Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 4 Issue 3 (2012.), Pages 71-77

ON AN INTEGRAL-TYPE OPERATOR FROM MIXED NORM SPACES TO ZYGMUND-TYPE SPACES

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Abstract. This paper studies the boundedness and compactness of an integral-type operator from mixed norm spaces to Zygmund-type spaces and little Zygmund-type spaces.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of all analytic functions in $\mathbb{D}$. Let $\beta > 0$. An $f \in H(\mathbb{D})$ is said to belong to the Zygmund-type space, denoted by $\mathcal{Z}_\beta$, if $\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|f''(z)| < \infty$. It is easy to see that $\mathcal{Z}_\beta$ is a Banach space with the norm

$$\|f\|_{\mathcal{Z}_\beta} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|f''(z)|.$$

Let $\mathcal{Z}_{\beta,0}$ denote the subspace of $\mathcal{Z}_\beta$ consisting of those $f \in \mathcal{Z}_\beta$ such that $\lim_{|z|\to1}(1-|z|^2)^\beta|f''(z)| = 0$. We call $\mathcal{Z}_{\beta,0}$ the little Zygmund-type space. When $\beta = 1$, the induced spaces $\mathcal{Z}_1$ and $\mathcal{Z}_{1,0}$ becomes the classical Zygmund space and the little Zygmund space respectively (see [2]).

If $0 < r < 1$ and $f \in H(\mathbb{D})$, we set

$$M_q^q(f,r) = \frac{1}{2\pi}\int_0^{2\pi}|f(re^{it})|^q dt,\ 0 < q < \infty.$$

Let ν be a normal function on $[0,1)$ (see [7]). For $0 < p, q < \infty$, the mixed norm space $H(p,q,\nu) = H(p,q,\nu)(\mathbb{D})$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{H(p,q,\nu)} = \left(\int_0^1 M_q^p(f,r)\frac{\nu^p(r)}{1-r}dr\right)^{1/p} < \infty.$$

[0]2000 Mathematics Subject Classification: 47B38, 32A37.
Keywords and phrases. Mixed norm space, Zygmund-type space, integral-type operator.

Submitted 31 May, 2012. Accepted July 17, 2012.

Let $g \in H(\mathbb{D})$, n be a nonnegative integer and φ be an analytic self-map of $\mathbb{D}$. In [11], Zhu defined a new integral-type operator as follows.

$$(C^n_{\varphi,g}f)(z) = \int_0^z f^{(n)}(\varphi(\xi))g(\xi)d\xi, \ z \in \mathbb{D}, \ f \in H(\mathbb{D}).$$

When $n = 1$, then $C^1_{\varphi,g}$ is the generalized composition operator C^g_φ, which firstly defined and studied in [4]. See some related results about the generalized composition operator C^g_φ and the operator $C^n_{\varphi,g}$ in [3, 4, 5, 6, 8, 9, 10]. The purpose of this paper is to study the operator $C^n_{\varphi,g}$. The boundedness and compactness of the operator $C^n_{\varphi,g}$ from $H(p,q,\nu)$ to Zygmund-type spaces and little Zygmund-type spaces are completely characterized, which generalized the results of [8].

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $A \asymp B$ means that there is a positive constant C such that $B/C \le A \le CB$.

2. Main results and proofs

In this section we give our main results and proofs. For this purpose, we need some auxiliary results. The following lemma can be proved in a standard way (see, e.g., Proposition 3.11 in [1]).

Lemma 2.1. *Assume that $0 < p, q, \beta < \infty$, ν is a normal function, φ is an analytic self-map of $\mathbb{D}$ and n is a nonnegative integer. Then $C^n_{\varphi,g} : H(p,q,\nu) \to \mathcal{Z}_\beta$ is compact if and only if $C^n_{\varphi,g} : H(p,q,\nu) \to \mathcal{Z}_\beta$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $H(p,q,\nu)$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, we have $\|C^n_{\varphi,g}f_k\|_{\mathcal{Z}_\beta} \to 0$ as $k \to \infty$.*

Lemma 2.2. ([8]) *A closed set G in $\mathcal{Z}_{\beta,0}$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in G} (1 - |z|^2)^\beta |f''(z)| = 0.$$

Lemma 2.3. ([8]) *Assume that $0 < p, q < \infty$ and ν is a normal function. If $f \in H(p,q,\nu)$, then there is a positive constant C independent of f such that*

$$|f^{(n)}(z)| \le C \frac{\|f\|_{H(p,q,\nu)}}{\nu(z)(1 - |z|^2)^{\frac{1}{q}+n}}, \qquad z \in \mathbb{D}. \tag{2.1}$$

Now we are in a position to state and prove the main results of this paper.

Theorem 2.1. *Assume that $0 < p, q, \beta < \infty$, ν is a normal function, φ is an analytic self-map of $\mathbb{D}$ and n is a nonnegative integer. Then $C^n_{\varphi,g} : H(p,q,\nu) \to \mathcal{Z}_\beta$ is bounded if and only if*

$$M_1 := \sup_{z\in\mathbb{D}} \frac{(1 - |z|^2)^\beta |g'(z)|}{\nu(\varphi(z))(1 - |\varphi(z)|^2)^{\frac{1}{q}+n}} < \infty \tag{2.2}$$

and

$$M_2 := \sup_{z\in\mathbb{D}} \frac{(1 - |z|^2)^\beta |g(z)||\varphi'(z)|}{\nu(\varphi(z))(1 - |\varphi(z)|^2)^{\frac{1}{q}+n+1}} < \infty. \tag{2.3}$$

Proof. Suppose that (2.2) and (2.3) hold. First it is easy to see that $(C^n_{\varphi,g}f)(0) = 0$ and $(C^n_{\varphi,g}f)'(z) = f^{(n)}(\varphi(z))g(z)$ for every $f \in H(\mathbb{D})$. For any $z \in \mathbb{D}$ and $f \in H(p,q,\nu)$, by Lemma 2.3 we have

$$\begin{aligned} & (1-|z|^2)^\beta |(C^n_{\varphi,g}f)''(z)| = (1-|z|^2)^\beta |(f^{(n)} \circ \varphi \cdot g)'(z)| \\ \leq\ & (1-|z|^2)^\beta |f^{(n+1)}(\varphi(z))g(z)\varphi'(z)| + (1-|z|^2)^\beta |f^{(n)}(\varphi(z))g'(z)| \\ \leq\ & C\frac{(1-|z|^2)^\beta |g(z)||\varphi'(z)|\|f\|_{H(p,q,\nu)}}{\nu(\varphi(z))(1-|\varphi(z)|^2)^{\frac{1}{q}+n+1}} + C\frac{(1-|z|^2)^\beta |g'(z)|\|f\|_{H(p,q,\nu)}}{\nu(\varphi(z))(1-|\varphi(z)|^2)^{\frac{1}{q}+n}}. \end{aligned} \tag{2.4}$$

Moreover $|(C^n_{\varphi,g}f)'(0)| \leq \frac{C|g(0)|\|f\|_{H(p,q,\nu)}}{\nu(\varphi(0))(1-|\varphi(0)|^2)^{\frac{1}{q}+n}}$. Taking the supremum in (2.4) for $z \in \mathbb{D}$, then employing (2.2) and (2.3) we see that $C^n_{\varphi,g} : H(p,q,\nu) \to \mathcal{Z}_\beta$ is bounded.

Conversely, suppose that $C^n_{\varphi,g} : H(p,q,\nu) \to \mathcal{Z}_\beta$ is bounded, i.e., there exists a constant C such that $\|C^n_{\varphi,g}f\|_{\mathcal{Z}_\beta} \leq C\|f\|_{H(p,q,\nu)}$ for all $f \in H(p,q,\nu)$. Taking the functions $f(z) \equiv z^n$ and $f(z) \equiv z^{n+1}$, which belongs to $H(p,q,\nu)$, we get

$$M_3 := \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |g'(z)| < \infty,\ M_4 := \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |g(z)||\varphi'(z)| < \infty. \tag{2.5}$$

For $a \in \mathbb{D}$, set $f_a(z) = \frac{(1-|a|^2)^{t+1}}{\nu(a)(1-\overline{a}z)^{\frac{1}{q}+t+1}}$. From [8] we see that $f_a \in H(p,q,\nu)$. Moreover there is a positive constant C such that $\sup_{a\in\mathbb{D}}\|f_a\|_{H(p,q,\nu)} \leq C$,

$$|f_a^{(n)}(a)| = \frac{\prod_{k=1}^n(\frac{1}{q}+t+k)|a|^n}{\nu(a)(1-|a|^2)^{\frac{1}{q}+n}},\ |f_a^{(n+1)}(a)| = \frac{\prod_{k=1}^{n+1}(\frac{1}{q}+t+k)|a|^{n+1}}{\nu(a)(1-|a|^2)^{\frac{1}{q}+n+1}}.$$

Hence,

$$\begin{aligned} \infty > \|C^n_{\varphi,g}f_{\varphi(\lambda)}\|_{\mathcal{Z}_\beta} \quad \geq \quad & \frac{\prod_{k=1}^n(\frac{1}{q}+t+k)(1-|\lambda|^2)^\beta |g'(\lambda)||\varphi(\lambda)|^n}{\nu(\varphi(\lambda))(1-|\varphi(\lambda)|^2)^{\frac{1}{q}+n}} - \\ & \frac{\prod_{k=1}^{n+1}(\frac{1}{q}+t+k)(1-|\lambda|^2)^\beta |g(\lambda)||\varphi'(\lambda)||\varphi(\lambda)|^{n+1}}{\nu(\varphi(\lambda))(1-|\varphi(\lambda)|^2)^{\frac{1}{q}+n+1}} \end{aligned} \tag{2.6}$$

for each $\lambda \in \mathbb{D}$.

For $a \in \mathbb{D}$, set

$$h_a(z) = \frac{(1-|a|^2)^{t+1}}{\nu(a)(1-\overline{a}z)^{\frac{1}{q}+t+1}} - \frac{\frac{1}{q}+t+1}{\frac{1}{q}+t+n+1}\frac{(1-|a|^2)^{t+2}}{\nu(a)(1-\overline{a}z)^{\frac{1}{q}+t+2}}.$$

Then

$$\sup_{a\in\mathbb{D}}\|h_a\|_{H(p,q,\nu)} \leq C,\ |h_a^{(n)}(a)| = 0,\ |h_a^{(n+1)}(a)| = \frac{\prod_{k=1}^n(\frac{1}{q}+t+k)|a|^{n+1}}{\nu(a)(1-|a|^2)^{\frac{1}{q}+n+1}}.$$

Hence,

$$\infty > \|C^n_{\varphi,g}h_{\varphi(\lambda)}\|_{\mathcal{Z}_\beta} \geq \frac{\prod_{k=1}^n(\frac{1}{q}+t+k)(1-|\lambda|^2)^\beta |g(\lambda)||\varphi'(\lambda)||\varphi(\lambda)|^{n+1}}{\nu(\varphi(\lambda))(1-|\varphi(\lambda)|^2)^{\frac{1}{q}+n+1}} \tag{2.7}$$

for each $\lambda \in \mathbb{D}$. Therefore, we obtain

$$\sup_{\lambda\in\mathbb{D}}\frac{(1-|\lambda|^2)^\beta |g(\lambda)||\varphi'(\lambda)||\varphi(\lambda)|^{n+1}}{\nu(\varphi(\lambda))(1-|\varphi(\lambda)|^2)^{\frac{1}{q}+n+1}} \leq C\|C^n_{\varphi,g}\|_{H(p,q,\nu)\to\mathcal{Z}_\beta} < \infty. \tag{2.8}$$

From (2.8), we have

$$\begin{aligned}&\sup_{|\varphi(\lambda)|>\frac{1}{2}}\frac{(1-|\lambda|^2)^\beta|g(\lambda)||\varphi'(\lambda)|}{\nu(\varphi(\lambda))(1-|\varphi(\lambda)|^2)^{\frac{1}{q}+n+1}}\\ <&\sup_{|\varphi(\lambda)|>\frac{1}{2}}\frac{2^{n+1}(1-|\lambda|^2)^\beta|g(\lambda)||\varphi'(\lambda)||\varphi(\lambda)|^{n+1}}{\nu(\varphi(\lambda))(1-|\varphi(\lambda)|^2)^{\frac{1}{q}+n+1}}<\infty. \qquad (2.9)\end{aligned}$$

Inequality (2.5) and the normality of ν give

$$\sup_{|\varphi(\lambda)|\le\frac{1}{2}}\frac{(1-|\lambda|^2)^\beta|g(\lambda)||\varphi'(\lambda)|}{\nu(\varphi(\lambda))(1-|\varphi(\lambda)|^2)^{\frac{1}{q}+n+1}}\le\frac{CM_4}{\nu(\frac{1}{2})(\frac{3}{4})^{\frac{1}{q}+n+1}}<\infty. \qquad (2.10)$$

Therefore, (2.3) follows from (2.9) and (2.10). From (2.6) and (2.7), we obtain

$$\sup_{\lambda\in\mathbb{D}}\frac{(1-|\lambda|^2)^\beta|g'(\lambda)||\varphi(\lambda)|^n}{\nu(\varphi(\lambda))(1-|\varphi(\lambda)|^2)^{\frac{1}{q}+n}}<\infty. \qquad (2.11)$$

Using (2.5) and (2.11), similarly to the above proof we obtain that (2.2) holds. The proof is completed. □

Theorem 2.2. *Assume that $0<p,q,\beta<\infty$, ν is a normal function, φ is an analytic self-map of $\mathbb{D}$ and n is a nonnegative integer. Then $C^n_{\varphi,g}:H(p,q,\nu)\to\mathcal{Z}_\beta$ is compact if and only if $C^n_{\varphi,g}:H(p,q,\nu)\to\mathcal{Z}_\beta$ is bounded and*

$$\lim_{|\varphi(z)|\to1}\frac{(1-|z|^2)^\beta|g'(z)|}{\nu(\varphi(z))(1-|\varphi(z)|^2)^{\frac{1}{q}+n}}=\lim_{|\varphi(z)|\to1}\frac{(1-|z|^2)^\beta|g(z)||\varphi'(z)|}{\nu(\varphi(z))(1-|\varphi(z)|^2)^{\frac{1}{q}+n+1}}=0. \qquad (2.12)$$

Proof. Suppose that $C^n_{\varphi,g}:H(p,q,\nu)\to\mathcal{Z}_\beta$ is bounded and (2.12) hold. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $H(p,q,\nu)$ such that $\sup_{k\in\mathbb{N}}\|f_k\|_{H(p,q,\nu)}\le C$ and f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$. By the assumption, for any $\varepsilon>0$, there exists a $\delta\in(0,1)$ such that

$$\frac{(1-|z|^2)^\beta|g'(z)|}{\nu(\varphi(z))(1-|\varphi(z)|^2)^{\frac{1}{q}+n}}<\varepsilon,\ \frac{(1-|z|^2)^\beta|g(z)||\varphi'(z)|}{\nu(\varphi(z))(1-|\varphi(z)|^2)^{\frac{1}{q}+n+1}}<\varepsilon \qquad (2.13)$$

when $\delta<|\varphi(z)|<1$. Since $C^n_{\varphi,g}:H(p,q,\nu)\to\mathcal{Z}_\beta$ is bounded, then from the proof of Theorem 2.1 we have $M_3<\infty,\ M_4<\infty$. Let $K=\{z\in\mathbb{D}:|\varphi(z)|\le\delta\}$. Then, by $M_3<\infty,\ M_4<\infty$ and (2.13) we have

$$\begin{aligned}&\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|(C^n_{\varphi,g}f_k)''(z)|\\ \le&\sup_K(1-|z|^2)^\beta|g'(z)||f_k^{(n)}(\varphi(z))|+\sup_K(1-|z|^2)^\beta|g(z)||\varphi'(z)||f_k^{(n+1)}(\varphi(z))|\\ +&C\sup_{\mathbb{D}\setminus K}\frac{(1-|z|^2)^\beta|g'(z)|\|f_k\|_{H(p,q,\nu)}}{\nu(\varphi(z))(1-|\varphi(z)|^2)^{\frac{1}{q}+n}}+C\sup_{\mathbb{D}\setminus K}\frac{(1-|z|^2)^\beta|g(z)||\varphi'(z)|\|f_k\|_{H(p,q,\nu)}}{\nu(\varphi(z))(1-|\varphi(z)|^2)^{\frac{1}{q}+n+1}}\\ \le&M_3\sup_{|w|\le\delta}|f_k^{(n)}(w)|+M_4\sup_{|w|\le\delta}|f_k^{(n+1)}(w)|+C\varepsilon\|f_k\|_{H(p,q,\nu)},\end{aligned}$$

i.e.,

$$\begin{aligned}&\|C^n_{\varphi,g}f_k\|_{\mathcal{Z}_\beta}=|f_k^{(n)}(\varphi(0))g(0)|+\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|(C^n_{\varphi,g}f_k)''(z)|\\ \le&|f_k^{(n)}(\varphi(0))g(0)|+M_3\sup_{|w|\le\delta}|f_k^{(n)}(w)|+M_4\sup_{|w|\le\delta}|f_k^{(n+1)}(w)|+C\varepsilon\|f_k\|_{H(p,q,\nu)}.\end{aligned}$$

From Cauchy's estimate and the assumption that $f_k \to 0$ as $k \to \infty$ on compact subsets of $\mathbb{D}$, we see that $f_k^{(n)} \to 0$ as $k \to \infty$ on compact subsets of $\mathbb{D}$. Letting $k \to \infty$ in the last inequality and using the fact that ε is an arbitrary positive number, we obtain $\lim_{k\to\infty} \|C^n_{\varphi,g} f_k\|_{\mathcal{Z}_\beta} = 0$. Applying Lemma 2.1, the result follows.

Conversely, suppose that $C^n_{\varphi,g} : H(p,q,\nu) \to \mathcal{Z}_\beta$ is compact. Then it is clear that $C^n_{\varphi,g} : H(p,q,\nu) \to \mathcal{Z}_\beta$ is bounded. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$ (if such a sequence does not exist then conditions in (2.12) are vacuously satisfied). Let

$$h_k(z) = \frac{(1-|\varphi(z_k)|^2)^{t+1}}{\nu(\varphi(z_k))(1-\overline{\varphi(z_k)}z)^{\frac{1}{q}+t+1}} - \frac{\frac{1}{q}+t+1}{\frac{1}{q}+t+n+1}\frac{(1-|\varphi(z_k)|^2)^{t+1}}{\nu(\varphi(z_k))(1-\overline{\varphi(z_k)}z)^{\frac{1}{q}+t+1}}.$$

Then

$$\sup_{k\in\mathbb{N}} \|h_k\|_{H(p,q,\nu)} < \infty,\ h_k^{(n+1)}(\varphi(z_k)) = \frac{\prod_{j=1}^{n}(\frac{1}{q}+t+j)|\varphi(z_k)|^{n+1}}{\nu(\varphi(z_k))(1-|\varphi(z_k)|^2)^{\frac{1}{q}+n+1}},$$

and h_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. Since $C^n_{\varphi,g} : H(p,q,\nu) \to \mathcal{Z}_\beta$ is compact, by Lemma 2.1 we have $\lim_{k\to\infty} \|C^n_{\varphi,g} h_k\|_{\mathcal{Z}_\beta} = 0$. On the other hand, we have

$$\|C^n_{\varphi,g} h_k\|_{\mathcal{Z}_\beta} \geq \frac{\prod_{j=1}^{n}(\frac{1}{q}+t+j)(1-|z_k|^2)^\beta |g(z_k)||\varphi'(z_k)||\varphi(z_k)|^{n+1}}{\nu(\varphi(z_k))(1-|\varphi(z_k)|^2)^{\frac{1}{q}+n+1}}$$

which together with $\lim_{k\to\infty} \|C^n_{\varphi,g} h_k\|_{\mathcal{Z}_\beta} = 0$ implies that

$$\lim_{|\varphi(z_k)|\to 1} \frac{(1-|z_k|^2)^\beta |g(z_k)||\varphi'(z_k)|}{\nu(\varphi(z_k))(1-|\varphi(z_k)|^2)^{\frac{1}{q}+n+1}} = 0. \tag{2.14}$$

Then the second equality of (2.12) follows.

Let

$$f_k(z) = \frac{(1-|\varphi(z_k)|^2)^{t+1}}{\nu(\varphi(z_k))(1-\overline{\varphi(z_k)}z)^{\frac{1}{q}+t+1}}.$$

Then $f_k \in H(p,q,\nu)$ and f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. Since $C^n_{\varphi,g} : H(p,q,\nu) \to \mathcal{Z}_\beta$ is compact, by Lemma 2.1 we have $\lim_{k\to\infty} \|C^n_{\varphi,g} f_k\|_{\mathcal{Z}_\beta} = 0$. On the other hand, we have

$$\begin{aligned}\|C^n_{\varphi,g} f_k\|_{\mathcal{Z}_\beta} \geq\ & \frac{\prod_{j=1}^{n}(\frac{1}{q}+t+j)(1-|z_k|^2)^\beta |g'(z_k)||\varphi(z_k)|^{n}}{\nu(\varphi(z_k))(1-|\varphi(z_k)|^2)^{\frac{1}{q}+n}} - \\ & \frac{\prod_{j=1}^{n+1}(\frac{1}{q}+t+j)(1-|z_k|^2)^\beta |g(z_k)||\varphi'(z_k)||\varphi(z_k)|^{n+1}}{\nu(\varphi(z_k))(1-|\varphi(z_k)|^2)^{\frac{1}{q}+n+1}}.\end{aligned} \tag{2.15}$$

Therefore, by (2.14) and (2.15) we get

$$\lim_{k\to\infty} \frac{(1-|z_k|^2)^\beta |g'(z_k)||\varphi(z_k)|^n}{\nu(\varphi(z_k))(1-|\varphi(z_k)|^2)^{\frac{1}{q}+n}} = 0. \tag{2.16}$$

Then the first equality of (2.12) follows from (2.16). □

Using the condition that polynomials is dense in $H(p,q,\nu)$ and similarly to the proof of Theorem 9 of [4], we get the following result. We omit the details.

Theorem 2.3. *Assume that $0 < p, q < \infty$, ν is a normal function, φ is an analytic self-map of $\mathbb{D}$ and n is a nonnegative integer. Then $C^n_{\varphi,g} : H(p,q,\nu) \to \mathcal{Z}_{\beta,0}$ is bounded if and only if $C^n_{\varphi,g} : H(p,q,\nu) \to \mathcal{Z}_{\beta}$ is bounded,*

$$\lim_{|z|\to 1}(1-|z|^2)^\beta |g'(z)| = \lim_{|z|\to 1}(1-|z|^2)^\beta |g(z)||\varphi'(z)| = 0. \tag{2.17}$$

Theorem 2.4. *Assume that $0 < p, q < \infty$, ν is a normal function, φ is an analytic self-map of $\mathbb{D}$ and n is a nonnegative integer. Then $C^n_{\varphi,g} : H(p,q,\nu) \to \mathcal{Z}_{\beta,0}$ is compact if and only if*

$$\lim_{|z|\to 1}\frac{(1-|z|^2)^\beta |g'(z)|}{\nu(\varphi(z))(1-|\varphi(z)|^2)^{\frac{1}{q}+n}} = \lim_{|z|\to 1}\frac{(1-|z|^2)^\beta |g(z)||\varphi'(z)|}{\nu(\varphi(z))(1-|\varphi(z)|^2)^{\frac{1}{q}+n+1}} = 0. \tag{2.18}$$

Proof. Assume that (2.18) holds. Let $f \in H(p,q,\nu)$. By the proof of Theorem 2.1 we have

$$\begin{aligned}&(1-|z|^2)^\beta |(C^n_{\varphi,g}f)''(z)|\\ \le\ & \frac{C(1-|z|^2)^\beta |g(z)||\varphi'(z)|\|f\|_{H(p,q,\nu)}}{\nu(\varphi(z))(1-|\varphi(z)|^2)^{\frac{1}{q}+n+1}} + \frac{C(1-|z|^2)^\beta |g'(z)|\|f\|_{H(p,q,\nu)}}{\nu(\varphi(z))(1-|\varphi(z)|^2)^{\frac{1}{q}+n}}\end{aligned} \tag{2.19}$$

Taking the supremum in (2.19) over all $f \in H(p,q,\nu)$ such that $\|f\|_{H(p,q,\nu)} \le 1$, then letting $|z| \to 1$, we get

$$\lim_{|z|\to 1}\sup_{\|f\|_{H(p,q,\nu)}\le 1}(1-|z|^2)^\beta |(C^n_{\varphi,g}f)''(z)| = 0.$$

From which by Lemma 2.2 we see that $C^n_{\varphi,g} : H(p,q,\nu) \to \mathcal{Z}_{\beta,0}$ is compact.

Conversely, suppose that $C^n_{\varphi,g} : H(p,q,\nu) \to \mathcal{Z}_{\beta,0}$ is compact. Then $C^n_{\varphi,g} : H(p,q,\nu) \to \mathcal{Z}_{\beta,0}$ is bounded and by Theorem 2.3 we get

$$\lim_{|z|\to 1}(1-|z|^2)^\beta |g'(z)| = \lim_{|z|\to 1}(1-|z|^2)^\beta |g(z)||\varphi'(z)| = 0. \tag{2.20}$$

If $\|\varphi\|_\infty < 1$, from (2.20), we obtain that

$$\lim_{|z|\to 1}\frac{(1-|z|^2)^\beta |g'(z)|}{\nu(\varphi(z))(1-|\varphi(z)|^2)^{\frac{1}{q}+n}} \le \frac{C\lim_{|z|\to 1}(1-|z|^2)^\beta |g'(z)|}{\nu(\|\varphi\|_\infty)(1-\|\varphi\|^2_\infty)^{\frac{1}{q}+n}} = 0$$

and

$$\lim_{|z|\to 1}\frac{(1-|z|^2)^\beta |g(z)||\varphi'(z)|}{\nu(\varphi(z))(1-|\varphi(z)|^2)^{\frac{1}{q}+n+1}} \le \frac{C\lim_{|z|\to 1}(1-|z|^2)^\beta |g(z)||\varphi'(z)|}{\nu(\|\varphi\|_\infty)(1-\|\varphi\|^2_\infty)^{\frac{1}{q}+n+1}} = 0,$$

from which the result follows in this case.

Now we assume that $\|\varphi\|_\infty = 1$. Let $(\varphi(z_k))_{k\in\mathbb{N}}$ be a sequence such that $\lim_{k\to\infty}|\varphi(z_k)| = 1$. From the compactness of $C^n_{\varphi,g} : H(p,q,\nu) \to \mathcal{Z}_{\beta,0}$ we see that the operator $C^n_{\varphi,g} : H(p,q,\nu) \to \mathcal{Z}_{\beta}$ is compact. From Theorem 2.2 we get

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\beta |g'(z)|}{\nu(\varphi(z))(1-|\varphi(z)|^2)^{\frac{1}{q}+n}} = \lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\beta |g(z)||\varphi'(z)|}{\nu(\varphi(z))(1-|\varphi(z)|^2)^{\frac{1}{q}+n+1}} = 0. \tag{2.21}$$

Using (2.20) and (2.21) we easily get the desired result. The proof is completed. □

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