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COMMON FIXED POINT THEOREMS IN MENGER PROBABILISTIC QUASI-METRIC SPACES

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Abstract. In 1989, Kent and Richardson [Ordered probabilistic metric spaces, J. Austral. Math. Soc. Ser. A 46(1) (1989), 88-99, MR0966286 (90b:54022)] introduced the class of probabilistic quasi-metric spaces which offers a wider framework than that of metric spaces. The aim of this paper is to prove common fixed point theorems for single-valued and set-valued weakly compatible mappings in Menger probabilistic quasi-metric spaces.

1. Introduction

Menger [12] introduced the notion of a probabilistic metric space (shortly, PM-space) in 1942. The study of this space received much attention after the pioneering work of Schweizer and Sklar [22] (also see [1]). In 1989, Kent and Richardson [8] introduced and studied the class of probabilistic quasi-metric spaces (shortly, PQM-spaces) and proved common fixed point theorems. Many mathematicians weakened the notion of commutativity by introducing the notions of weak commutativity [24], compatibility [6] and weak compatibility [7] in metric spaces and proved a number of fixed point theorems using these notions. It is worth to mention that each pair of commuting self mappings is weakly commuting, each pair of weakly commuting self mappings is compatible and each pair of compatible self mappings is weak compatible but the converse is not always true. Many authors formulated the definitions of weakly commuting [27], compatible [16] and weakly compatible mappings [26] in probabilistic settings and proved a number of fixed point theorems.

Fixed point theorems for single-valued mappings have appeared in PQM-spaces (see [2, 14, 15, 17, 19, 20, 21, 23, 25]. Recently, Cho [3] proved common fixed point theorems for set-valued mappings in quasi-metric spaces. The theory of quasi-metric spaces can be used as an efficient tool to solve so many several problems like theoretical computer science, approximation theory and topological algebra (see [4, 10, 17]).

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In the present paper, we prove common fixed point theorems for single-valued and set-valued weakly compatible mappings in Menger PQM-spaces.

2. Preliminaries

Definition 2.1. [22] *A mapping* $\mathrm{T} : [0,1]\times[0,1] \to [0,1]$ *is t-norm if* T *is satisfying the following conditions:*

(1) T *is commutative and associative;*
(2) $\mathrm{T}(a,1) = a$ *for all* $a \in [0,1]$*;*
(3) $\mathrm{T}(a,b) \le \mathrm{T}(c,d)$ *whenever* $a \le c$ *and* $b \le d$*, for all* $a,b,c,d \in [0,1]$.

The following are the some basic t-norms:

$$\begin{aligned}\mathrm{T}_M(a,b) &= \min\{a,b\};\\ \mathrm{T}_P(a,b) &= ab;\\ \mathrm{T}_L(a,b) &= \max\{a+b-1,0\}.\end{aligned}$$

Each t-norm T can be extended [9] (by associativity) in a unique way taking for $(x_1,\ldots,x_n) \in [0,1]^n (n \in N)$ the values $\mathrm{T}^1(x_1,x_2) = \mathrm{T}(x_1,x_2)$ and $\mathrm{T}^n(x_1,\ldots,x_{n+1})$ $= \mathrm{T}(\mathrm{T}^{n-1}(x_1,\ldots,x_n),x_{n+1})$ for $n \ge 2$ and $x_i \in [0,1]$, for all $i \in \{1,2,\ldots,n+1\}$.

Definition 2.2. [22] *A mapping* $F : \mathbb{R} \to \mathbb{R}^+$ *is called a distribution function if it is non-decreasing and left continuous with* $\inf\{F(t) : t \in \mathbb{R}\} = 0$ *and* $\sup\{F(t) : t \in \mathbb{R}\} = 1$.

We shall denote by $\Im$ *the set of all distribution functions defined on* $[-\infty,\infty]$ *while* ϵ_0 *will always denote the specific distribution function defined by*

$$\epsilon_0(t) = \begin{cases} 0, & \text{if } t \le 0;\\ 1, & \text{if } t > 0.\end{cases}$$

If X *is a non-empty set,* $\mathcal{F} : X \times X \to \Im$ *is called a probabilistic distance on* X *and the value of* $\mathcal{F}$ *at* $(x,y) \in X \times X$ *is represented by* $F_{x,y}$.

Definition 2.3. [8] *A Menger PQM-space is a triplet* $(X,\mathcal{F},\mathrm{T})$*, where* X *is a non-empty set,* T *is a continuous t-norm and* $\mathcal{F}$ *is a probabilistic distance satisfying the following conditions: for all* $x,y,z \in X$ *and* $t,s > 0$*:*

(1) $F_{x,y}(t) = \epsilon_0(t)$ *and* $F_{y,x}(t) = \epsilon_0(t)$ *then* $x = y$*;*
(2) $F_{x,z}(t+s) \ge \mathrm{T}(F_{x,y}(t),F_{y,z}(s))$.

A Menger PQM-space is called a Menger PM-space if it satisfies the symmetry condition, i.e. $F_{x,y}(t) = F_{y,x}(t)$, for all $x,y \in X$.

The notion of a Menger PQM-space is a generalization of the notion of a metric space. So Menger PQM-spaces offer a wider framework than that of metric spaces and are better suited to cover even wider statistical situations. In [11], Marcus gave an example of a PQM-space based on stationary Markov chains which is not a PM-space.

Definition 2.4. [1] *Let* $(X,\mathcal{F},\mathrm{T})$ *be a Menger PQM-space and* A *be a non-empty subset of* X*. Then* A *is said to be probabilistically bounded if*

$$\sup_{t>0} \inf_{x,y\in A} F_{x,y}(t) = 1.$$

If X *itself is probabilistically bounded, then* X *is said to be a probabilistically bounded space.*

Throughout this paper, $\mathcal{B}(X)$ will denote the family of non-empty bounded subsets of a Menger PQM-space X. For all $A, B \in \mathcal{B}(X)$ and for every $t > 0$, we define

$$_DF_{A,B}(t) \quad = \quad \sup\{F_{a,b}(t); a \in A, b \in \mathcal{B}\}$$

and

$$_\delta F_{A,B}(t) \quad = \quad \inf\{F_{a,b}(t); a \in A, b \in \mathcal{B}\}.$$

If set A consists of a single point a, we write

$$_\delta F_{A,B}(t) \quad = \quad {}_\delta F_{a,B}(t).$$

If set B also consists of a single point b, we write

$$_\delta F_{A,B}(t) \quad = \quad F_{a,b}(t).$$

It follows immediately from the definition that $_\delta F_{A,B}(t) = 1$ for all $t > 0$. Thus we conclude that $A = B = \{a\}$, for some $a \in X$.

Definition 2.5. [22] *Let $(X, \mathcal{F}, \mathrm{T})$ be a Menger PQM-space.*

(1) *A sequence $\{x_n\}$ is said to be convergent to $x \in X$ if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $\mathbb{N}$ such that $F_{x_n,x}(\epsilon) > 1 - \lambda$ whenever $n \geq \mathbb{N}$.*
(2) *A sequence $\{x_n\}$ in X is said to be Cauchy if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $\mathbb{N}$ such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$ whenever $n, m \geq \mathbb{N}$.*
(3) *A Menger PQM-space in which every Cauchy sequence is convergent is said to be complete.*

Definition 2.6. [5] *A t-norm T is of Hadžić-type (H-type in short) and $\mathrm{T} \in \mathcal{H}$ if the family $\{\mathrm{T}^n\}_{n\in\mathbb{N}}$ of its iterates defined, for each x in $[0, 1]$, by $\mathrm{T}^0(x) = 1, \mathrm{T}^{n+1}(x) = \mathrm{T}(\mathrm{T}^n(x), x)$, for all $n \geq 0$ is equicontinuous at $x = 1$, that is*
$\epsilon \in (0, 1)\ \exists\ \delta \in (0, 1) : x > 1 - \delta \Rightarrow \mathrm{T}^n(x) > 1 - \epsilon$ *for all $n \geq 1$.*

There is a nice characterization of continuous t-norm T of the class $\mathcal{H}$ [18].

The t-norm T_M is an trivial example of a t-norm of H-type, but there are t-norms T of Hadžić-type with $\mathrm{T} \neq \mathrm{T}_M$ (see e.g., [5]).

Definition 2.7. [5] *If T is a t-norm and $(x_1, x_2, \ldots, x_n) \in [0, 1]^n (n \in \mathbb{N})$, then $\mathrm{T}_{i=1}^n x_i$ is defined recurrently by 1, if $n = 0$ and $\mathrm{T}_{i=1}^n x_i = \mathrm{T}(\mathrm{T}_{i=1}^{n-1} x_i, x_n)$ for all $n \geq 1$. If $(x_i)_{i\in\mathbb{N}}$ is a sequence of numbers from $[0, 1]$ then $\mathrm{T}_{i=1}^\infty x_i$ is defined as $\lim_{n\to\infty} \mathrm{T}_{i=1}^n x_i$ (this limit always exists) and $\mathrm{T}_{i=n}^\infty x_i$ as $\mathrm{T}_{i=1}^\infty x_{n+i}$.*

In fixed point theory in probabilistic metric spaces there are of particular interest the t-norms T and sequences $(x_n) \subset [0, 1]$ such that $\lim_{n\to\infty} x_n = 1$ and $\lim_{n\to\infty} \mathrm{T}_{i=1}^\infty x_{n+i} = 1$.

Proposition 2.1. [5]

(1) *If $\mathrm{T} \geq \mathrm{T}_L$ then the following implication holds:*

$$\lim_{n\to\infty} \mathrm{T}_{i=1}^\infty x_{n+i} = 1 \Leftrightarrow \sum_{n=1}^\infty (1 - x_n) < \infty.$$

(2) *If $\mathrm{T} \in \mathcal{H}$ then for every sequence $(x_n)_{n\in N}$ in $[0, 1]$ such that $\lim_{n\to\infty} x_n = 1$, one has $\lim_{n\to\infty} \mathrm{T}_{i=1}^\infty x_{n+i} = 1$.*

Note that if T is a t-norm for which there exists $(x_n) \subset [0,1]$ such that $\lim_{n\to\infty} x_n = 1$ and $\lim_{n\to\infty} \mathrm{T}_{i=1}^{\infty} x_{n+i} = 1$, then $\sup_{t<1} \mathrm{T}(t,t) = 1$.

Proposition 2.2. [5] *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of numbers from $[0,1]$ such that $\lim_{n\to\infty} x_n = 1$ and t-norms T is of H-type. Then*

$$\lim_{n\to\infty} \mathrm{T}_{i=n}^{\infty} x_i = \lim_{n\to\infty} \mathrm{T}_{i=1}^{\infty} x_{n+i} = 1.$$

Lemma 2.1. [19] *If a Menger PQM-space $(X, \mathcal{F}, \mathrm{T})$ satisfies the following condition*

$$F_{x,y}(t) = C, \text{ for all } t > 0 \text{ with fixed } x, y \in X.$$

Then we have $C = 1$ and $x = y$.

Lemma 2.2. [5] *Let the function $\phi(t)$ satisfy the following condition (Φ) : $\phi(t) : [0,\infty) \to [0,\infty)$ is non-decreasing and $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for all $t > 0$, when $\phi^n(t)$ denotes the n^{th} iterative function of $\phi(t)$. Then $\phi(t) < t$ for all $t > 0$.*

Definition 2.8. [26] *The mappings $f : X \to X$ and $g : X \to \mathcal{B}(X)$ are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, that is $gu = \{fu\}$ for some $u \in X$ then $fgu = gfu$ (Note that the term $gu = \{fu\}$ implies that gu is a singleton).*

3. Results

Theorem 3.1. *Let $(X, \mathcal{F}, \mathrm{T})$ be a complete Menger PQM-space. Further, let $f : X \to X$ be a single valued and $g : X \to \mathcal{B}(X)$ be a set-valued functions, let the following conditions are satisfied:*

(i) T is Hadžić-type;

(ii) Every convergent sequence in X has a unique limit;

(iii) $g(X) \subset f(X)$;

(iv) ${}_{\delta}F_{g(x),g(y)}(\phi(t)) \geq F_{f(x),f(y)}(t)$,

for all $x, y \in X$ and $t > 0$ where the function $\phi(t) : [0,\infty) \to [0,\infty)$ is onto, strictly increasing and satisfies condition (Φ);

(v) $f(X)$ is a closed subset of X, then

(a) g and f have a coincidence point;

(vi) The pair (g, f) is weakly compatible.

Then there exists a unique common fixed point $z \in X$ such that $\{z\} = \{fz\} = gz$.

Proof. Let x_0 be an arbitrary point in X. By (iii) we can find x_1 such that $f(x_1) \in g(x_0)$. By induction, we can find the sequences $\{x_n\}$ and $\{y_n\}$ such that $y_{2n} = f(x_{2n+1}) \in g(x_{2n})$ for $n \in \mathbb{N}$.

Putting $x = x_{2n}$ and $y = x_{2n+1}$ in (iv) we have

$$\begin{aligned} {}_{\delta}F_{g(x_{2n}),g(x_{2n+1})}((\phi(t)) &\geq F_{f(x_{2n}),f(x_{2n+1})}(t), \\ F_{y_{2n},y_{2n+1}}((\phi(t)) &\geq F_{y_{2n-1},y_{2n}}(t). \end{aligned}$$

Similarly, we can also prove that for $n \in \mathbb{N}$ and for all $t > 0$,

$$F_{y_{2n+1},y_{2n+2}}((\phi(t)) \geq F_{y_{2n},y_{2n+1}}(t).$$

So, we have

$$\begin{aligned} F_{y_n,y_{n+1}}((\phi(t)) &\geq F_{y_{n-1},y_n}(t), \\ F_{y_n,y_{n+1}}(t) &\geq F_{y_{n-1},y_n}(\phi^{-1}(t)), \\ &\geq \ldots \geq F_{y_0,y_1}(\phi^{-n}(t)). \end{aligned}$$

We show that $\{y_n\}$ is a Cauchy sequence.

Let $\epsilon > 0$ be given and $\lambda \in (0,1)$ be such that $\mathrm{T}^{m-1}(1-\lambda,\dots,1-\lambda) > 1-\epsilon$. Also let $t > 0$ be such that $F_{y_0,y_1}(t) > 1-\lambda$, ψ be a positive number and $n_1 \in \mathbb{N}$ be such that $\sum_{n_1}^{\infty} \phi^i(t) \leq \psi$. Then, for every $n \geq n_1$ and $m \in \mathbb{N}$ we have

$$\begin{aligned} F_{y_n,y_{n+m}}(\psi) &\geq F_{y_n,y_{n+m}}\left(\sum_{i=n}^{n+m-1} \phi^i(t)\right) \\ &\geq T^{m-1}\left(F_{y_n,y_{n+1}}(\phi^n(t)),\dots,F_{y_{n+m-1},y_{n+m}}(\phi^{n+m-1}(t))\right) \\ &\geq T^{m-1}(1-\lambda,\dots,1-\lambda) \\ &> 1-\epsilon. \end{aligned}$$

Hence $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, $\{y_n\}$ converges to z in X. Thus

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} fx_{2n+1} = z \in \lim_{n\to\infty} gx_{2n}.$$

Since $f(X)$ is a closed subset of X, there exists a point $v \in X$ such that $z = fv \in f(X)$.

Putting $x = x_{2n}$ and $y = v$ in (iv), we get

$$\begin{aligned} {}_{\delta}F_{g(x_{2n}),g(v)}(\phi(t)) &\geq F_{f(x_{2n}),f(v)}(t), \\ {}_{\delta}F_{y_{2n},g(v)}(\phi(t)) &\geq F_{y_{2n-1},z}(t), \end{aligned}$$

now taking limit $n \to \infty$, we have

$$ {}_{\delta}F_{z,g(v)}(\phi(t)) \geq F_{z,z}(t) = 1.$$

Hence, ${}_{\delta}F_{z,g(v)}(\phi(t)) = 1$ we obtain $g(v) = z$. It shows that v is a coincidence point of f and g. Since the pair (g,f) is weakly compatible, we have $gf(v) = fg(v)$, hence $g(z) = \{f(z)\}$.

Putting $x = x_{2n}$ and $y = z$ in (iv), we get

$$\begin{aligned} {}_{\delta}F_{g(x_{2n}),g(z)}(\phi(t)) &\geq F_{f(x_{2n}),f(z)}(t), \\ {}_{\delta}F_{y_{2n},g(z)}(\phi(t)) &\geq {}_{\delta}F_{y_{2n-1},g(z)}(t), \end{aligned}$$

taking limit $n \to \infty$, we get

$$ {}_{\delta}F_{z,g(z)}(\phi(t)) \geq {}_{\delta}F_{z,g(z)}(t).$$

On the other hand, since F is non-decreasing, we get

$$ {}_{\delta}F_{z,g(z)}(\phi(t)) \leq {}_{\delta}F_{z,g(z)}(t).$$

Hence ${}_{\delta}F_{g(z),z}(t) = C$ for all $t > 0$. From Lemma 2.10 we conclude that $C = 1$, that is $g(z) = \{z\}$. Now combine all the results, we get $g(z) = \{f(z)\} = \{z\}$. It implies z is a common fixed point of f and g in X.

Uniqueness: Let $w(\neq z)$ be another common fixed point of f and g. Taking $x = z$ and $y = w$ in (iv), we have

$$\begin{aligned} {}_{\delta}F_{g(z),g(w)}(\phi(t)) &\geq F_{f(z),f(w)}(t) \\ F_{z,w}(\phi(t)) &\geq F_{z,w}(t). \end{aligned}$$

Since F is non-decreasing, we get $F_{z,w}(\phi(t)) \leq F_{z,w}(t)$. Hence $F_{z,w}(t) = C$ for all $t > 0$. From Lemma 2.10 we conclude that $C = 1$, that is $z = w$ and so the uniqueness of the common fixed point. $\square$

Now we extend our result to finite number of mappings in Menger PQM-space.

Theorem 3.2. *Let $(X, \mathcal{F}, \mathrm{T})$ be a complete Menger PQM-space. Let $f_1, f_2, \ldots, f_n : X \to X$ be a family of single-valued functions and let $g : X \to \mathcal{B}(X)$ be set-valued function. If the following conditions are satisfied:*

(i) T *is Hadžić-type;*

(ii) Every convergent sequence in X has a unique limit;

(iii) $g(X) \subset f_1 f_2 \ldots f_n(X)$*;*

(iv) ${}_\delta F_{g(x),g(y)}(\phi(t)) \geq F_{f_1 f_2 \ldots f_n(x), f_1 f_2 \ldots f_n(y)}(t)$,

for all $x, y \in X$ and $t > 0$ where the function $\phi(t) : [0, \infty) \to [0, \infty)$ is onto, strictly increasing and satisfies condition (Φ)*;*

(v) $f_1 f_2 \ldots f_n(X)$ is a closed subset of X, then

(a) g and $f_1 f_2 \ldots f_n$ have a coincidence point.

Further if,

(vi) $g(f_2 \ldots f_n) = (f_2 \ldots f_n)g$,

$g(f_3 \ldots f_n) = (f_3 \ldots f_n)g$,

$\vdots$

$g f_n = f_n g$,

$f_1(f_2 \ldots f_n) = (f_2 \ldots f_n) f_1$,

$f_1 f_2 (f_3 \ldots f_n) = (f_3 \ldots f_n) f_1 f_2$,

$\vdots$

$f_1 \ldots f_{n-1}(f_n) = (f_n) f_1 \ldots f_{n-1}$*;*

(vii) The pair $(g, f_1 f_2 \ldots f_n)$ is weakly compatible.

Then there exists a unique common fixed point $z \in X$ such that $\{z\} = \{f_1 z\} = \{f_2 z\} = \ldots = \{f_n z\} = gz$.

Proof. If we put $f_1 f_2 \ldots f_n = f$ in Theorem 3.1 then we get $g(z) = \{f_1 f_2 \ldots f_n(z)\} = \{z\}$. Now we show that z is a common fixed point of all the component mappings, by putting $x = z, y = f_2 \ldots f_n z$ and $f_1^{'} = f_1 f_2 \ldots f_n$ in (iv), we get

$$\begin{aligned} {}_\delta F_{g(z), g(f_2 \ldots f_n z)}(\phi(t)) &\geq F_{f_1^{'}(z), f_1^{'} f_2 \ldots f_n z}(t), \\ F_{z, f_2 \ldots f_n(z)}(\phi(t)) &\geq F_{z, f_2 \ldots f_n(z)}(t). \end{aligned}$$

Since F is non-decreasing, we get $F_{z, f_2 \ldots f_n(z)}(\phi(t)) \leq F_{z, f_2 \ldots f_n(z)}(t)$. Hence $F_{z, f_2 \ldots f_n(z)}$ $(t) = C$ for all $t > 0$. From Lemma 2.10 we conclude that $C = 1$, that is $f_2 \ldots f_n(z) = z$. Thus, $f_1 z = f_1(f_2 \ldots f_n) z = z$. Similarly, we have $f_2 z = f_3 z = \ldots = f_n z = z$. So there exists a common fixed point $z \in X$ such that $\{z\} = \{f_1 z\} = \{f_2 z\} = \ldots = \{f_n z\} = g(z)$. Uniqueness of the common fixed point follows easily from (iv). □

It should be noticed (see Theorem 3.3 in [13] for the case $g(x) = x$) that the condition T is of Hadžić-type in Theorem 3.1 and Theorem 3.2 may be replaced by $\lim_{n\to\infty} \mathrm{T}_{i=n}^{\infty}\ {}_\delta F_{f(x),g(x)}\left(\frac{1}{\mu^i}\right) = 1$ and $\lim_{n\to\infty} \mathrm{T}_{i=n}^{\infty}\ {}_\delta F_{f_1 f_2 \ldots f_n(x), gx}\left(\frac{1}{\mu^i}\right) = 1$, for some $x \in X$ and some $\mu \in (0, 1)$. Taking into account Proposition (2.1), we get the following results:

Corollary 3.1. *Let $(X, \mathcal{F}, \mathrm{T}_L)$ be a complete Menger PQM-space. Let $f : X \to X$ be a single-valued function and let $g : X \to \mathcal{B}(X)$ be set-valued function. If the following conditions are satisfied:*

(i) Every convergent sequence in X has a unique limit;

(ii) $g(X) \subset f(X)$*;*

(iii) ${}_{\delta}F_{g(x),g(y)}(\phi(t)) \geq F_{f(x),f(y)}(t)$,
for all $x, y \in X$ *and* $t > 0$ *where the function* $\phi(t) : [0, \infty) \to [0, \infty)$ *is onto, strictly increasing and satisfies condition* (Φ)*;*
(iv) $f(X)$ *is a closed subset of* X*. then*
(*a*) g *and* f *have a coincidence point.*
Further if,
(v) the pair (g, f) *is weakly compatible.*
Then there exists a unique common fixed point $z \in X$ *such that* $\{z\} = \{fz\} = gz$ *provided that*

$$\sum_{i=1}^{\infty} \left(1 -_{\delta} F_{f(x),g(x)}\left(\frac{1}{\mu^i}\right)\right) < \infty,$$

for some $x \in X$ *and some* $\mu \in (0, 1)$.

Corollary 3.2. *Let* $(X, \mathcal{F}, \mathrm{T}_L)$ *be a complete Menger PQM-space. Let* $f_1, f_2, \ldots, f_n : X \to X$ *be a family of single-valued functions and let* $g : X \to \mathcal{B}(X)$ *be set-valued function. If the following conditions are satisfied:*
(i) Every convergent sequence in X *has a unique limit;*
(ii) $g(X) \subset f_1 f_2 \ldots f_n(X)$*;*
(iii) ${}_{\delta}F_{g(x),g(y)}(\phi(t)) \geq F_{f_1 f_2 \ldots f_n(x), f_1 f_2 \ldots f_n(y)}(t)$,
for all $x, y \in X$ *and* $t > 0$ *where the function* $\phi(t) : [0, \infty) \to [0, \infty)$ *is onto, strictly increasing and satisfies condition* (Φ)*;*
(iv) $f_1 f_2 \ldots f_n(X)$ *is a closed subset of* X*. then*
(*a*) g *and* $f_1 f_2 \ldots f_n$ *have a coincidence point.*
Further if,
(v) $g(f_2 \ldots f_n) = (f_2 \ldots f_n)g$,
$g(f_3 \ldots f_n) = (f_3 \ldots f_n)g$,
$\vdots$
$g f_n = f_n g$,
$f_1(f_2 \ldots f_n) = (f_2 \ldots f_n)f_1$,
$f_1 f_2(f_3 \ldots f_n) = (f_3 \ldots f_n)f_1 f_2$,
$\vdots$
$f_1 \ldots f_{n-1}(f_n) = (f_n)f_1 \ldots f_{n-1}$*;*
(vi) The pair $(g, f_1 f_2 \ldots f_n)$ *is weakly compatible.*
Then there exists a unique common fixed point $z \in X$ *such that* $\{z\} = \{f_1 z\} = \{f_2 z\} = \ldots = \{f_n z\} = gz$ *provided that*

$$\sum_{i=1}^{\infty} \left(1 -_{\delta} F_{f_1 f_2 \ldots f_n(x), gx}\left(\frac{1}{\mu^i}\right)\right) < \infty,$$

for some $x \in X$ *and some* $\mu \in (0, 1)$.

Remark 3.1. *The conclusions of Theorem 3.1, Theorem 3.2, Corollary 3.1 and Corollary 3.2 remain true for* $\phi(t) = kt$*, where* $k \in (0, 1)$.

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