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CLOSE-TO-CONVEXITY AND STARLIKENESS OF CERTAIN ANALYTIC FUNCTIONS DEFINED BY A LINEAR OPERATOR

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Abstract. The main object of the present paper is to derive some results for multivalent analytic functions defined by a linear operators. As a special case of these results, we obtain several sufficient conditions for close-to-convexity and starlikeness of certain analytic functions.

1. Introduction

Let $\mathcal{A}(p,n)$ denote the class of functions f in the form

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k \quad (p, n \in \mathbb{N} = \{1, 2, ...\}) \tag{1}$$

which are analytic and p-valent in the open unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. We write $\mathcal{A}(p,1) = \mathcal{A}(p), \mathcal{A}(1,n) = \mathcal{A}_n$ and $\mathcal{A}_1 = \mathcal{A}$. A function $f \in \mathcal{A}(p,n)$ is said to be p-valent starlike of order α $(0 \le \alpha < p)$ in Δ if

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in \Delta,$$

and we denote by $S_p^*(\alpha)$ the class of all such functions. A function $f \in \mathcal{A}(p,n)$ is said to be p-valent convex of order α $(0 \le \alpha < p)$ in Δ if

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad z \in \Delta,$$

and we denote by $K_p^*(\alpha)$ the class of all such functions. Further a function $f \in \mathcal{A}$ is said to be close-to-convex if there exists a (not necessarily normalized) convex function g such that

$$\mathrm{Re}\left(\frac{f'(z)}{g'(z)}\right) > 0, \; z \in \Delta.$$

We shall denote by $\mathcal{C}$ the class of close-to-convex functions in Δ.

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For two functions f given by (1) and g given by

$$g(z) = z^p + \sum_{k=n+p}^{\infty} b_k z^k \quad (p, n \in \mathbb{N}),$$

their Hadamard product (or convolution) is defined by

$$(f * g)(z) = z^p + \sum_{k=n+p}^{\infty} b_k a_k z^k.$$

Define the function $\phi_p(a, c; z)$ by

$$\phi_p(a, c; z) := z^p + \sum_{k=n}^{\infty} \frac{(a)_k}{(c)_k} z^{k+p} \quad (c \neq 0, -1, -2, ..., z \in \Delta),$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n := \begin{cases} 1, & (n = 0); \\ a(a+1)(a+2)\dots(a+n-1), & (n \in \mathbb{N} := \{1, 2, 3\dots\}). \end{cases}$$

Corresponding to the function $\phi_p(a, c; z)$, Saitoh [7] introduced a linear operator $\mathcal{L}_p(a, c)$ which is defined by means of the following Hadamard product:

$$\mathcal{L}_p(a, c)f(z) = \phi_p(a, c) * f(z) \qquad (f \in \mathcal{A}(p, n)),$$

or, equivalently, by

$$\mathcal{L}_p(a, c)f(z) = z^p + \sum_{k=n}^{\infty} \frac{(a)_k}{(c)_k} a_{k+p} z^{k+p}, \ z \in \Delta. \tag{2}$$

It follows from (2) that

$$z(\mathcal{L}_p(a, c)f(z))' = a\mathcal{L}_p(a+1, c)f(z) - (a-p)\mathcal{L}_p(a, c)f(z) \tag{3}$$

Note that $\mathcal{L}_p(a, a)f(z) = f(z)$, $\mathcal{L}_p(p+1, p)f(z) = \frac{zf'(z)}{p}$, $\mathcal{L}_1(3, 1)f(z) = zf'(z) + \frac{1}{2}z^2 f''(z)$ and $\mathcal{L}_p(\delta+1, 1)f(z) = D^{\delta+p}f(z)$, where $D^{\delta+p}f$ is the Ruscheweyh derivative of order $\delta + p$.

Many properties of analytic functions defined by the linear operator $\mathcal{L}_p(a, c)f(z)$ were studied by (among others), Aghalary and Ebadian [1], Owa and Srivastava [6], Cho et al. [3], and Carlson and Shaffer [2].

In the present paper we aim to find simple sufficient conditions for close-to-convexity and starlikeness of multivalent analytic functions. The following lemma will be required in our present investigations.

Lemma 1.1. (see[4]) Let the (nonconstant) function ω be analytic in Δ with $\omega(z) = \omega_n z^n + \cdots$. If $|\omega|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \Delta$, then

$$z_0 \omega'(z_0) = c\omega(z_0),$$

where c is a real number and $c \geq n$.

2. Main Results

Theorem 2.1. *Let* $a \in \mathbb{C}$, $|a| > 0$, $\beta \geq 0, \gamma \geq 0$, *and* $0 \leq \alpha < p$. *If the function* $f \in \mathcal{A}(p,n)$ *satisfies*

$$\left|\frac{\mathcal{L}_p(a,c)f(z)}{z^p} - 1\right|^{\gamma} \left|\frac{\mathcal{L}_p(a+1,c)f(z) - \mathcal{L}_p(a,c)f(z)}{z^p}\right|^{\beta} < \frac{1}{|a|^{\beta}}(1-\frac{\alpha}{p})^{\gamma+\beta} n^{\beta},\ z \in \Delta, \tag{4}$$

then

$$Re\left(\frac{\mathcal{L}_p(a,c)f(z)}{z^p}\right) > \frac{\alpha}{p},\ z \in \Delta. \tag{5}$$

Proof. Define the function ω by

$$\frac{\mathcal{L}_p(a,c)f(z)}{z^p} = \frac{1 + (1 - \frac{2\alpha}{p})\omega(z)}{1 - \omega(z)}, \quad (\omega(z) \neq -1,\ z \in \Delta). \tag{6}$$

Then, clearly, $\omega(z) = \omega_n z^n + \cdots$ is analytic in Δ. By a simple computation and by making use of the familiar identity (3), we find from (6) that

$$\begin{aligned}&\left|\frac{\mathcal{L}_p(a,c)f(z)}{z^p} - 1\right|^{\gamma} \left|\frac{\mathcal{L}_p(a+1,c)f(z) - \mathcal{L}_p(a,c)f(z)}{z^p}\right|^{\beta} \\ &= \frac{1}{|a|^{\beta}} \frac{2^{\gamma+\beta}(1-\frac{\alpha}{p})^{\gamma+\beta}}{|1-\omega(z)|^{\gamma+\beta}} |z\omega'(z)|^{\beta}.\end{aligned}$$

Suppose now that there exists a point $z_0 \in \Delta$ such that

$$|\omega(z_0)| = 1 \quad \textit{and} \quad |\omega(z)| < 1, \quad \textit{when} \quad |z| < |z_0|.$$

Then by using Lemma 1.1, we have $\omega(z_0) = e^{i\theta}, 0 < \theta \leq 2\pi$ and $z_0\omega'(z_0) = \xi\omega(z_0), \xi \geq n$. Therefore

$$\begin{aligned}&\left|\frac{\mathcal{L}_p(a,c)f(z_0)}{z_0^p} - 1\right|^{\gamma} \left|\frac{\mathcal{L}_p(a+1,c)f(z_0) - \mathcal{L}_p(a,c)f(z_0)}{z_0^p}\right|^{\beta} \\ &= \frac{1}{|a|^{\beta}} \frac{2^{\gamma+\beta}(1-\frac{\alpha}{p})^{\gamma+\beta}}{|1-e^{i\theta}|^{\gamma+\beta}} |\xi|^{\beta} \\ &> \frac{1}{|a|^{\beta}}(1-\frac{\alpha}{p})^{\gamma+\beta} n^{\beta},\end{aligned}$$

which contradicts our hypothesis (4). Thus, we have

$$|w(z)| < 1,\ z \in \Delta,$$

and the proof is complete. □

By letting $a = c = 1$ and $p = n = 1$ in Theorem 2.1 we obtain Theorem 3 of [5] that is:

Corollary 2.2. *Let* $\gamma \geq 0, \beta \geq 0$ *and* $0 \leq \alpha < 1$. *If the function* $f \in \mathcal{A}$ *satisfies*

$$|f'(z) - 1|^{\gamma} |zf''(z)|^{\beta} < 2^{\beta}(1-\alpha)^{\gamma+\beta},\ z \in \Delta,$$

then

$$\operatorname{Re} f'(z) > \alpha,\ z \in \Delta,$$

i.e. f *is close-to-convex function.*

Theorem 2.3. *Let $a \in \mathbb{C}$ with $\operatorname{Re} a > 0$, let $\beta \geq 0, \gamma > 0$ and $0 \leq \alpha < p$. If $f \in \mathcal{A}(p)$ satisfies the inequality*

$$\left|\frac{\mathcal{L}_p(a,c)f(z)}{z^p} - 1\right|^{\gamma} \left|\frac{\mathcal{L}_p(a+1,c)f(z)}{z^p} - 1\right|^{\beta} \leq \frac{(1-\frac{\alpha}{p})^{\gamma+\beta}}{\mid a \mid^{\beta}}(Re\, a+\frac{n}{2})^{\beta},\ z \in \Delta,\ (7)$$

then

$$\operatorname{Re}\left(\frac{\mathcal{L}_p(a,c)f(z)}{z^p}\right) > \frac{\alpha}{p},\ z \in \Delta. \qquad (8)$$

Proof. Let define the function ω by

$$\frac{\mathcal{L}_p(a,c)f(z)}{z^p} = \frac{1 + \left(1 - \frac{2\alpha}{p}\right)\omega(z)}{1-\omega(z)}, \qquad (\omega(z) \neq -1,\ z \in \Delta).$$

Then ω is analytic in Δ, $\omega(z) = \omega_n z^n + \cdots$. By making use of the identity (3), we obtain

$$\begin{aligned} &\left|\frac{\mathcal{L}_p(a,c)f(z)}{z^p} - 1\right|^{\gamma} \left|\frac{\mathcal{L}_p(a+1,c)f(z)}{z^p} - 1\right|^{\beta} \\ &= \left|\frac{2(1-\frac{\alpha}{p})\omega(z)}{1-\omega(z)}\right|^{\gamma} \left|\frac{1}{a}\frac{2(1-\frac{\alpha}{p})z\omega'(z)}{(1-\omega(z))^2} + \frac{2(1-\frac{\alpha}{p})\omega(z)}{1-\omega(z)}\right|^{\beta} \\ &= \frac{2^{\gamma+\beta}(1-\frac{\alpha}{p})^{\gamma+\alpha}}{\mid a \mid^{\beta}} \left|\frac{\omega(z)}{1-\omega(z)}\right|^{\gamma+\beta} \left|a + \frac{z\omega'(z)}{(1-\omega(z))\omega(z)}\right|^{\beta}. \end{aligned}$$

Suppose that there exists a point $z_0 \in \Delta$ such that $\max|\omega(z)| = |\omega(z_0)| = 1$ ($|z| \leq |z_0|$). Then by using Lemma 1.1, we have $\omega(z_0) = e^{i\theta}$, $0 < \theta \leq 2\pi$ and $z_0\omega'(z_0) = \xi\omega(z_0)$, $\xi \geq n$. Therefore

$$\begin{aligned} &\left|\frac{\mathcal{L}_p(a,c)f(z_0)}{z_0^p} - 1\right|^{\gamma} \left|\frac{\mathcal{L}_p(a+1,c)f(z_0)}{z_0^p} - 1\right|^{\beta} \\ &= \frac{2^{\gamma+\beta}(1-\frac{\alpha}{p})^{\gamma+\beta}}{\mid a \mid^{\beta}} \frac{1}{\mid 1-\omega(z_0) \mid^{\gamma+\beta}} \left|a + \frac{\xi}{(1-e^{i\theta})}\right|^{\beta} \\ &\geq \frac{(1-\frac{\alpha}{p})^{\gamma+\beta}}{\mid a \mid^{\beta}}(Re\, a + \frac{n}{2})^{\beta}. \end{aligned}$$

Which contradicts obviously our hypothesis (7). Thus, we have $\mid \omega(z) \mid < 1$ for all $z \in \Delta$, and hence (8) holds true. □

By letting $c = a - 1 = 1$, $\gamma = \beta = \frac{1}{2}$ and $p = n = 1$ in Theorem 2.1, we obtain the following Corollary:

Corollary 2.4. *If the function $f \in \mathcal{A}$ satisfies the inequality*

$$|f'(z) - 1|^{\frac{1}{2}} \left|f'(z) + \frac{1}{2}zf''(z) - 1\right|^{\frac{1}{2}} < \frac{(1-\alpha)}{\sqrt{2}}(2 + \frac{1}{2})^{\frac{1}{2}},\ z \in \Delta,$$

then

$$\operatorname{Re} f'(z) > \alpha,\ z \in \Delta,$$

i.e. f is close-to-convex function.

By letting $c = a = 1$, $\gamma = \beta = \frac{1}{2}$ and $p = 1$ in Theorem 2.3, we conclude the following result:

Corollary 2.5. *If the function $f \in \mathcal{A}$ satisfies the inequality*

$$|f(z)-1|^{\frac{1}{2}}\,|f'(z)-1|^{\frac{1}{2}} < \frac{3}{2}(1-\alpha),\ z \in \Delta,$$

then

$$\mathrm{Re}\left(\frac{f(z)}{z}\right) > \alpha,\ z \in \Delta.$$

Finally we prove:

Theorem 2.6. *Suppose that $a \in \mathbb{C}$, $Re\, a \geq 0$, $\beta \geq 0, \gamma \geq 0$ and $0 \leq \alpha < p$. If the function $f \in \mathcal{A}(p,n)$ satisfies the inequality*

$$\left|\frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)}-1\right|^{\gamma}\left|\frac{\mathcal{L}_p(a+2,c)f(z)}{\mathcal{L}_p(a+1,c)f(z)}-1\right|^{\beta} < N(\alpha,p,n,\gamma,\beta),\ z \in \Delta, \tag{9}$$

where

$$N(\alpha,p,n,\gamma,\beta) = \begin{cases} \frac{(1-\frac{\alpha}{p})^{\gamma}\left((\mathrm{Re}a)(1-\frac{\alpha}{p})+\frac{n}{2}\right)^{\beta}}{|a+1|^{\beta}}, & 0 \leq \alpha \leq \frac{p}{2}, \\ \frac{(1-\frac{\alpha}{p})^{\gamma+\beta}}{|a+1|^{\beta}}(\mathrm{Re}a+n)^{\beta}, & \frac{p}{2} \leq \alpha < p. \end{cases} \tag{10}$$

Then

$$\mathrm{Re}\left(\frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)}\right) > \frac{\alpha}{p},\ z \in \Delta.$$

Proof. Define the function M by

$$M(z) = \frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)}.$$

Then by a simple computation and by making use of the identity (3), we have

$$\begin{aligned} &\left|\frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)}-1\right|^{\gamma}\left|\frac{\mathcal{L}_p(a+2,c)f(z)}{\mathcal{L}_p(a+1,c)f(z)}-1\right|^{\beta} \\ &=\mid M(z)-1\mid^{\gamma}\left|\frac{1}{a+1}\left(\frac{zM'(z)}{M(z)}+a(M(z)-1)\right)\right|^{\beta}. \end{aligned} \tag{11}$$

Now we distinguish two cases,
Case(i). If $0 \leq \alpha \leq \frac{p}{2}$, define a function ω

$$M(z) = \frac{1+\left(1-\frac{2\alpha}{p}\right)\omega(z)}{1-\omega(z)},\ z \in \Delta.$$

Then ω is analytic in Δ, $\omega(z) = \omega_n z^n + \cdots$ and $\omega(z) \neq 1$ in Δ. we find from (11) that

$$\begin{aligned} &\left|\frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)}-1\right|^{\gamma}\left|\frac{\mathcal{L}_p(a+2,c)f(z)}{\mathcal{L}_p(a+1,c)f(z)}-1\right|^{\beta} \\ &= \frac{2^{\gamma+\beta}\left(1-\frac{\alpha}{p}\right)^{\gamma+\beta}}{\mid a+1\mid^{\beta}}\left|\frac{\omega(z)}{1-\omega(z)}\right|^{\gamma+\beta}\left|a+\frac{z\omega'(z)}{\left[1+\left(1-\frac{2\alpha}{p}\right)\omega(z)\right]\omega(z)}\right|^{\beta}. \end{aligned} \tag{12}$$

Suppose now that there exists a point $z_0 \in \Delta$ such that $\max|\omega(z)| = |\omega(z_0)| = 1$ $(|z| \leq |z_0|)$. Then by using Lemma 1.1, we have $\omega(z_0) = e^{i\theta}$, $0 < \theta \leq 2\pi$ and $z_0\omega'(z_0) = m\omega(z_0)$, $m \geq n$. Therefore from (12), we obtain

$$\begin{aligned}
&\left|\frac{\mathcal{L}_p(a+1,c)f(z_0)}{\mathcal{L}_p(a,c)f(z_0)} - 1\right|^{\gamma} \left|\frac{\mathcal{L}_p(a+2,c)f(z_0)}{\mathcal{L}_p(a+1,c)f(z_0)} - 1\right|^{\beta} \\
&\geq \frac{\left(1-\frac{\alpha}{p}\right)^{\gamma+\beta}}{|\ a+1\ |^{\beta}} \left(\operatorname{Re} a + \frac{n}{2\left(1-\frac{\alpha}{p}\right)}\right)^{\beta} \\
&= \frac{1}{|\ a+1\ |^{\beta}} \left(1-\frac{\alpha}{p}\right)^{\gamma} \left(\operatorname{Re}\left(1-\frac{\alpha}{p}\right) + \frac{n}{2}\right)^{\beta},
\end{aligned}$$

which contradicts (10) for $0 \leq \alpha \leq \frac{p}{2}$, Hence, we must have $|\ \omega(z)\ | < 1$ for all $z \in \Delta$, and the first part of theorem complete.
Case(ii). When $\frac{p}{2} \leq \alpha < p$, let a function ω be defined by

$$M(z) = \frac{\frac{\alpha}{p}}{\frac{\alpha}{p} - \left(1-\frac{\alpha}{p}\right)\omega(z)},\ z \in \Delta.$$

Then ω is analytic in Δ and $\omega(z) = \omega_n z^n + \cdots$ proceeding the same as case(i). We find from (12) that

$$\begin{aligned}
&\left|\frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)} - 1\right|^{\gamma} \left|\frac{\mathcal{L}_p(a+2,c)f(z)}{\mathcal{L}_p(a+1,c)f(z)} - 1\right|^{\beta} \\
&= \frac{\left(1-\frac{\alpha}{p}\right)^{\gamma+\beta}}{|\ a+1\ |^{\beta}} \left|\frac{\omega(z)}{\frac{\alpha}{p} - \left(1-\frac{\alpha}{p}\right)\omega(z)}\right|^{\gamma+\beta} \left|a + \frac{z\omega'(z)}{\omega(z)}\right|^{\beta}. \qquad (13)
\end{aligned}$$

Suppose that there exists a point $z_0 \in \Delta$ such that $\max|\omega(z)| = |\omega(z_0)| = 1(|z| \leq |z_0|)$. Then by using Lemma 1.1, we have obtain $\omega(z_0) = e^{i\theta}$, $0 \leq \theta \leq 2\pi$ and $z_0\omega'(z_0) = m\omega(z_0)$, $m \geq n$. Now from (13) we have

$$\begin{aligned}
&\left|\frac{\mathcal{L}_p(a+1,c)f(z_0)}{\mathcal{L}_p(a,c)f(z_0)} - 1\right|^{\gamma} \left|\frac{\mathcal{L}_p(a+2,c)f(z_0)}{\mathcal{L}_p(a+1,c)f(z_0)} - 1\right|^{\beta} \\
&= \frac{\left(1-\frac{\alpha}{p}\right)^{\gamma+\beta}}{|\ a+1\ |^{\beta}} \left|\frac{\omega(z_0)}{\frac{\alpha}{p} - \left(1-\frac{\alpha}{p}\right)\omega(z_0)}\right|^{\gamma+\beta} \left|a + \frac{z_0\omega'(z_0)}{\omega(z_0)}\right|^{\beta} \\
&\geq \frac{\left(1-\frac{\alpha}{p}\right)^{\gamma+\beta}}{|\ a+1\ |^{\gamma+\beta}} \left(\operatorname{Re} a + n\right)^{\beta},
\end{aligned}$$

which contradicts (9) for $\frac{p}{2} \leq \alpha < p$. Therefore, we must have $|\ \omega(z)\ | < 1$ for all $z \in \Delta$, and the proof is complete. □

By letting $c = a = 1$ and $p = 1$ in the theorem 2.6, we have:

Corollary 2.7. *If the function $f \in \mathcal{A}_n$ satisfies the inequality*

$$\left|\frac{zf'(z)}{f(z)} - 1\right|^{\gamma} \left|\frac{zf''(z)}{f'(z)}\right|^{\beta} \leq M(\alpha,\beta,\gamma,n),\ z \in \Delta,$$

where

$$M(\alpha,\beta,\gamma,n)=\begin{cases}(1-\alpha)^{\gamma}\left(1+\frac{n}{2}-\alpha\right)^{\beta}, & 0\le\alpha\le\frac{1}{2},\\ (1-\alpha)^{\gamma+\beta}(1+n)^{\beta}, & \frac{1}{2}\le\alpha<1.\end{cases}$$

Then

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha,\ z\in\Delta.$$

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