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ALGEBRABILITY OF SPACE OF QUASI-EVERYWHERE SURJECTIVE FUNCTIONS

(COMMUNICATED BY D. KALAJ)

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Abstract. Existence of an infinitely generated algebra in a certain set, is called algebrability. In this paper we will show that the space of quasi-everywhere surjective functions is algebrable.

1. Introduction

The main theorem of this paper concerns the existence of an infinitely generated algebra in the space of quasi-everywhere surjective functions that are not everywhere surjective. This is a contribution to a very new area of research in mathematical analysis, that is to search for large algebraic structures (linear spaces or algebras) in the space of functions that are enjoying a special property. It has become a usual notation to call a subset M of a topological vector space X, μ-lineable (respectively, μ-spaceable) if $M \cup \{0\}$ contains a vector space (respectively, closed vector space) of dimension μ. If M contains an infinite-dimensional (closed) vector space, then M will be shortly called lineable (spaceable).

The origin of lineability and spaceability is due to Gurariy ([20, 21]) that showed that there exists an infinite dimensional linear space such that every non-zero element of which is a continuous nowhere differentiable function on $\mathcal{C}[0;1]$. Many examples of vector spaces of functions on $\mathbb{R}$ or $\mathbb{C}$ enjoying certain special properties have been constructed in the recent years. More recently, many authors got interested in this subject and gave a wide range of examples. For instance, in [4] it was shown that the set of everywhere surjective functions in $\mathbb{R}$ is $2^{\mathfrak{c}}$-lineable (where $\mathfrak{c}$ denotes the cardinality of $\mathbb{R}$) and that the set of differentiable functions on $\mathbb{R}$ which are nowhere monotone is lineable in $\mathcal{C}(\mathbb{R})$. These behaviors occur, sometimes, in particularly interesting ways. For example, in [22], Hencl showed that any separable Banach space is isometrically isomorphic to a subspace of $\mathcal{C}[0;1]$ whose non-zero elements are nowhere approximately differentiable and nowhere Holder. We refer

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the interested reader to [2, 3, 7, 8, 9, 11, 12, 13, 14, 16, 18, 19] for a wider range of results in lineability and spaceability.

Of course, one could go further and not just consider linear spaces but, instead, larger or more complex structures. For instance, in [1] the authors showed that there exists an uncountably generated algebra, that every non-zero element of which is an everywhere surjective function on $\mathbb{C}$, and in [5] it was shown that, if $E \subset \mathbb{T}$, the unit circle, is a set of measure zero, and if $\mathcal{F}(\mathbb{T})$ denotes the subset of $\mathcal{C}(\mathbb{T})$ of continuous functions whose Fourier series expansion diverges at every point of E, then $\mathcal{F}(\mathbb{T})$ contains an infinitely generated and dense subalgebra. One of the newest result in this area ([8]) proves the existence of uncountably generated algebras inside the following sets of special functions: Sierpinski-Zygmund functions, perfectly everywhere surjective functions, and nowhere continuous Darboux functions. That a space contains an infinitely generated algebra is called algebrability. It is clear that algebrability implies lineability but studying the algebrability of a space is sometimes far harder than lineability.

The notation of everywhere surjective functions that was mentioned before was first introduced by Lebesgue ([23]) in 1904 by showing the existence of a function $f : \mathbb{R} \to \mathbb{R}$ with the property that

$$f((a, b)) = \mathbb{R},$$

for every non-trivial interval (a, b). Space of everywhere surjective functions on $\mathbb{R}$ will be denoted by $\mathcal{ES}$ in this paper. These sort of functions was not taken in to account until recently that Aron and Seoane-Sepulveda [6] has investigated the algebraic structure contained in the space of these functions. This trend of research was continued in [17] that they defined perfectly everywhere surjective and strongly everywhere surjective functions and showed some pathological properties of space of such functions. The author proved the existence of infinite dimensional vector space in the space quasi-everywhere surjective functions that are not everywhere surjective([15]).

Our main concept in this paper is to expand the theory of everywhere surjective functions by defining quasi-everywhere surjective functions and investigating the pathological properties of those spaces.

2. Quasi-everywhere surjective functions

This section is devoted to defining quasi-everywhere surjective functions and studying their pathological properties.

Definition 2.1. *Let X and Y be two topological vector spaces. A function $f : X \to Y$ is called quasi-everywhere surjective if $f(U)$ is dense in Y for every open subset U of X. We will show the collection of all quasi-everywhere surjective functions by $\mathcal{QES}(X, Y)$ and if $X = Y$, it will be shown by $\mathcal{QES}(X)$.*

Here we provide the reader with an example of a quasi-everywhere surjective function $f : \mathbb{C} \to \mathbb{C}$ that is not everywhere surjective.

Example 2.2. *A functions $f : \mathbb{C} \to \mathbb{C}$ that is quasi-everywhere surjective but not everywhere surjective.*

To construct this function let $\{Q_j, j = 1, 2, \ldots\}$ *be the collection of all open rectangles in* $\mathbb{R}^2$ *whose vertexes have rational components, i. e.*

$$Q_j = \{x + iy,\ a_j < x < b_j,\ c_j < y < d_j\}$$

where a_j, b_j, c_j, d_j *are rational numbers for all* $j = 1, 2, \ldots$. *We construct a collection* $\{C_j\}$ *of uncountable nowhere dense subsets of* $\mathbb{R}$ *with the following properties.*

- $C_j \subset (a_j, b_j)$;
- $C_{j+1} \cap \left(\cup_{n=1}^{j} C_n\right) = \emptyset$.

To construct this collection, let $C_1 \subset (a_1, b_1)$ *be any uncountable nowhere dense subset and assume that* $C_2, \ldots, C_j$ *have been constructed. Since* $\cup_{n=1}^{j} C_j$ *is nowhere dense, we can take* $C_{j+1} \subset (a_{j+1}, b_{j+1})\backslash \left(\cup_{n=1}^{j} C_j\right)$ *to be uncoutable and nowhere dense set. This completes the construction of the collection* $\{C_j\}$.

Now let h_j *be a bijection between* (c_j, d_j) *and* $\mathbb{C}\backslash \left(\cup_{j=1}^{\infty} C_j\right)$, *and* Φ_j *be a bijection between* $[0, 1]$ *and* C_j. *Let* $\alpha \in [0, 1]$ *be fixed and define* $f_\alpha : \mathbb{C} \to \mathbb{C}$ *as follow.*

$$f_\alpha(z) = \begin{cases} h_j(Im\ z) & if\ Re\ z = \Phi_j(\alpha)\ and\ Im\ z \in (c_j, d_j)\ for\ some\ j \in \mathbb{N}, \\ 1 & otherwise. \end{cases}$$

We claim that $f_\alpha \in \mathcal{QES}(\mathbb{C})\backslash\mathcal{ES}(\mathbb{C})$. *To prove the claim, let* U *be an open subset of plane.* $Q_j \subset U$ *for some* $j \in \mathbb{N}$. $Ran(f_\alpha|_{Q_j}) = \mathbb{C}\backslash \left(\cup_{j=1}^{\infty} C_j\right)$. *This shows that* $f_\alpha \in \mathcal{QES}(\mathbb{C})\backslash\mathcal{ES}(\mathbb{C})$.

Note 2.3. *We have used the fact that if* X *is a nowhere dense subset of* $\mathbb{R}$ *then its complement is uncountable. This can be easily proved by Bair's theorem.*

The following lemma help us creating an infinitely generated algebra contained in the space of all quasi-everywhere surjective functions that are not everywhere surjective.

Lemma 2.4. *If* $\alpha_1, \alpha_2 \in [0, 1]$ *with* $\alpha_1 \neq \alpha_2$, *then* $f_{\alpha_1} f_{\alpha_2} \in \mathcal{QES}(\mathbb{C})\backslash\mathcal{ES}(\mathbb{C})$.

Proof. Let U be an open subset of plane. $Q_j \subset U$ for some $j \in \mathbb{N}$. We claim that for each $z \in \mathbb{C}$, $f_{\alpha_1}(z) = 1$ or $f_{\alpha_2}(z) = 1$. Let $f_{\alpha_1}(z) \neq 1$. By the definition of f_{α_1}, there exists $j_1 \in \mathbb{N}$ such that $Im\ z \in (c_{j_1}, d_{j_1})$ and $Re\ z = \Phi_{j_1}(\alpha_1)$. This shows that $Re\ z \in C_{j_1}$. Now by contradiction, let $f_{\alpha_2}(z) \neq 1$ then there there exists $j_2 \in \mathbb{N}$ such that $Im\ z \in (c_{j_2}, d_{j_2})$ and $Re\ z = \Phi_{j_2}(\alpha_2)$. This shows that $Re\ z \in C_{j_2}$. Since $\{C_j\}$ are mutually disjoint, so $j_1 = j_2$. Let $j = j_1 = j_2$. But Φ_j is a bijection between $[0, 1]$ and C_j. Since $Re\ z = \Phi_j(\alpha_1) = \Phi_j(\alpha_2)$, so $\alpha_1 = \alpha_2$, that is a contradiction to the hypothesis. Therefore $Ran(f_{\alpha_1} f_{\alpha_2}|_{Q_j}) = \{f_{\alpha_1}(z) f_{\alpha_2}(z), z \in Q_j\} = \mathbb{C}\backslash \left(\cup_{j=1}^{\infty} C_j\right)$. □

Theorem 2.5. $\mathcal{QES}(\mathbb{C})\backslash\mathcal{ES}(\mathbb{C})$ *is* $\mathfrak{c}$*-algebrable.*

Proof. Fix $\alpha_0 \in [0, 1]$ and assume that A is the algebra generated by $\{f_{\alpha_0} f_\alpha : \alpha_0 \neq \alpha, \alpha \in [0, 1]\}$. For each f in A, we have that

$$f = P(f_{\alpha_0} f_{\alpha_1}, \ldots, f_{\alpha_0} f_{\alpha_n}),$$

for some $n \in \mathbb{N}$, where $P \in \mathbb{C}[z_1, \dots, z_n]$ (the set of polynomials in $\mathbb{C}$ of n variables) with $P(0) = 0$. In order to show that $f \in \mathcal{QES}(\mathbb{C}) \backslash \mathcal{ES}(\mathbb{C})$, define $q(z) := P(z, \dots, z)$. Two cases can be occurred.

Case 1: $q(z)$ is non-constant. Let $\varepsilon > 0$ and $z \in \mathbb{C}$ are given. There exists $w \in \mathbb{C}$ such that $q(w) = z$. For every non empty open set U of $\mathbb{C}$, there exists $j \in \mathbb{N}$ such that $Q_j \subset U$. By continuity of q and the properties of h_j,

$$|q(h_j(t)) - q(w)| < \varepsilon,$$

for some $t \in (c_j, d_j)$. Put

$$w' = \Phi_j(\alpha_0) + it.$$

It is clear that $w' \in U$ and $f_{\alpha_0}(w') = h_j(t)$ and $f_\alpha(w') = 1$ for all $\alpha \in [0,1]$ that $\alpha \neq \alpha_0$. Therefore

$$f(w') = P(f_{\alpha_0} f_{\alpha_1}, \dots, f_{\alpha_0} f_{\alpha_n})(w') = P(h_j(t), \dots, h_j(t)) = q(h_j(t)).$$

Thus

$$|f(w') - z| < \varepsilon.$$

This completes the proof of case one.

Case2: q is constant. Proof of this part is from [1]. For the sake of completeness, we adapt the proof and take it here.

This implies that $q = 0$. For each $k = 1, \dots, n$, we can decompose P as $z_k p_k + q_k$, where $p_k \in \mathbb{C}[z_1, \dots, z_n]$, and q_k is a $(n-1)$-variable polynomial depending on z_j, $j \neq k$. If we fix all variables in P and p_k as 1, except the k-th variable , equal to z, we obtain polynomials $r_k(z)$ and $s_k(z)$ respectively. Easily, $r_k(z)$ is constant if and only if $s_k(z) = 0$. If for some k the corresponding $r_k(z)$ is non-constant, we proceed as in case 1, with $r_k(z)$ and α_k, to get that, given arbitrary $z \in \mathbb{C}$ and $U \subset \mathbb{C}$ open and $\varepsilon > 0$, there are $\tilde{z} \in \mathbb{C}$ and $z' \in U$ with $|r_k(\tilde{z}) - z| < \varepsilon$ and $f_{\alpha_k}(z') = \tilde{z}$. Therefore $|f(z') - z| = |r_k(\tilde{z}) - z| < \varepsilon$ and this shows that $f \in \mathcal{QES}$. If this is not the case, then $s_k(z) = 0$, $k = 1, \dots, n$. We will show that this yields a contradiction. Indeed, given $z \in \mathbb{C}$, we either have $f_{\alpha_k}(z) = 1$, $k = 1, \dots, n$, which implies $f(z) = q(f_{\alpha_0}(z)) = 0$, or there is some j so that $z' := f_{\alpha_j}(z) \neq 1$. Thus $f_{\alpha_k}(z) = 1$ for $k \neq j$ and

$$\begin{aligned} f(z) &= r_j(z') = z' S_j(z') + q_j(1, \dots, 1) = q_j(1, \dots, 1) \\ &= s_j(1) + q_j(1, \dots, 1) = r_j(1) = q(1) = 0. \end{aligned}$$

That is, $f = 0$, which is a contradiction. This completes the proof of case two.

To show that A is uncountably generated, we just have to show that $f_{\alpha_0} f\alpha \neq P(f_{\alpha_0} f\alpha_1, \dots, f_{\alpha_0} f\alpha_n)$ for any $n \in \mathbb{N}$, where $P \in \mathbb{C}[z_1, \dots, z_n]$, if $\alpha \neq \alpha_k$, $k = 1, \dots, n$.

If this is not the case then there exists $z \in \mathbb{C}$ such that $f_\alpha(z) \notin \{1, q(1)\}$. This shows that $Re(z) = \Phi_j(\alpha)$ for some $j \in \mathbb{N}$. Thus $Re(z) \neq \Phi_m(\alpha_i)$, $i = 0, 1, \dots, n$ and $m \in \mathbb{N}$, which gives that $f_{\alpha_i}(z) = 1$, $i = 0, 1, \dots, n$. That is

$$f_\alpha(z) = f_{\alpha_0}(z) f\alpha(z) \neq P(1, \dots, 1) = P(f_{\alpha_0} f\alpha_1, \dots, f_{\alpha_0} f\alpha_n)(z)$$

□

Note 2.6. *One can continue this trend of research by finding the maximum cardinal number μ that $\mathcal{QES}\backslash\mathcal{ES}$ is μ-algebrable. A new concept of strong and densely algebrability was recently expanded in* [10]*. The other question that can be asked here is that is $\mathcal{QES}\backslash\mathcal{ES}$ strongly of densely algebrable?*

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