\|f\right\|. \tag{3.18}$$

In view of (3.12) and (3.18), the equality (3.11) implies that

$$\begin{aligned} \left|\left(K_{n,q}f\right)(x,t)-f(x)\right| &\leq \left|\left(L_{n,q}(f-g)\right)(x,t)\right|+\left|(f-g)(x)\right|+\left|\left(L_{n,q}g-g(x)\right)(x,t)\right| \\ &+\left|f\left(x-\frac{tx}{[n]\left(1-bq^{n+1}\right)}+\frac{1}{[2][n]}\right)-f(x)\right| \\ &\leq 4\left\|f-g\right\|+\left|\left(L_{n,q}g\right)(x,t)-g(x)\right| \\ &+\omega\left(f;\left|-\frac{tx}{[n]\left(1-bq^{n+1}\right)}+\frac{1}{[2][n]}\right|\right) \\ &\leq 4\left\|f-g\right\|+\left\{\frac{-t\left(3x^2+x\right)}{[n]\left(1-bq^{n+1}\right)}-\frac{2tx}{[2][n]^2\left(1-bq^{n+1}\right)}\right. \\ &+\left(1+\frac{4}{[2]}\right)\frac{x}{[n]}+\frac{1}{[3][n]^2} \\ &\left.+\left(\frac{-tx}{[n]\left(1-bq^{n+1}\right)}+\frac{1}{[2][n]}\right)^2\right\}\left\|g''\right\| \\ &+\omega\left(f;\left|-\frac{tx}{[n]\left(1-bq^{n+1}\right)}+\frac{1}{[2][n]}\right|\right) \\ &\leq 4\left\|f-g\right\|+4\delta_n(x)\left\|g''\right\|+\omega\left(f;\left|\frac{-tx}{[n]\left(1-bq^{n+1}\right)}+\frac{1}{[2][n]}\right|\right). \end{aligned}$$

Hence taking infimum on two-hand side of above inequality over all $g \in \Omega^2$ and considering (3.10), we get

$$\begin{aligned} \left|\left(K_{n,q}f\right)(x,t_0) - f(x)\right| &\leq 4K_2\left(f;\delta_n(x)\right) + \omega\left(f;\left|\frac{-tx}{[n]\left(1-bq^{n+1}\right)} + \frac{1}{[2]\,[n]}\right|\right) \\ &\leq C\omega_2\left(f;\sqrt{\delta_n(x)}\right) + \omega\left(f;\left|\frac{-tx}{[n]\left(1-bq^{n+1}\right)} + \frac{1}{[2]\,[n]}\right|\right) \end{aligned}$$

which is the desired result. ∎

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