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MAPS PRESERVING THE CLOSEDNESS OF OPERATOR RANGES ON A HILBERT H^*-MODULE

(COMMUNICATED BY M.MOSLEHIAN)

ESMAEIL ANSARI-PIRI, REZA SANATI

Abstract. Let E be a Hilbert H^*-module over an H^*-algebra A, and $\mathcal{B}_A(E)$ be the C^*-algebra of all bounded A-linear operators on E. In this paper, we show that the set of all semi A-Fredholm operators on E equals to the set of all regular operators on E and we apply this fact to show that if the linear map φ on $\mathcal{B}_A(E)$ is surjective up to compact operators and preserves closedness of operator ranges in both directions, then φ preserves semi A-Fredholm operators in both directions.

1. Introduction

An H^*-algebra is a complex associative Banach algebra A with involution, whose underlying space is a Hilbert space $(A, \langle , \rangle)$ which has an involution $a \to a^*$ such that $\langle ab, c\rangle = \langle b, a^*c\rangle = \langle a, cb^*\rangle$ for all $a, b, c \in A$. The trace-class in H^*-algebra A is defined as the set $\tau(A) = \{ab : \ a, b \in A\}$. There is a continuous linear form tr on $\tau(A)$ which is defined by $tr(ab) = \langle a, b^*\rangle$ for all $a, b \in A$.

An H^*-algebra whose annihilator is zero is called proper H^*-algebra. A Hilbert H^*-module is a left module E over a proper H^*-algebra A provided with a mapping $[.,.] : \ E \times E \to \tau(A)$ which satisfies the following conditions:

$i.$ $[\alpha x, y] = \alpha[x, y]$ $\quad \forall \alpha \in \mathbb{C}, \forall x, y \in E,$

$ii.$ $[x + y, z] = [x, z] + [y, z]$ $\quad \forall x, y, z \in E,$

$iii.$ $[ax, y] = a[x, y]$ $\quad \forall a \in A, \forall x, y \in E,$

$iv.$ $[x, y] = [y, x]^*$ $\quad \forall x, y \in E,$

$v.$ for every nonzero $x \in E$ there is a nonzero $a \in A$ such that $[x, x] = aa^*$,

$vi.$ E is a Hilbert space with the inner product $(x, y) = tr([x, y])$.

For more details about H^*-algebras and Hilbert H^*-modules, see [4, 5, 8]. Let E be a Hilbert H^*-module over an H^*-algebra A. An operator $T : E \to E$ is called A-linear, if it is linear and for each $a \in A$ and $x \in E$, $T(ax) = aT(x)$. The set of all bounded A-linear operators on E is denoted by $\mathcal{B}_A(E)$. It is well known ([8]) that each $T \in \mathcal{B}_A(E)$ has an adjoint $T^* \in \mathcal{B}_A(E)$ in the sense that $[Tx, y] = [x, T^*y]$ for all $x, y \in E$. Note that $\mathcal{B}_A(E)$ is a C^*-algebra contained in the algebra of all

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bounded linear operators on E which is denoted by $B(E)$. For each $x, y \in E$, we define the elementary operators $\theta_{x,y}$ by $\theta_{x,y}(z) = [z, y]x$ $(z \in E)$. Here, by a finite rank operator we mean an operator T which is a linear combination of the elementary operators and by $\mathcal{F}_A(E)$ we mean the set of all finite rank operators. In fact, the image of such a finite rank operator on E is a finitely generated submodule of E. The closed linear span of the set $\{\theta_{x,y} : x, y \in E\}$ is denoted by $\mathcal{K}_A(E)$ and its elements are called compact operators.
Unlike Hilbert C^*-modules, it is well known that every Hilbert H^*-module has an orthonormal basis and all orthonormal bases in a Hilbert H^*-module have the same cardinal number [5; Corollary 1.10, Proposition 1.11]. The same cardinal number of all orthonormal bases for a Hilbert H^*-module over an H^*-algebra A is denoted by $dim_A(E)$. Let e be a minimal projection in A in the sense that $eAe = \mathbb{C}e$. Then the set $E_e = \{ex : x \in E\}$ is a closed subspace of the Hilbert space $(E, (.,.))$ and also E_e generates a dense submodule in E [4; Lemma 2.7]. Moreover, if A is a simple H^*-algebra (that is, an H^*-algebra without nonterivial closed two-sided ideal), we have the following theorem:

Theorem 1.1. *Let E be a Hilbert H^*-module over a simple H^*-algebra A and e be a minimal projection in A. Then*
i. $dim_A(E) = dim(E_e)$;
ii. the map $\psi : \mathcal{B}_A(E) \to B(E_e)$, $\psi(T) = T|_{E_e}$ is an isomorphism between C^*-algebras.

Proof. See [4; Corollary 2.9, Theorem 2.10]. □

Let $\mathcal{C}_A(E)$ be the quotient algebra $\mathcal{B}_A(E)/\mathcal{K}_A(E)$ and $\pi : \mathcal{B}_A(E) \to \mathcal{C}_A(E)$ be the canonical quotient map. An operator $T \in \mathcal{B}_A(E)$ is called A-Fredholm (resp. semi A-Fredholm) if and only if $\pi(T)$ is invertible (resp. right or left invertible). The set of all A-Fredholm operators (resp. semi A-Fredholm operators) is denoted by $\mathcal{FR}_A(E)$ (resp. $\mathcal{SF}_A(E)$).

In [4], D. Bakić and B. Guljaš apply Theorem 1.1 and show that $T \in \mathcal{B}_A(E)$ is A-Fredholm (resp. semi A-Fredholm) if and only if the range of T is closed and both $dim_A Ker(T)$ and $dim_A Ker(T^*)$ are finite (resp. $dim_A Ker(T)$ or $dim_A Ker(T^*)$ is finite). However, in the case that W is a Hilbert C^*-module over a C^*-algebra B, even it is not necessary that the range of every B-Fredholm operator is closed [1; Example 2.2]. Although the Hilbert C^*-modules are the extensions of Hilbert spaces, there are many differences between these two categories. Let $B(H)$ be the set of all bounded linear operators on a Hilbert space H. If $G(H)$ denotes the set of all operators on H with closed range, then Lemma 3.1. of [6] shows that $R(H) = SF(H)$ where,

$$R(H) = \{T \in B(H) |\ \forall A \notin G(H),\ \exists \lambda \in \mathbb{C} \text{ such that } A + \lambda T \in G(H) \setminus \{0\}\}.$$

But the similar conclusion is not necessarily valid, if we replace the Hilbert space H by the Hilbert C^*-module W over a C^*-algebra B [1; Example 2.2]. T. Aghasizadeh and H. Hejazian show that if $W = \mathcal{H}_B$ (standard Hilbert module over the C^*-algebra B), then the self-adjoint part of $R(W)$ is contained in $SF(W)$ [1; Proposition 2.5].

An important consequence of this note is that if W is a Hilbert C^*-module over the C^*- algebra of all compact operators on a Hilbert space, then $SF(W) = R(W)$.

Now let E be a Hilbert H^*-module over an H^*-algebra A. An operator $T \in \mathcal{B}_A(E)$ is called generalized invertible, if there is an operator $R \in \mathcal{B}_A(E)$ such that

$TRT = T$. We denote the set of all generalized invertible operators on E by $\mathcal{G}_A(E)$. The set of regular operators on E is denoted by $\mathcal{R}(E)$ and define as follows:

$$\mathcal{R}(E) = \{T \in \mathcal{B}_A(E) |\ \forall A \notin \mathcal{G}_A(E),\ \exists \lambda \in \mathbb{C} \textit{ such that } A + \lambda T \in \mathcal{G}_A(E) \setminus \{0\}\}.$$

In this paper, first we show that $\mathcal{R}(E) = \mathcal{SF}_A(E)$ and we apply this equality to study a certain class of maps on $\mathcal{B}_A(E)$. For example, we show that if $\varphi : \mathcal{B}_A(E) \to \mathcal{B}_A(E)$ is a surjective map up to finite rank operators and φ preserves the closedness of operator ranges in both directions, then φ preserves semi A-Fredholm operators in both directions. Recall that we say φ preserves the property p in both directions, whenever $\varphi(T)$ has the property p if and only if T has this property.

2. Semi A-Fredholm operators on a Hilbert H^*-module

Throughout this section A denotes a simple H^*-algebra, e is a minimal projection in A, ψ denotes the isomorphism given in Theorem 1.1, and $G(E_e)$ is the set of all generalized invertible operators on the Hilbert space E_e.

Lemma 2.1. *Let $T \in \mathcal{B}_A(E)$. The range of $\psi(T)$ is closed if and only if the range of T is closed.*

Proof. Since $Im(\psi(T)) = eIm(T)$, it is clear that the closedness of $Im(T)$ implies the closedness of $Im(\psi(T))$. For the converse, we show that $T|_{Ker(T)^\perp}$ is bounded below. If this is not the case, there is a sequence (x_n) in $Ker(T)^\perp$ such that for each n, $\|x_n\| = 1$ and $lim_n T(x_n) = 0$. Therefore, $lim_n eT(x_n) = lim_n T(ex_n) = 0$. But (ex_n) is a sequence in the $Ker(\psi(T))^\perp$ which is a contradiction with the closedness of $Im(\psi(T))$. □

As an immediate consequence of this lemma, we have the following corollary:

Corollary 2.2. *Let E be a Hilbert H^*-module over A. Then $T \in \mathcal{G}_A(E)$ if and only if the range of T is closed.*

Proof. Let $T \in \mathcal{G}_A(E)$. Then there is an operator $R \in \mathcal{B}_A(E)$ such that $TRT = T$. Since ψ is a homomorphism, $\psi(T)\psi(R)\psi(T) = \psi(T)$. This shows that $\psi(T) \in G(E_e)$. Hence, $Im(\psi(T))$ is closed and so $Im(T)$ is closed. For the converse, let $Im(T)$ be closed. Since $Im(\psi(T)) = eIm(T)$, $Im(\psi(T))$ is closed and so there is a bounded linear operator R on the Hilbert space E_e such that $\psi(T)R\psi(T) = \psi(T)$. Now there is an operator $U \in \mathcal{B}_A(E)$ such that $\psi(U) = R$, hence $\psi(TUT) = \psi(T)$ which implies that $TUT = T$. Thus $T \in \mathcal{G}_A(E)$. □

Let H be a Hilbert space, $T \in G(H)$ and F be a finite rank opertaor on H. Then it is well known that $T + F \in G(H)$.

Remark 2.3. *Let $e \in A$ be a minimal projection, $T \in \mathcal{G}_A(E)$ and $L \in \mathcal{F}_A(E)$. Then $\psi(L) \in F(E_e)$ and Corollary 2.2 implies that $\psi(T) \in G(E_e)$. Hence, $\psi(T + L) = \psi(T) + \psi(L) \in G(E_e)$. By Lemma 2.1, $T + L$ is closed and so $T + L \in \mathcal{G}_A(E)$.*

Theorem 2.4. *Let E be a Hilbert H^*-module over A, then $\mathcal{R}(E) = \mathcal{SF}_A(E)$.*

Proof. Let $T \in \mathcal{R}(E)$ and $R \notin G(E_e)$. There is $U \in \mathcal{B}_A(E)$ such that $\psi(U) = R$. Since $R \notin G(E_e)$ and $Im(R) = eIm(U)$, $Im(U)$ can not be closed and so $U \notin \mathcal{G}_A(E)$. Hence, there is $\lambda \in \mathbb{C}$ such that $U + \lambda T \in \mathcal{G}_A(E) \setminus \{0\}$. This shows that $\psi(U + \lambda T) = \psi(U) + \lambda\psi(T) \in G(E_e) \setminus \{0\}$. Hence, $\psi(T) \in R(E_e)$. Since E_e is a Hilbert space, we have $R(E_e) = SF(E_e)$. So $Im(\psi(T))$ is closed and one of $dim(Ker(\psi(T)))$

or $dim(Ker(\psi(T)^*))$ is finite. Lemma 2.1 implies that $Im(T)$ is closed. On the other hand, $Ker(\psi(T)) = eKer(T)$, $Ker(\psi(T)^*) = eKer(T^*)$. Hence, part (i) of Theorem 1.1 implies that one of the $dim_A(Ker(T)$ or $dim_A(Ker(T^*)$ is finite. Therefore $T \in \mathcal{SF}_A(E)$.

Now let $T \in \mathcal{SF}_A(E)$. A similar argument shows that $\psi(T) \in SF(E_e) = R(E_e)$. Let $R \notin \mathcal{G}_A(E)$, then $\psi(R) \notin G(E_e)$. So there is $\lambda \in \mathbb{C}$ such that

$$\psi(R + \lambda T) = \psi(R) + \lambda\psi(T) \in G(E_e) \setminus \{0\}.$$

Therefore, $R + \lambda T \in \mathcal{G}_A(E)$ and so $T \in \mathcal{R}(E)$. □

Remark 2.5. (i). *We know that $\psi(T)$ is compact if and only if T is compact [See 4]. Also the Calkin algebra $C(E_e) = B(E_e)/K(E_e)$ is a prime C^*-algebra of real rank zero. So the isomorphism map ψ in Theorem 1.1 implies that the algebra $\mathcal{C}_A(E) = \mathcal{B}_A(E)/\mathcal{K}_A(E)$ is a prime C^*-algebra of real rank zero.*
(ii). *An operator $T \in \mathcal{B}_A(E)$ is semi A-Fredholm if and only if $\psi(T)$ is a semi Fredholm operator on the Hilbert space E_e. Since $Ker(\psi(T)) = eKer(T)$ and $T \in \mathcal{G}_A(E)$ if and only if $\psi(T) \in G(E_e)$.*

Let H be an infinite dimensional Hilbert space. Then Lemma 2.3 of [7] shows that a bounded linear operator T on H is compact if and only if for every semi Fredholm operator S on H, $T + S$ is again semi Fredholm. Now we have:

Proposition 2.6. *An operator $T \in \mathcal{B}_A(E)$ is compact if and only if for every semi A-Fredholm operator S, $T + S$ is semi A-Fredholm.*

Proof. Let $\pi : \mathcal{B}_A(E) \to \mathcal{B}_A(E)/\mathcal{K}_A(E)$ be the canonical quotient map, $T \in \mathcal{K}_A(E)$ and $S \in \mathcal{SF}_A(E)$. Then $\pi(T) = 0$ and $\pi(S)$ has a right or left inverse. So $\pi(T+S) = \pi(S)$ has a right or left inverse, that is $T+S$ is semi A-Fredholm operator. For the converse, let $T \in \mathcal{B}_A(E)$ such that for each $S \in \mathcal{SF}_A(E)$, $T+S \in \mathcal{SF}_A(E)$. Let $R \in SF(E_e)$, Since ψ is onto, there is an operator $S \in \mathcal{B}_A(E)$ such that $\psi(S) = R$. But $Ker(R) = eKer(S)$ and $Ker(R^*) = eKer(S^*)$, so S must be semi A-Fredholm. By assumption, $T + S \in \mathcal{SF}_A(E)$. Hence, $\psi(T+S) = \psi(T)+\psi(S) = \psi(T) + R \in SF(E_e)$. Therefore, $\psi(T) \in K(E_e)$ and so $T \in \mathcal{K}_A(E)$. □

Definition 2.7. *Let E be a Hilbert H^*-module over A. The map $\varphi : \mathcal{B}_A(E) \to \mathcal{B}_A(E)$ is said to be surjective up to compact operators (resp. finite rank operators), if for each $T \in \mathcal{B}_A(E)$ there is $A \in \mathcal{B}_A(E)$ and $M \in \mathcal{K}_A(E)$ (resp. $L \in \mathcal{F}_A(E)$) such that $T = \varphi(A) + M$ (resp. $T = \varphi(A) + L$).*

Theorem 2.8. *Let E be a Hilbert H^*-module over A and $\varphi : \mathcal{B}_A(E) \to \mathcal{B}_A(E)$ be a linear map.*
(i) *If φ is surjective up to finite rank operators and preserves the closedness of operator ranges in both directions, then φ preserves semi A-Fredholm operators in both directions.*
(ii) *If φ is surjective up to compact operators and preserves semi A-Fredholm operators in both directions, then φ preserves compact operators in both directions.*

Proof. For (i), by Theorem 2.4, it is enough to show that $T \in \mathcal{R}(E)$ if and only if $\varphi(T) \in \mathcal{R}(E)$. Let $T \in \mathcal{SF}_A(E) = \mathcal{R}(E)$ and $B \notin \mathcal{G}_A(E)$. Since φ is surjective up to finite rank operators, there is $U \in \mathcal{B}_A(E)$ and $L \in \mathcal{F}_A(E)$ such that $B = \varphi(U) + L$. This shows that the range of U is not closed. (otherwise, $\varphi(U)$ also has closed range and since $L \in \mathcal{F}_A(E)$, Remark 2.3 implies that B has closed range which is a contradiction). Since $T \in \mathcal{R}(E)$, there is $\lambda \in \mathbb{C}$ such that $0 \neq U + \lambda T \in \mathcal{G}_A(E)$ and so $\varphi(U) + \lambda\varphi(T) = \varphi(U + \lambda T) \in \mathcal{G}_A(E)$. Apply again Remark 2.3, we have

$$B + \lambda\varphi(T) = \varphi(U) + L + \lambda\varphi(T) \in \mathcal{G}_A(E).$$

Since $T \in \mathcal{SF}_A(E) = \mathcal{R}(E)$, the range of T is closed and so is the range of $\varphi(T)$. This shows that $B + \lambda\varphi(T) \neq 0$. For if $B = -\lambda\varphi(T)$, then B has the closed range and this is a contradiction. Thus $\varphi(T) \in \mathcal{R}(E)$.
Now let $\varphi(T) \in \mathcal{R}(E) = \mathcal{SF}_A(E)$ and $R \notin \mathcal{G}_A(E)$, so $\varphi(R) \notin \mathcal{G}_A(E)$. Hence, there is $\lambda \in \mathcal{C}$ such that $\varphi(R + \lambda T) = \varphi(R) + \lambda\varphi(T) \in \mathcal{G}_A(E) \setminus \{0\}$. This shows that $R + \lambda T \in \mathcal{G}_A(E) \setminus \{0\}$. Thus $T \in \mathcal{R}(E)$.
For (ii), let $T \in \mathcal{K}_A(E)$ and $S \in \mathcal{SF}_A(E)$. Since φ is surjective up to compact operators, there is $U \in \mathcal{B}_A(E)$ and $M \in \mathcal{K}_A(E)$ such that $\varphi(U) = S + M$. Proposition 2.6 implies that $\varphi(U) \in \mathcal{SF}_A(E)$ and so $U \in \mathcal{SF}_A(E)$. Therefore, $T + U \in \mathcal{SF}_A(E)$ and so $\varphi(T) + \varphi(U) = \varphi(T + U) \in \mathcal{SF}_A(E)$. Since M is compact, $\varphi(T) + S = \varphi(T) + \varphi(U) - M \in \mathcal{SF}_A(E)$. So $\varphi(T) \in \mathcal{K}_A(E)$. On the other hand, if $\varphi(T) \in \mathcal{K}_A(E)$ and S is an arbitrary semi A-Fredholm operator, then $\varphi(S)$ is semi A-Fredholm and so $\varphi(T + S) = \varphi(T) + \varphi(S) \in \mathcal{SF}_A(E)$. Therefore, $T + S \in \mathcal{SF}_A(E)$. This shows that $T \in \mathcal{K}_A(E)$. □

Remark 2.9. *Let E be a Hilbert H^*-module over A and $\varphi : \mathcal{B}_A(E) \to \mathcal{B}_A(E)$ be a linear map preserving the closedness of operator ranges(resp. semi A-Fredholm operators) on E in both directions. It is straightforward to check that the map $\psi\varphi\psi^{-1} : B(E_e) \to B(E_e)$ preserves the closedness of the range of operators (resp. semi Fredholm operators) on the Hilbert space E_e. Moreover, if φ is surjective up to compact operators, then so is $\psi\varphi\psi^{-1}$. Also if φ is surjective up to finite rank operators, then $\psi\varphi\psi^{-1}$ is surjective up to finite rank operators.*

Corollary 2.10.
(i). *If φ is surjective up to finite rank operators and preserves the closedness of operator ranges in both directions, then $\psi\varphi\psi^{-1}$ preserves both finite rank operators and semi Fredholm operators in both directions on the Hilbert space E_e.*
(ii). *If φ is surjective up to compact operators and preserves semi A-Fredholm operators in both directions, $\psi\varphi\psi^{-1}$ preserves compact operators in both directions on the Hilbert space E_e.*

Proof. Apply Remark 2.9, (ii) and (ν) of Theorem 2.2 of [2]. □

Remark 2.11. *Let H be an infinite dimensional Hilbert space and W be a Hilbert C^*-module over the C^*-algebra $K(H)$ of all compact operators on H. Consider the space $\mathcal{HS} \subsetneqq K(H)$ of all Hilbert-Schmidt operators on H. Let $\mathcal{W}^0_{\mathcal{HS}}$ denote the linear span of the set $\mathcal{HS}W$. The submodule $\mathcal{W}^0_{\mathcal{HS}}$ of W can be made to a pre Hilbert H^*-module over the H^*-algebra $\mathcal{HS}$ with the inner product $(x, y) = tr(\langle x, y\rangle)$. Let us denotes by $\|.\|_{\mathcal{HS}}$ the resulting norm:*

$$\|x\|_{\mathcal{HS}} = \sqrt{tr(\langle x, x\rangle)} \quad \forall x \in \mathcal{W}^0_{\mathcal{HS}}.$$

The completion of $\mathcal{W}^0_{\mathcal{HS}}$ in the norm $\|.\|_{\mathcal{HS}}$, is a Hilbert H^*-module over the H^*-algebra $\mathcal{HS}$ which is a dense submodule in W with respect to the original norm [See 3]. D. Bakić and B. Guljaš apply this fact and obtain a similar result to the Theorem 1.1 in the case that W is a Hilbert C^*-module over the C^*-algebra $K(H)$, instead of an Hilbert H^*-module [3; Theorem 5]. Therefore the results of this paper *valid in the case that W is a Hilbert C^*-module over the C^*-algebra $K(H)$. In particular, $SF(W) = R(W)$.*

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Esmaeil Ansari-Pir

Department of Mathematics, Faculty of Science, University of Guilan, P.O.Box 1914, RASHT, IRAN.

E-mail address: e_ansari@guilan.ac.ir

Reza Sanati

Department of Mathematics, Faculty of Science, University of Guilan, P.O.Box 1914, RASHT, IRAN.

E-mail address: re_gaba64@guilan.ac.ir