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GENERALIZED q-BESSEL OPERATOR

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Abstract. In this paper we attempt to build a coherent q-harmonic analysis attached to a new type of q-difference operator which can be considered as a generalized of the q-Bessel operator.

1. Introduction

This paper deals with the increasing relevance of q -Bessel Fourier analysis [3, 10, 16]. We introduce a generalized q-Bessel operator of index (α, β), which is a generalization of the well-known q-Bessel operator [10, 3, 4].

This operator satisfy some various identities and admits generalized q-Bessel functions as eigenfunction, in the same way for the q-Bessel functions. We establish the orthogonality relation and Sonine representation.

Second , we study a generalized q-Bessel transform and we use the work in [4] to establish inversion formula , Plancherel formula, generalized q-Bessel translation operator and generalized q-convolution product. Often we use the crucial properties namely the positivity of the q-Bessel translation operator in [9] to prove the positivity of the generalized q-Bessel translation operator.

As application, we give the Heisenberg uncertainty inequality for functions in $\mathcal{L}_{q,2,\nu}$ space and the Hardy's inequality which give an information about how a function and its generalized q-Bessel Fourier transform are linked.

Finally, we study a generalized version of the q-Modified Bessel functions and we establish some of its properties.

2. The generalized q-Bessel operator

For $\alpha, \beta \in \mathbb{R}$, we put

$$\nu = (\alpha, \beta), \quad \overline{\nu} = (\beta, \alpha),$$

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and

$$\nu+1=(\alpha+1,\beta), \quad |\nu|=\alpha+\beta.$$

Throughout this paper, we will assume that $0<q<1$ and $\alpha+\beta>-1$. We refer to [13] for the definitions, notations, properties of the q-shifted factorials, the Jackson's q-derivative and the Jackson's q-integrals.

The q-shifted factorial are defined by

$$(a;q)_0=1, \quad (a;q)_n=\prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k),$$

and

$$\mathbb{R}_q^+=\{q^n \ : \ n\in\mathbb{Z}\}.$$

The q-derivative of a function f is given by

$$D_qf(x)=\frac{f(x)-f(qx)}{(1-q)x} \quad \text{if} \quad x\neq 0.$$

and $D_qf(0)=f'(0)$ provided $f'(0)$ exists. Note that when f is differentiable, at x, then $D_qf(x)$ tends to $f'(x)$ as q tends to 1^-.
The q-Jackson integrals from 0 to a and from 0 to ∞ are defined by [15]

$$\int_0^a f(x)d_qx=(1-q)a\sum_{n=0}^{\infty}f(aq^n)q^n,$$

$$\int_0^\infty f(x)d_qx=(1-q)\sum_{n=-\infty}^{\infty}f(q^n)q^n,$$

provided the sums converge absolutely. Note that

$$\int_a^b D_qf(x)d_qx=f(b)-f(a), \quad \forall a,b\in\mathbb{R}_q^+.$$

The space $\mathcal{L}_{q,p,\nu}$, $1\leq p<\infty$ denotes the set of functions on $\mathbb{R}_q^+$ such that

$$\|f\|_{q,p,\nu}=\left[\int_0^\infty |f(x)|^p x^{2|\nu|+1}d_qx\right]^{1/p}<\infty.$$

Similarly $\mathcal{C}_{q,0}$ is the space of functions defined on $\mathbb{R}_q^+$, continuous in 0 and vanishing at infinity, equipped with the induced topology of uniform convergence such that

$$\|f\|_{q,\infty}=\sup_{x\in\mathbb{R}_q^+}|f(x)|<\infty,$$

and $\mathcal{C}_{q,b}$ the space of continuous functions at 0 and bounded on $\mathbb{R}_q^+$.

The normalized q-Bessel function is given by

$$\begin{aligned} j_\alpha(x,q^2) &= \sum_{n=0}^{\infty}(-1)^n\frac{q^{n(n+1)}}{(q^{2\alpha+2},q^2)_n(q^2,q^2)_n}x^{2n} \\ &= {}_1\phi_1\left(0,q^{2\alpha+2},q^2;q^2x^2\right). \end{aligned}$$

The q-Bessel operator is defined as follows

$$\Delta_{q,\alpha}f(x)=\frac{f(q^{-1}x)-(1+q^{2\alpha})f(x)+q^{2\alpha}f(qx)}{x^2}, \quad x\neq 0.$$

One can see that $x \mapsto j_\alpha(\lambda x, q^2),\ \lambda \in \mathbb{C}$ is eigenfunction for the operator $\Delta_{q,\alpha}$ with $-\lambda^2$ as eigenvalue .

Let now introduce the following generalized q-Bessel operator:

$$\widetilde{\Delta}_{q,\nu} f(x) = \widetilde{\Delta}_{q,(\alpha,\beta)} f(x) = \frac{f(q^{-1}x) - (q^{2\alpha} + q^{2\beta})f(x) + q^{2\alpha+2\beta} f(qx)}{x^2},$$

which can be factorized as follows

$$\widetilde{\Delta}_{q,\nu} f(x) = \partial^*_{q,\alpha} \partial_{q,\beta},$$

where we have put

$$\partial_{q,\beta} f(x) = \frac{f(q^{-1}x) - q^{2\beta} f(x)}{x}$$

$$\partial^*_{q,\alpha} f(x) = \frac{f(x) - q^{2\alpha+1} f(qx)}{x}.$$

When $\nu = (\alpha, 0)$, the last operator is reduced to the q-Bessel operator $\Delta_{q,\alpha}$ (see[3, 4, 9]).

Remark 1. *We have*

$$\begin{aligned}
\widetilde{\Delta}_{q,\nu} f(x) &= \frac{f(q^{-1}x) - (q^{2\alpha} + q^{2\beta})f(x) + q^{2\alpha+2\beta} f(qx)}{x^2} \\
&= \frac{f(q^{-1}x) - (1 + q^{2|\nu|})f(x) + q^{2|\nu|} f(qx)}{x^2} + V(x)f(x) \\
&= \Delta_{q,|\nu|} f(x) + V(x)f(x),
\end{aligned}$$

where

$$V(x) = \frac{(1 + q^{2|\nu|}) - (q^{2\alpha} + q^{2\beta})}{x^2}.$$

In the rest of this paper, we denote by

$$\begin{aligned}
\widetilde{j}_{q,\nu}(x, q^2) &= \widetilde{j}_{q,(\alpha,\beta)}(x, q^2) \\
&= x^{-2\beta} j_{\alpha-\beta}(q^{-\beta}x, q^2), \quad \nu = (\alpha, \beta).
\end{aligned}$$

Proposition 1. *The functions $\widetilde{j}_{q,\nu}(., q^2)$ and $\widetilde{j}_{q,\overline{\nu}}(., q^2)$ span the space of solutions of the following q-differential equation*

$$\widetilde{\Delta}_{q,\nu} f(x) = -f(x).$$

Proof. We have

$$\begin{aligned}
\widetilde{\Delta}_{q,\nu} \widetilde{j}_{q,\nu}(x, q^2) &= x^{-2\beta} \frac{q^{2\beta} j_{\alpha-\beta}(q^{-\beta-1}x, q^2) - (q^{2\alpha} + q^{2\beta}) j_{\alpha-\beta}(q^{-\beta}x, q^2) + q^{2\alpha+2\beta} q^{-2\beta} j_{\alpha-\beta}(q^{-\beta+1}x, q^2)}{x^2} \\
&= q^{2\beta} x^{-2\beta} \frac{j_{\alpha-\beta}(q^{-\beta-1}x, q^2) - (1 + q^{2(\alpha-\beta)}) j_{\alpha-\beta}(q^{-\beta}x, q^2) + q^{2(\alpha-\beta)} j_{\alpha-\beta}(q^{-\beta+1}x, q^2)}{x^2} \\
&= -q^{2\beta} x^{-2\beta} q^{-2\beta} j_{\alpha-\beta}(q^{\beta}x, q^2) \\
&= -j_{q,(\alpha,\beta)}(x, q^2),
\end{aligned}$$

and the result follows. □

Proposition 2. *We have*

$$\partial_{q,\beta}\widetilde{j}_{q,\nu}(x,q^2) = -\frac{q^{\beta+1}}{1-q^{2(\alpha-\beta)+2}}\ x\ \widetilde{j}_{q,\nu+1}(x,q^2), \qquad (1)$$

and

$$\partial^*_{q,\alpha}\left[\frac{q^{\beta+1}}{1-q^{2(\alpha-\beta)+2}}\ x\ \widetilde{j}_{q,\nu+1}(x,q^2)\right] = \widetilde{j}_{q,\nu}(x,q^2). \qquad (2)$$

Proof. We have

$$\begin{aligned}\partial_{q,\beta}\widetilde{j}_{q,\nu}(x,q^2) &= q^{2\beta}x^{-2\beta}\frac{j_{\alpha-\beta}(q^{-\beta-1}x,q^2)-j_{\alpha-\beta}(q^{-\beta}x,q^2)}{x}\\ &= q^{2\beta}x^{-2\beta}(1-q)D_q[j_{\alpha-\beta}(q^{-\beta-1}x)].\end{aligned}$$

From the following formula (see [5])

$$D_q[j_\alpha(t,q^2)] = -\frac{q^2}{(1-q)(1-q^{2\nu+2})}tj_{\alpha+1}(qt,q^2),$$

we obtain

$$\begin{aligned}\partial_{q,\beta}\widetilde{j}_{q,\nu}(x,q^2) &= -\frac{q^{\beta+1}}{1-q^{2(\alpha-\beta)+2}}x^{-2\beta+1}j_{\alpha+1-\beta}(q^{-\beta}x,q^2)\\ &= -\frac{q^{\beta+1}}{1-q^{2(\alpha-\beta)+2}}x\widetilde{j}_{q,\nu+1}(x,q^2).\end{aligned}$$

The relation

$$\widetilde{\Delta}_{q,\nu} = \partial^*_{q,\alpha}\partial_{q,\beta},$$

leads to the second result (2). □

Proposition 3. *Let f and g be two linearly independent solutions of the following q-differential equation*

$$\widetilde{\Delta}_{q,\nu}y(x) = \pm\ \lambda^2 y(x).$$

Then there exists a constant $c(f,g)\neq 0$, such that

$$x^{2|\nu|}\Big[f(x)g(qx)-f(qx)g(x)\Big] = c(f,g), \quad \forall x\in\mathbb{R}_q^+.$$

Proof. The q-wronskian of two functions f and g is defined by

$$w_y(f,g) = (1-q)\Big[\partial_{q,\beta}f(y)g(y)-f(y)\partial_{q,\beta}g(y)\Big].$$

The fact that

$$D_q\left[y\mapsto y^{2|\nu|+1}w_y(f,g)\right](x) = \Big[\widetilde{\Delta}_{q,\nu}f(x)g(x)-f(x)\widetilde{\Delta}_{q,\nu}g(x)\Big]x^{2|\nu|+1},$$

leads to

$$\int_a^b\Big[\widetilde{\Delta}_{q,\nu}f(x)g(x)-f(x)\widetilde{\Delta}_{q,\nu}g(x)\Big]x^{2|\nu|+1}d_qx = b^{2|\nu|+1}w_b(f,g)-a^{2|\nu|+1}w_a(f,g),$$

which prove the result. □

Proposition 4. *Let $f,g\in\mathcal{L}_{q,2,\nu}$ such that $\widetilde{\Delta}_{q,\nu}f\in\mathcal{L}_{q,2,\nu}$. Then*

$$\langle\widetilde{\Delta}_{q,\nu}f,g\rangle = \langle f,\widetilde{\Delta}_{q,\nu}g\rangle,$$

if and only if

$$w_x(f,g) = o(x^{-2|\nu|-1}) \quad \textit{as} \quad x\downarrow 0. \qquad (3)$$

Proof. In fact

$$\int_a^b \widetilde{\Delta}_{q,\nu} f(x)g(x)x^{2|\nu|+1}d_qx - \int_a^b f(x)\widetilde{\Delta}_{q,\nu} g(x)x^{2|\nu|+1}d_qx = b^{2|\nu|+1}w_b(f,g) - a^{2|\nu|+1}w_a(f,g).$$

Since $\widetilde{\Delta}_{q,\nu} f, g \in \mathcal{L}_{q,2,\nu}$ we obtain that

$$\lim_{a\downarrow 0}\lim_{b\to\infty}\int_a^b \widetilde{\Delta}_{q,\nu} f(x)g(x)x^{2|v|+1}d_qx \quad < \infty.$$

On the other hand $f(x) = o(x^{-|\nu|-1})$ and $g(x) = o(x^{-|\nu|-1})$ when $x \to \infty$, then we have

$$\lim_{b\to\infty} b^{2|\nu|+1}w_b(f,g) = 0.$$

This implies that

$$\lim_{a\downarrow 0} a^{2|\nu|+1}w_a(f,g) = 0 \Rightarrow \langle \widetilde{\Delta}_{q,\nu} f, g\rangle = \langle f, \widetilde{\Delta}_{q,\nu} g\rangle.$$

The converse is true. □

In the rest of this paper, we put

$$\nu = (\alpha, -n), \quad n \in \mathbb{N}.$$

Proposition 5. *The function $\widetilde{j}_{q,\nu}(x,q^2)$ has the following Sonine integral representation*

$$\widetilde{j}_{q,\nu}(x,q^2) = x^{2n}\int_0^1 W_\nu(t,q^2)j_\alpha(q^nxt,q^2)t^{2\alpha+1}d_qt,$$

where

$$W_\nu(t,q^2) = \frac{(q^{2n},q^2)_\infty(q^{2\alpha+2},q^2)_\infty}{(q^2,q^2)_\infty(q^{2(\alpha+n)+2},q^2)_\infty}\frac{(q^2t^2,q^2)_\infty}{(q^{2n}t^2,q^2)_\infty}. \tag{4}$$

Proof. Using the following identity (see [5])

$$c_{q,\alpha+n}j_{\alpha+n}(\lambda,q^2) = \frac{(q^{2n},q^2)_\infty}{(q^2,q^2)_\infty}\ c_{q,\alpha}\int_0^1 \frac{(q^2t^2,q^2)_\infty}{(q^{2n}t^2,q^2)_\infty}j_\alpha(\lambda t,q^2)t^{2\alpha+1}d_qt,$$

where

$$c_{q,\alpha} = \frac{1}{1-q}\frac{(q^{2\alpha+2},q^2)_\infty}{(q^2,q^2)_\infty}.$$

The definition of the function $\widetilde{j}_{q,\nu}(x,q^2)$ leads to the result. □

Proposition 6. *The generalized q-Bessel function $\widetilde{j}_{q,\nu}(.,q^2)$ satisfies the following estimate*

$$|\widetilde{j}_{q,\nu}(q^k,q^2)| \le q^{2kn}\frac{(-q^2;q^2)_\infty(-q^{2\alpha+2};q^2)_\infty(q^{2\alpha+2},q^2)_n}{(-q^{2\alpha+2};q^2)_n(q^{2\alpha+2},q^2)_\infty}$$
$$\times\begin{cases} q^{2nk} & if \quad n+k \ge 0 \\ q^{(n+k)^2-(n+k)(2\alpha+1)-2n^2} & if \quad n+k < 0 \end{cases}.$$

Proof. For all $n, k \in \mathbb{N}$, we have

$$\widetilde{j}_{q,\nu}(q^k,q^2) = q^{2kn}j_{\alpha+n}(q^{n+k},q^2).$$

Using the following identity (see [4, 3])

$$|j_{\alpha+n}(q^{n+k},q^2)| \leq \frac{(-q^2;q^2)_\infty(-q^{2(\alpha+n)+2};q^2)_\infty}{(q^{2(\alpha+n)+2},q^2)_\infty} \times \begin{cases} 1 & \text{if} \quad k \geq -n \\ q^{(n+k)^2-(2(\alpha+n)+1)(n+k)} & \text{if} \quad k < -n \end{cases},$$

we obtain the result. □

Proposition 7. *Let $x \in \mathbb{C}^*\backslash\mathbb{R}_q^+$, then the kernel $\widetilde{j}_{q,\nu}(.,q^2)$ has the following asymptotic expansion as $|x| \to \infty$*

$$\widetilde{j}_{q,\nu}(x,q^2) \sim x^{2n}\frac{(q^2x^2,q^2)_\infty(q^{2\alpha+2},q^2)_n}{(q^2x^2,q^2)_n(q^{2\alpha+2},q^2)_\infty}.$$

Proof. Let $x \in \mathbb{C}^*\backslash\mathbb{R}_q^+$, the function $j_\alpha(.,q^2)$ has the following asymptotic expansion as $|x| \to \infty$ (see [6])

$$j_\alpha(x,q^2) \sim \frac{(x^2q^2,q^2)_\infty}{(q^{2\alpha+2},q^2)_\infty}.$$

Then for all $x \in \mathbb{C}^*\backslash\mathbb{R}_q^+$, we have

$$\widetilde{j}_{q,\nu}(x,q^2) \sim x^{2n}\frac{(x^2q^{2+2n},q^2)_\infty}{(q^{2(\alpha+n)+2},q^2)_\infty} = x^{2n}\frac{(q^2x^2,q^2)_\infty(q^{2\alpha+2},q^2)_n}{(q^2x^2,q^2)_n(q^{2\alpha+2},q^2)_\infty},$$

which achieves the proof. □

Definition 1. *We define the following delta by*

$$\delta_{q,\nu}(x,y) = \begin{cases} 0 & if \quad x \neq y \\ \frac{1}{(1-q)x^{2(|\nu|+1)}} & if \quad x = y \end{cases}.$$

So that for any function f defined on $\mathbb{R}_q^+$, we have

$$\int_0^\infty f(y)\delta_{q,\nu}(x,y)y^{2|\nu|+1}d_qy = f(x).$$

Proposition 8. *The following orthogonality holds relation*

$$c_{q,\nu}^2\int_0^\infty \widetilde{j}_{q,\nu}(tx,q^2)\widetilde{j}_{q,\nu}(ty,q^2)t^{2|\nu|+1}d_qt = \delta_{q,\nu}(x,y),$$

where

$$c_{q,\nu} = \frac{q^{n(\alpha+n)}}{(1-q)}\frac{(q^{2\alpha+2},q^2)_\infty}{(q^2,q^2)_\infty(q^{2\alpha+2},q^2)_n}. \tag{5}$$

Proof. $\forall x,y \in \mathbb{R}_q^+$, we have

$$\begin{aligned}\int_0^\infty \widetilde{j}_{q,\nu}(tx,q^2)\widetilde{j}_{q,\nu}(ty,q^2)t^{2|\nu|+1}d_qt &= (xy)^{2n}\int_0^\infty j_{\alpha+n}(xq^nt,q^2)j_{\alpha+n}(yq^nt,q^2)t^{2(\alpha+n)+1}d_qt \\ &= (xy)^{2n}q^{-2n(\alpha+n)}\int_0^\infty j_{\alpha+n}(xu,q^2)j_{\alpha+n}(yu,q^2)u^{2(\alpha+n)+1}d_qu.\end{aligned}$$

Using the following formula (see [3])

$$c_{q,\alpha}^2\int_0^\infty j_\alpha(xu,q^2)j_\alpha(yu,q^2)u^{2\alpha+1}d_qu = \delta_{q,\alpha}(x,y).$$

Then the result follows. □

3. Generalized q-Bessel Fourier transform

Definition 2. *The generalized q-Bessel Fourier transform $\mathcal{F}_{q,\nu}$ is defined as follows*

$$\mathcal{F}_{q,\nu}f(x) = c_{q,\nu}\int_0^\infty f(t)\ \widetilde{j}_{q,\nu}(tx,q^2)\ t^{2|\nu|+1}d_qt, \tag{6}$$

where $c_{q,\nu}$ is given by (5).

Proposition 9. *The generalized q-Bessel Fourier transform*

$$\mathcal{F}_{q,\nu}:\mathcal{L}_{q,1;\nu}\to\mathcal{C}_{q,0},$$

satisfies

$$\|\mathcal{F}_{q,\nu}f\|_{q,\infty}\leq B_{q,\nu}\|f\|_{\nu,1,q},$$

where

$$B_{q,\nu}=\frac{q^{n(\alpha+n+2k)}}{(1-q)}\frac{(-q^2,q^2)_\infty(-q^{2\alpha+2},q^2)_\infty}{(q^2,q^2)_\infty(-q^{2\alpha+2},q^2)_n}.$$

Proof. Use Proposition 6. □

Theorem 1. (1) *Let f be a function in the $\mathcal{L}_{\nu,p,q}$ space where $p\geq 1$ then*

$$\mathcal{F}^2_{q,\nu}f=f. \tag{7}$$

(2) *If $f\in\mathcal{L}_{q,1,\nu}$ with $\mathcal{F}_{q,\nu}f\in\mathcal{L}_{q,1,\nu}$ then*

$$\|\mathcal{F}_{q,\nu}f\|_{q,2,\nu}=\|f\|_{q,2,\nu}.$$

(3) *Let f be a function in the $\mathcal{L}_{q,1,\nu}\cap\mathcal{L}_{q,p,\nu}$, where $p>2$ then*

$$\|\mathcal{F}_{q,\nu}f\|_{q,2,\nu}=\|f\|_{q,2,\nu}.$$

(4) *Let f be a function in the $\mathcal{L}_{q,2,\nu}$ then*

$$\|\mathcal{F}_{q,\nu}f\|_{q,2,\nu}=\|f\|_{q,2,\nu}.$$

(5) *Let $1\leq p\leq 2$. If $f\in\mathcal{L}_{q,p,\nu}$ then $f\in\mathcal{L}_{q,\overline{p},\nu}$.*

$$\|\mathcal{F}_{q,\nu}f\|_{q,\overline{p},\nu}\leq B_{q,\nu}^{\frac{2}{p}-1}\|f\|_{q,p,\nu}, \tag{8}$$

where the numbers p and $\overline{p}$ above are conjugate exponents

$$\frac{1}{p}=1-\frac{1}{\overline{p}}\ .$$

Proof. The following proof is identical to the proof of Theorems 1,2 and 3 in [4]. □

Proposition 10. *Let $f\in\mathcal{L}_{q,2,\nu}$ then*

$$\mathcal{F}_{q,\nu}\widetilde{\Delta}_{q,\nu}f(\xi)=-\xi^2\mathcal{F}_{q,\nu}f(\xi),\quad\forall\xi\in\mathbb{R}_q^+, \tag{9}$$

if and only if

$$w_x(f,\psi_\xi)=o(x^{-2|\nu|-1})\quad as\quad x\downarrow 0,\quad\forall\xi\in\mathbb{R}_q^+. \tag{10}$$

In particular this is true if we have

$$\partial_{q,\beta}f(x)=O(x^{-|\nu|})\quad as\quad x\downarrow 0.$$

Proof. Indeed we have (9) if and only if

$$\langle\widetilde{\Delta}_{q,\nu}f,\psi_\xi\rangle=\langle f,\widetilde{\Delta}_{q,\nu}\psi_\xi\rangle.$$

By Proposition (4) this is equivalent to (10). □

The q-Schwartz space $S_{q,\nu}$ denote the set of functions f defined on $\mathbb{R}_q^+$ such that

$$|\widetilde{\Delta}_{q,\nu}^k f(x)| \leq \frac{c_{n,k}}{1+x^{2n}}, \quad \forall n,k \in \mathbb{N}, \forall x \in \mathbb{R}_q^+.$$

For some constant $c_{n,k} > 0$ and

$$\partial_{q,\beta}\widetilde{\Delta}_{q,\nu}^k f(x) = O(x^{-|\nu|}), \quad \text{as } x \downarrow 0.$$

Corollary 1. *The generalized q-Bessel transform*

$$\mathcal{F}_{q,\nu} : S_{q,\nu} \to S_{q,\nu}$$

define an isomorphism.

3.1. **Generalized q-Bessel Translation Operator.** We introduce the generalized q-Bessel translation operator associated via the generalized q-Bessel transform as follows:

$$T_{q,x}^\nu f(y) = c_{q,\nu}\int_0^\infty \mathcal{F}_{q,\nu}f(t)\ \widetilde{j}_{q,\nu}(yt,q^2)\widetilde{j}_{q,\nu}(xt,q^2)t^{2|\nu|+1}d_qt,\ \forall x,y \in \mathbb{R}_q^+.$$

Proposition 11. *For any function $f \in \mathcal{L}_{q,1,\nu}$, we have*

$$T_{q,x}^\nu f(y) = T_{q,y}^\nu f(x).$$

and

$$T_{q,x}^\nu f(0) = f(x).$$

Theorem 2. *Let $f \in \mathcal{L}_{q,p,\nu}$ then $T_{q,x}^\nu f$ exists and we have*

$$T_{q,x}^\nu f(y) = \int_0^\infty f(z)\mathcal{D}_{q,\nu}(x,y,z)z^{2|\nu|+1}d_qz,$$

where

$$\begin{aligned}\mathcal{D}_{q,\nu}(x,y,z) &= {c_{q,\nu}}^2\int_0^\infty \widetilde{j}_{q,\nu}(xs,q^2)\ \widetilde{j}_{q,\nu}(ys,q^2)\ \widetilde{j}_{q,\nu}(zs,q^2)s^{2|\nu|+1}d_qs\\ &= (xyz)^{2n}\ {c_{q,\alpha+n}}^2\int_0^\infty j_{\alpha+n}(xs,q^2)\ j_{\alpha+n}(ys,q^2)\ j_{\alpha+n}(zs,q^2)s^{2(\alpha+2n)+1}d_qs.\end{aligned}$$

Proof. We write the operator $T_{q,x}^\nu$ in the following form

$$\begin{aligned}T_{q,x}^\nu f(y) &= c_{q,\nu}\int_0^\infty \mathcal{F}_{q,\nu}f(z)\ \widetilde{j}_{q,\nu}(yz,q^2)\ \widetilde{j}_{q,\nu}(xz,q^2)z^{2|\nu|+1}d_qz\\ &= c_{q,\nu}\int_0^\infty\left[c_{q,\nu}\int_0^\infty f(t)\widetilde{j}_{q,\nu}(tz,q^2)t^{2|\nu|+1}d_qt\right]\widetilde{j}_{q,\nu}(yz,q^2)\ \widetilde{j}_{q,\nu}(xz,q^2)z^{2|\nu|+1}d_qz\\ &= \int_0^\infty f(t)\left[{c_{q,\nu}}^2\int_0^\infty \widetilde{j}_{q,\nu}(yz,q^2)\ \widetilde{j}_{q,\nu}(xz,q^2)\widetilde{j}_{q,\nu}(tz,q^2)z^{2|\nu|+1}d_qz\right]t^{2|\nu|+1}d_qt\\ &= \int_0^\infty f(t)\mathcal{D}_{q,\nu}(x,y,t)t^{2|\nu|+1}d_qt.\end{aligned}$$

The computation is justified by the Fubuni's theorem

$$\begin{aligned}&\int_0^\infty\left[\int_0^\infty |f(t)||\widetilde{j}_{q,\nu}(tz,q^2)|t^{2|\nu|+1}d_qt\right]|\widetilde{j}_{q,\nu}(yz,q^2)\ \widetilde{j}_{q,\nu}(xz,q^2)|z^{2|\nu|+1}d_qz\\ \leq\ &\|f\|_{q,p,\nu}\int_0^\infty\left[\int_0^\infty |\widetilde{j}_{q,\nu}(tz,q^2)|^{\overline{p}}t^{2|\nu|+1}d_qt\right]^{1/\overline{p}}|\widetilde{j}_{q,\nu}(yz,q^2)\ \widetilde{j}_{q,\nu}(xz,q^2)|z^{2|\nu|+1}d_qz\\ \leq\ &\|f\|_{q,p,\nu}\|\widetilde{j}_{q,\nu}(.,q^2)\|_{q,\overline{p},\nu}\int_0^\infty |\widetilde{j}_{q,\nu}(yz,q^2)\ \widetilde{j}_{q,\nu}(xz,q^2)|z^{2(|\nu|+1)(1-\frac{1}{\overline{p}})-1}d_qz < \infty,\end{aligned}$$

the result follows. $\square$

Recall that the generalized q-Bessel translation operator $T^{\nu}_{q,x}$ is said to be positive if it satisfies :

$$\text{If} \quad f \geq 0 \quad \text{then} \quad T^{\nu}_{q,x} f \geq 0, \forall\, x \in \mathbb{R}^+_q.$$

Obviously, the positivity of the generalized q-Bessel translation operator $T^{\nu}_{q,x}$ is related to the positivity of the kernel $\mathcal{D}_{q,\nu}(x,y,t)$.
Let us denote by $Q_{q,\nu}$ the domain of positivity of the generalized q-Bessel translation operator given by:

$$Q_{q,\nu} = \{q \in]0,1[, \text{ if } f \geq 0 \quad \text{then} \quad T^{\nu}_{q,x} f \geq 0, \forall\, x \in \mathbb{R}^+_q\}.$$

Lemma 1. *We have*

$$\mathcal{D}_{q,\nu}(x,y,t) \geq 0, \quad \forall x,y,t \in \mathbb{R}^+_q.$$

Proof. From Lemma 5 in [9], we have when

$$D_{\nu,q}(x,y,t) = c^2_{q,\nu} \int_0^{\infty} j_\nu(zx,q^2) j_\nu(zy,q^2) j_\nu(zt,q^2) z^{2\nu+1} d_q z \geq 0,$$

that

$$D_{\nu+\mu,q}(x,y,t) = c^2_{q,\nu+\mu} \int_0^{\infty} j_{\nu+\mu}(zx,q^2) j_{\nu+\mu}(zy,q^2) j_{\nu+\mu}(zt,q^2) z^{2(\nu+\alpha)+1} d_q z \geq 0,$$

where

$$0 < \mu < \alpha < 1.$$

Put $\alpha = 2\mu$, we obtain

$$D_{\nu+\mu,q}(x,y,t) = c^2_{q,\nu+\mu} \int_0^{\infty} j_{\nu+\mu}(zx,q^2) j_{\nu+\mu}(zy,q^2) j_{\nu+\mu}(zt,q^2) z^{2(|\nu|+2\mu)+1} d_q z \geq 0.$$

Then for all $k \in \mathbb{N}, 0 < \mu < 1$, we have

$$D_{\nu+k\mu,q}(x,y,t) = c^2_{q,\nu+k\mu} \int_0^{\infty} j_{\nu+k\mu}(zx,q^2) j_{\nu+k\mu}(zy,q^2) j_{\nu+k\mu}(zt,q^2) z^{2(|\nu|+2k\mu)+1} d_q z \geq 0.$$

For $k\mu = n$ and the definition of the kernel $\mathcal{D}_{q,\nu}(x,y,t)$ lead to the result. $\square$

3.2. **Generalized q-Convolution Product.**

Definition 3. *The Generalized q-convolution product is defined by*

$$f *_q g = \mathcal{F}_{q,\nu}[\mathcal{F}_{q,\nu} f \times \mathcal{F}_{q,\nu} g].$$

Theorem 3. *let* $1 \leq p, r, s$ *such that*

$$\frac{1}{p} + \frac{1}{r} - 1 = \frac{1}{s}.$$

Given two functions $f \in \mathcal{L}_{q,p,\nu}$ *and* $g \in \mathcal{L}_{q,r,\nu}$ *then* $f *_q g$ *exists and we have*

$$f *_q g(x) = c_{q,\nu} \int_0^{\infty} T^{\nu}_{q,x} f(y) g(y) y^{2|\nu|+1} d_q y,$$

and

$$\begin{aligned} f *_q g &\in \mathcal{L}_{q,s,\nu}, \\ \mathcal{F}_{q,\nu}[f *_q g] &= \mathcal{F}_{q,\nu} f \times \mathcal{F}_{q,\nu} g. \end{aligned}$$

If $s \geq 2$,

$$\|f *_q g\|_{q,s,\nu} \leq B_{q,\nu} \ \|f\|_{q,p,\nu} \ \|g\|_{q,r,\nu}.$$

Proof. We have

$$\begin{aligned} f *_q g(x) &= \mathcal{F}_{q,\nu}[\mathcal{F}_{q,\nu} f \times \mathcal{F}_{q,\nu} g](x) \\ &= c_{q,\nu} \int_0^\infty \mathcal{F}_{q,\nu} f(y) \ \mathcal{F}_{q,\nu} g(y) \ \widetilde{j}_{q,\nu}(yx, q^2) y^{2|\nu|+1} d_q y \\ &= c_{q,\nu} \int_0^\infty \mathcal{F}_{q,\nu} f(y) \left[c_{q,\nu} \int_0^\infty g(t) \ \widetilde{j}_{q,\nu}(ty, q^2) t^{2|\nu|+1} d_q t \right] \widetilde{j}_{q,\nu}(yx, q^2) y^{2|\nu|+1} d_q y \\ &= \int_0^\infty c_{q,\nu} \left[c_{q,\nu} \int_0^\infty \mathcal{F}_{q,\nu} f(y) \ \widetilde{j}_{q,\nu}(ty, q^2) \widetilde{j}_{q,\nu}(yx, q^2) y^{2|\nu|+1} d_q y \right] g(t) t^{2|\nu|+1} d_q t \\ &= c_{q,\nu} \int_0^\infty T^\nu_{q,x} f(t) \ g(t) \ t^{2|\nu|+1} d_q t. \end{aligned}$$

The computation is justified by the Fubuni's Theorem

$$\begin{aligned} &\int_0^\infty |\mathcal{F}_{q,\nu} f(y)| \left[\int_0^\infty |g(t) \ \widetilde{j}_{q,\nu}(ty, q^2)| t^{2|\nu|+1} d_q t \right] |\widetilde{j}_{q,\nu}(yx, q^2)| y^{2|\nu|+1} d_q y \\ &\leq \|g\|_{q,r,\nu} \int_0^\infty |\mathcal{F}_{q,\nu} f(y)| \left[\int_0^\infty |\widetilde{j}_{q,\nu}(ty, q^2)|^{\overline{r}} t^{2|\nu|+1} d_q t \right]^{1/\overline{r}} |\widetilde{j}_{q,\nu}(yx, q^2)| y^{2|\nu|+1} d_q y \\ &\leq \|g\|_{q,r,\nu} \ \|\widetilde{j}_{q,\nu}(ty, q^2)\|_{q,r,\nu} \int_0^\infty |\mathcal{F}_{q,\nu} f(y)| \ \left[|\widetilde{j}_{q,\nu}(yx, q^2)| y^{-\frac{2|\nu|+2}{\overline{r}}} \right] y^{2|\nu|+1} d_q y \\ &\leq \|g\|_{q,r,\nu} \ \|\widetilde{j}_{q,\nu}(ty, q^2)\|_{q,r,\nu} \ \|\mathcal{F}_{q,\nu} f\|_{q,\overline{p},\nu} \left(\int_0^\infty |\widetilde{j}_{q,\nu}(ty, q^2)|^p \ y^{2(|\nu|+1)(1-\frac{p}{\overline{r}})-1} \right)^{1/p} < \infty. \end{aligned}$$

From (8), we deduce that

$$\mathcal{F}_{q,\nu} f \in \mathcal{L}_{q,\overline{p},\nu} \text{ and } \mathcal{F}_{q,\nu} g \in \mathcal{L}_{q,\overline{r},\nu}.$$

Hence, using the Hölder inequality and the fact that

$$\frac{1}{\overline{p}} + \frac{1}{\overline{r}} = \frac{1}{\overline{s}},$$

we conclude that

$$\mathcal{F}_{q,\nu} f \times \mathcal{F}_{q,\nu} g \in \mathcal{L}_{q,\overline{s},\nu}.$$

gives

$$f *_q g = \mathcal{F}_{q,\nu}[\mathcal{F}_{q,\nu} f \times \mathcal{F}_{q,\nu} g] \in \mathcal{L}_{q,s,\nu}.$$

From the inversion formula (7), we obtain

$$\mathcal{F}_{q,\nu}[f *_q g] = \mathcal{F}_{q,\nu} f \times \mathcal{F}_{q,\nu} g.$$

Suppose that $s \geq 2$, so $1 \leq \overline{s} \leq 2$ and we can write

$$\begin{aligned} \|f *_q g\|_{q,s,\nu} &= \|\mathcal{F}_{q,\nu}[\mathcal{F}_{q,\nu} f \times \mathcal{F}_{q,\nu} g]\|_{q,s,\nu} \\ &\leq B_{q,\nu}^{\frac{2}{\overline{s}}-1} \|\mathcal{F}_{q,\nu} f\|_{q,\overline{p},\nu} \|\mathcal{F}_{q,\nu} g\|_{q,\overline{r},\nu} \\ &\leq B_{q,\nu}^{\frac{2}{\overline{s}}-1} B_{q,\nu}^{\frac{2}{p}-1} B_{q,\nu}^{\frac{2}{r}-1} \|f\|_{q,p,\nu} \|g\|_{q,r,\nu} \\ &\leq B_{q,\nu} \|f\|_{q,p,\nu} \|g\|_{q,r,\nu}. \end{aligned}$$

□

4. Uncertainty principle

In the survey articles by Folland and Sitaram [12] and by Cowling and Price [2] , one find various uncertainty principles in the literature. In this section, the Heisenberg uncertainty inequality is established for functions in $\mathcal{L}_{q,2,\nu}$.

Proposition 12. *If* $\langle \partial_{q,\beta} f, g\rangle$ *exists and*

$$\lim_{\substack{a\to\infty\\ \varepsilon\to 0}} \left| a^{2|\nu|+1} f(q^{-1}a)g(a) - \varepsilon^{2|\nu|+1} f(q^{-1}\varepsilon)g(\varepsilon)\right| = 0,$$

then $\left\langle f, \partial^*_{q,\alpha} g\right\rangle$ *exists and we have*

$$\langle \partial_{q,\beta} f, g\rangle = -q^{2\beta}\left\langle f, \partial^*_{q,\alpha} g\right\rangle.$$

Proof. Let $\varepsilon \in \mathbb{R}_q^+$. The following computation

$$\begin{aligned}
&\int_\varepsilon^a \partial_{q,\beta} f(x) g(x) x^{2|\nu|+1} d_q x\\
&= \int_\varepsilon^a \frac{f(q^{-1}x) - q^{2\beta} f(x)}{x} g(x) x^{2|\nu|+1} d_q x\\
&= \int_\varepsilon^a \frac{f(q^{-1}x)}{x} g(x) x^{2|\nu|+1} d_q x - q^{2\beta}\int_\varepsilon^a \frac{f(x)}{x} g(x) x^{2|\nu|+1} d_q x\\
&= q^{2|\nu|+1}\int_{q^{-1}\varepsilon}^{q^{-1}a} \frac{f(x)}{x} g(qx) x^{2|\nu|+1} d_q x - q^{2\beta}\int_\varepsilon^a \frac{f(x)}{x} g(x) x^{2|\nu|+1} d_q x\\
&= q^{2|\nu|+1}\int_\varepsilon^a \frac{f(x)}{x} g(qx) x^{2|\nu|+1} d_q x - q^{2\beta}\int_\varepsilon^a \frac{f(x)}{x} g(x) x^{2|\nu|+1} d_q x + a^{2|\nu|+1} f(q^{-1}a)g(a)\\
&-\varepsilon^{2|\nu|+1} f(q^{-1}\varepsilon)g(\varepsilon)\\
&= -q^{2\beta}\int_\varepsilon^a f(x)\frac{g(x) - q^{2\alpha+1} g(qx)}{x} x^{2|\nu|+1} d_q x + a^{2|\nu|+1} f(q^{-1}a)g(a) - \varepsilon^{2|\nu|+1} f(q^{-1}\varepsilon)g(\varepsilon)\\
&= -q^{2\beta}\int_\varepsilon^a f(x)\partial^*_{q,\alpha} g(x) x^{2|\nu|+1} d_q x + a^{2|\nu|+1} f(q^{-1}a)g(a) - \varepsilon^{2|\nu|+1} f(q^{-1}\varepsilon)g(\varepsilon),
\end{aligned}$$

leads to the result. □

Corollary 2. *If* $f \in \mathcal{L}_{q,2,\nu}$ *such that* $x^2\mathcal{F}_{q,\nu} f \in \mathcal{L}_{q,2,\nu}$ *and*

$$\partial_{q,\beta} f(x) = O(x^{-|\nu|}) \quad \text{as} \quad x \downarrow 0.$$

Then $\partial_{q,\beta} f \in \mathcal{L}_{q,2,\nu}$ *and we have*

$$\|\partial_{q,\beta} f\|_2 = q^\beta \|x\mathcal{F}_{q,\nu} f\|_2.$$

Proof. In fact

$$\begin{aligned}
q^{2\beta}\|x\mathcal{F}_{q,\nu} f\|_2^2 &= q^{2\beta}\left\langle \mathcal{F}_{q,\nu} f, x^2\mathcal{F}_{q,\nu} f\right\rangle\\
&= -q^{2\beta}\left\langle \mathcal{F}_{q,\nu} f, \mathcal{F}_{q,\nu}\widetilde{\Delta}_{q,\nu} f\right\rangle\\
&= -q^{2\beta}\left\langle \mathcal{F}^2_{q,\nu} f, \mathcal{F}^2_{q,\nu}\widetilde{\Delta}_{q,\nu} f\right\rangle\\
&= -q^{2\beta}\left\langle f, \widetilde{\Delta}_{q,\nu} f\right\rangle\\
&= -q^{2\beta}\left\langle f, \partial^*_{q,\alpha}\partial_{q,\beta} f\right\rangle\\
&= \langle \partial_{q,\beta} f, \partial_{q,\beta} f\rangle = \|\partial_{q,\beta} f\|_2^2,
\end{aligned}$$

which prove the result. □

Theorem 4. *Assume that f belongs to the space $\mathcal{L}_{q,2,\nu}$ such that*

$$xf, x^2\mathcal{F}_{q,\nu}f \in \mathcal{L}_{q,2,\nu}$$

and

$$\partial_{q,\beta}f(x) = O(x^{-|\nu|}) \quad as \quad x \downarrow 0.$$

Then the generalized q-Bessel transform satisfies the following uncertainty principle

$$\|f\|_2^2 \le k_{q,\nu} \|xf\|_2 \|x\mathcal{F}_{q,\nu}f\|_2 ,$$

where

$$k_{q,\nu} = \frac{\left[q^{\beta} + \sqrt{q} \times q^{\alpha+1}\right]}{1 - q^{2(|\nu|+1)}}.$$

Proof. In fact

$$\partial^*_{q,\alpha}xf = f(x) - q^{2\alpha+2}f(qx),$$

$$x\partial_{q,\beta}f = f(q^{-1}x) - q^{2\beta}f(x).$$

We introduce the following operator

$$\Lambda_q f(x) = f(qx),$$

then

$$\langle \Lambda_q f, g\rangle = q^{-2(|\nu|+1)} \left\langle f, \Lambda_q^{-1}g\right\rangle .$$

So

$$\frac{1}{1 - q^{2(|\nu|+1)}} \left[q^{2\beta}\partial^*_{q,\alpha}xf(x) - q^{2\alpha+2}\Lambda_q x\partial_{q,\beta}f(x)\right] = f(x).$$

Assume that xf and $x^2\mathcal{F}_{q,\nu}f$ belong to the space $\mathcal{L}_{q,2,\nu}$, then we have

$$\langle f, f\rangle = -\frac{1}{1 - q^{2(|\nu|+1)}} \langle xf, \partial_{q,\beta}f\rangle - \frac{q^{-2\beta}}{1 - q^{2(|\nu|+1)}} \left\langle \partial_{q,\beta}f, x\Lambda_q^{-1}f\right\rangle .$$

Note that

$$\langle xf, \partial_{q,\beta}f\rangle \text{ and } \left\langle \partial_{q,\beta}f, x\Lambda_q^{-1}f\right\rangle \text{ exist}$$

and

$$\lim_{\varepsilon\to 0} \varepsilon^{2|\nu|+2}f(q^{-1}\varepsilon)f(\varepsilon) = 0.$$

By Cauchy-Schwartz inequality, we get

$$\langle f, f\rangle \le \frac{1}{1 - q^{2(|\nu|+1)}} \|xf\|_2 \|\partial_{q,\beta}f\|_2 + \frac{q^{-2\beta}}{1 - q^{2(|\nu|+1)}} \|\partial_{q,\beta}f\|_2 \left\|x\Lambda_q^{-1}f\right\|_2 .$$

On the other hand

$$\left\|x\Lambda_q^{-1}f\right\|_2 = \sqrt{q} \times q^{|\nu|+1} \|xf\|_2 .$$

Corollary 2 gives the result. □

5. Hardy's theorem

One of the famous formulations of the uncertainty principle is stated by the so-called Hardy's theorem [14], and many interesting results about this theorem was proved in the last years [18, 7, 11, 1]. In this section, we give Hardy's theorem for the generalized q-Bessel Fourier transform which its proof are the same as those in [4].

Theorem 5. *Suppose $f \in \mathcal{L}_{q,1,\nu}$ satisfying the following behaviour*

$$|f(x)| \leq Ce^{-\frac{1}{2}x^2}, \quad \forall x \in \mathbb{R}_q^+,$$

$$|\mathcal{F}_{q,\nu} f(x)| \leq Ce^{-\frac{1}{2}x^2}, \quad \forall x \in \mathbb{R},$$

where C is a positive constant. Then there exists $A \in \mathbb{R}$ such that

$$f(z) = Ac_{q,\nu}\mathcal{F}_{q,\nu}\left(e^{-\frac{1}{2}x^2}\right)(z), \quad \forall z \in \mathbb{C},$$

where $c_{q,\nu}$ is given by (5).

Corollary 3. *Suppose $f \in \mathcal{L}_{q,1,\nu}$ satisfying the following behaviour*

$$|f(x)| \leq Ce^{-px^2}, \quad \forall x \in \mathbb{R}_q^+,$$

$$|\mathcal{F}_{q,\nu} f(x)| \leq Ce^{-\sigma x^2}, \quad \forall x \in \mathbb{R},$$

where C, p, σ are positive constants with $p\ \sigma = \frac{1}{4}$. We suppose that there exists $a \in \mathbb{R}_q^+$ such that $a^2p = \frac{1}{2}$. Then there exists $A \in \mathbb{R}$ such that

$$f(z) = Ac_{q,\nu}\mathcal{F}_{q,\nu}\left(e^{-\sigma t^2}\right)(z), \quad \forall z \in \mathbb{C},$$

where $c_{q,\nu}$ is given by (5).

Corollary 4. *Suppose $f \in \mathcal{L}_{q,1,\nu}$ satisfying the following behaviour*

$$|f(x)| \leq Ce^{-px^2}, \quad \forall x \in \mathbb{R}_q^+,$$

$$|\mathcal{F}_{q,\nu} f(x)| \leq Ce^{-\sigma x^2}, \quad \forall x \in \mathbb{R},$$

where C, p, σ are positive constants with $p\sigma > \frac{1}{4}$. We suppose that there exists $a \in \mathbb{R}_q^+$ such that $a^2p = \frac{1}{2}$. Then $f \equiv 0$.

6. Generalized q-Macdonald function

Definition 4. *The generalized modified q-Bessel functions is defined by*

$$I_{q,\nu}^a(x, q^2) = \widetilde{j}_{q,\nu}(iax, q^2), \ i^2 = -1, \quad a > 0.$$

We put

$$\gamma_{q,\nu}^a(x, q^2) = \widetilde{j}_{q,\overline{\nu}}(ax, q^2), \quad \pi_{q,\nu}^a(x, q^2) = \gamma_{q,\nu}^a(ix, q^2).$$

The integral representation of Macdonald function in [20, p. 434] suggests that to define the generalized q-Macdonald function as follows:

$$\begin{aligned} K^a_{q,\nu}(x,q^2) &= c_{q,\nu}\int_0^\infty \left[1+\frac{t^2}{a^2}\right]^{-1} \widetilde{j}_{q,\nu}(tx,q^2)\; t^{2|\nu|+1}d_qt \\ &= x^{2n}c_{q,\nu}\int_0^\infty \left[1+\frac{t^2}{a^2}\right]^{-1} j_{\alpha+n}(q^ntx,q^2)\; t^{2\alpha+1}d_qt, \end{aligned}$$

where $c_{q,\nu}$ is given by (5).

Theorem 6. *The previous generalized q-Macdonald function $K^a_{q,\nu}$ is $\mathcal{L}_{q,1,\nu}$ and we have*

$$\mathcal{F}_{q,\nu}(K^a_{q,\nu})(x) = \left[1+\frac{x^2}{a^2}\right]^{-1}, \quad \forall x\in\mathbb{R}^+_q. \tag{11}$$

Proof. Since $\left[1+\frac{x^2}{a^2}\right]^{-1}\in\mathcal{L}_{q,p,\nu}$, the inversion formula of the generalized q-Bessel transform leads to the result. □

Proposition 13. *The functions $x\to I^a_{q,\nu}(\lambda x,q^2)$ and $x\to K^a_{q,\nu}(\lambda x,q^2)$ are two linearly independent solutions of the following equation*

$$\widetilde{\Delta}_{q,\nu}f(x) = \lambda^2 f(x). \tag{12}$$

Proof. In fact we have

$$\left[1-\frac{\widetilde{\Delta}_{q,\nu}}{a^2}\right]K^a_{q,\nu}(x,q^2) = c_{q,\nu}\int_0^\infty\left[1+\frac{t^2}{a^2}\right]^{-1}\left[1-\frac{\widetilde{\Delta}_{q,\nu}}{a^2}\right]\widetilde{j}_{q,\nu}(tx,q^2)t^{2|\nu|+1}d_qt = 0.$$

Note that

$$\left[1-\frac{\widetilde{\Delta}_{q,\nu}}{a^2}\right]\widetilde{j}_{q,\nu}(tx,q^2) = \left[1+\frac{t^2}{a^2}\right]\widetilde{j}_{q,\nu}(tx,q^2).$$

The function $K^a_{q,\nu}\in\mathcal{L}_{q,1,\nu}$ but $I^a_{q,\nu}\notin\mathcal{L}_{q,1,\nu}$. Hence, we conclude that they provide two linearly independent solutions. □

Lemma 2. *Let $\lambda\in\mathbb{C}$ such that $\lambda\notin\mathbb{R}^+_q\cup q^{|\nu|}\mathbb{R}^+_q$, then we have*

$$\lim_{k\to\infty} q^{2|\nu|k}\frac{\widetilde{j}_{q,-\nu}(q^{-k-|\nu|}\lambda,q^2)}{\widetilde{j}_{q,\nu}(q^{-k}\lambda,q^2)}$$
$$= (-1)^nq^{-n(n-1)}\lambda^{-2n}\frac{(q^{2(\alpha+n)+2},q^2)_\infty}{(q^{-2(\alpha+n)+2},q^2)_\infty}\frac{(q^{2n+2|\nu|}\lambda^{-2},q^2)_\infty(q^{2|\nu|}\lambda^{-2},q^2)_n(q^{2-2|\nu|}\lambda^2,q^2)_\infty}{(q^{-2n}\lambda^{-2},q^2)_\infty(q^{2+2n}\lambda^2,q^2)_\infty}. \tag{13}$$

Proof. Let $x\in\mathbb{C}^*\backslash\mathbb{R}^+_q$, then we have the following asymptotic expansion

$$\widetilde{j}_{q,\nu}(x,q^2)\sim x^{2n}\frac{(x^2q^{2+2n},q^2)_\infty}{(q^{2(\alpha+n)+2},q^2)_\infty}, \quad |x|\to\infty.$$

If $k\to\infty$, we have

$$\widetilde{j}_{q,\nu}(q^{-k}\lambda,q^2)\sim q^{-2kn}\lambda^{2n}\frac{(q^{2-2k+2n}\lambda^2,q^2)_\infty(q^2,q^2)_\infty}{(q^{2\alpha+2},q^2)_\infty(q^{2(\alpha+n)+2},q^2)_\infty}, \quad \forall\lambda\notin\mathbb{R}^+_q.$$

On the other hand

$$(q^{2-2k}q^{2n}\lambda^2,q^2)_\infty=(-1)^kq^{-k(k-1)}q^{2kn}\lambda^{2k}(q^{-2n}\lambda^{-2},q^2)_k(q^{2+2n}\lambda^2,q^2)_\infty.$$

Hence when $n\to\infty$

$$\widetilde{j}_{q,\nu}(q^{-k}\lambda,q^2)\sim\frac{(-1)^kq^{-k(k-1)}\lambda^{2n}\lambda^{2k}(q^{-2n}\lambda^{-2},q^2)_k(q^{2+2n}\lambda^2,q^2)_\infty}{(q^{2(\alpha+n)+2},q^2)_\infty},\quad\forall\lambda\notin\mathbb{R}_q^+,$$

and for all $\lambda\notin q^{|\nu|}\mathbb{R}_q^+$, we have

$$\widetilde{j}_{q,-\nu}(q^{-k-|\nu|}\lambda,q^2)\sim\frac{(-1)^{k+n}q^{-n(n-1)}q^{-k(k-1)}q^{-2|\nu|k}\lambda^{2k}(q^{2n+2|\nu|}\lambda^{-2},q^2)_k(q^{2|\nu|}\lambda^{-2},q^2)_n(q^{2-2|\nu|}\lambda^2,q^2)_\infty}{(q^{-2(\alpha+n)+2},q^2)_\infty}.$$

This implies

$$q^{2|\nu|k}\frac{\widetilde{j}_{q,-\nu}(q^{-k-|\nu|}\lambda,q^2)}{\widetilde{j}_{q,\nu}(q^{-k}\lambda,q^2)}$$

$$=(-1)^nq^{-n(n-1)}\lambda^{-2n}\frac{(q^{2(\alpha+n)+2},q^2)_\infty}{(q^{-2(\alpha+n)+2},q^2)_\infty}\frac{(q^{2n+2|\nu|}\lambda^{-2},q^2)_k(q^{2|\nu|}\lambda^{-2},q^2)_n(q^{2-2|\nu|}\lambda^2,q^2)_\infty}{(q^{-2n}\lambda^{-2},q^2)_k(q^{2+2n}\lambda^2,q^2)_\infty}.$$

Hence when $k\to\infty$ we obtain the result. □

Proposition 14. *We have*

$$K^a_{q,\nu}(x,q^2)=\sigma^a_\nu\left[\pi^a_{q,\nu}(x,q^2)-\theta^a_\nu I^a_{q,\nu}(x,q^2)\right],\tag{14}$$

where

$$\begin{aligned}\theta^a_\nu&=\lim_{k\to\infty}\frac{\pi^a_{q,\nu}(q^{-k},q^2)}{I^a_{q,\nu}(q^{-k},q^2)}\\&=a^{-2\alpha}q^{-n(n-1)}\frac{(q^{2(\alpha+n)+2},q^2)_\infty}{(q^{-2(\alpha+n)+2},q^2)_\infty}\frac{(-q^{2n+2|\nu|}a^{-2},q^2)_\infty(-q^{2|\nu|}a^{-2},q^2)_n(-q^{2-2|\nu|}a^2,q^2)_\infty}{(-q^{-2n}a^{-2},q^2)_\infty(-q^{2+2n}a^2,q^2)_\infty},\end{aligned}$$

and

$$\sigma^a_\nu=\begin{cases}\dfrac{(q^2,q^2)_\infty}{(q^{2|\nu|},q^2)_\infty} & \text{if}\quad|\nu|\geq0\\ -\dfrac{c_{q,\nu}}{\theta^a_\nu}\displaystyle\int_0^\infty\left[1+\frac{t^2}{a^2}\right]^{-1}t^{2|\nu|+1}d_qt & \text{if}\quad-1<|\nu|<0\end{cases}.$$

Proof. The functions $x\to I^a_{q,\nu}(\lambda x,q^2)$ and $x\to\pi^a_{q,\nu}(\lambda x,q^2)$ are two linearly independent solutions of (12). Then there exist two constants θ^a_ν and σ^a_ν such that (14) hold true. Now we can write

$$K^a_{q,\nu}(q^{-k},q^2)=\sigma^a_\nu\left[\frac{\pi^a_{q,\nu}(q^{-k},q^2)}{I^a_{q,\nu}(q^{-k},q^2)}-\theta^a_\nu\right]I^a_{q,\nu}(q^{-k},q^2).$$

On the other hand

$$\lim_{k\to\infty}I^a_{q,\nu}(q^{-k},q^2)=\infty.$$

Using Theorem 6, we have

$$\lim_{k\to\infty}K^a_{q,\nu}(q^{-k},q^2)=0.$$

Then it is necessary that

$$\lim_{k\to\infty}\left[\frac{\pi^a_{q,\nu}(q^{-k},q^2)}{I^a_{q,\nu}(q^{-k},q^2)}-\theta^a_\nu\right]=0.$$

Formula (13) with $\lambda = ia$ leads to the result. To estimate σ^a_ν, we consider two cases:

• If $|\nu| > 0$, we obtain

$$\sigma^a_\nu=\lim_{x\to 0}x^{2|\nu|}K^a_{q,\nu}(x,q^2)=c_{q,\nu}\int_0^\infty \widetilde{j}_{q,\nu}(t,q^2)t^{2|\nu|-1}d_qt.$$

Using an identity established in [16], with $(\theta = \alpha + n, \mu = 0, \lambda = 1 - \alpha, m \to \infty)$. We conclude that

$$\sigma^a_\nu=\frac{(q^2,q^2)_\infty}{(q^{2|\nu|},q^2)_\infty}.$$

• If $-1 < |\nu| < 0$, we see that

$$\sigma^a_\nu=-\frac{1}{\theta^a_\nu}\lim_{x\to 0}K^a_{q,\nu}(x,q^2)=-\frac{c_{q,\nu}}{\theta^a_\nu}\int_0^\infty\left[1+\frac{t^2}{a^2}\right]^{-1}t^{2|\nu|+1}d_qt.$$

□

Corollary 5. *As direct consequence, we have*

$$c(I^a_{q,\nu},K^a_{q,\nu})=\sigma^a_\nu(q^{-2|\nu|}-1).$$

Proposition 15. *The generalized q-Macdonald function $K^a_{q,\nu}(.,q^2)$ satisfies the following properties*

a. For all $x \in \mathbb{R}^+_q$ we have

$$\partial_{q,\beta}K^a_{q,\nu}(x,q^2)=-\frac{q^{1-n}}{1-q^{2(\alpha+n)+2}}\ x\ K^a_{\alpha+1,n}(x,q^2).$$

b. For all $x \in \mathbb{R}^+_q$ we have

$$K^a_{q,\nu}\in\mathcal{L}_{q,2,\nu}.$$

c. If $f \in \mathcal{L}_{q,1,\nu}$ and if $h(x) = K^a_{q,\nu} *_q g(x)$ then

$$\left[1-\frac{\Delta_{q,\nu}}{a^2}\right]h(x)=f(x).$$

d. There exist $c, \sigma > 0$ such that

$$|K^a_{q,\nu}(q^{-k},q^2)|<\sigma c^k q^{k^2},$$

and

$$\lim_{k\to\infty}\frac{K^a_{q,\nu}(q^{-k},q^2)}{K^a_{q,\nu}(q^{-k+1},q^2)}=0.$$

Proof. c.) From Theorem 6 and Theorem 1, we see that the generalized q-Macdonald function belongs to $\mathcal{L}_{q,2,\nu}$.

b.) By Theorem 3, we see that $h \in \mathcal{L}_{q,1,\nu}$ and we have

$$\mathcal{F}_{q,\nu}h(x)=\left[1+\frac{x^2}{a^2}\right]^{-1}\mathcal{F}_{q,\nu}g(x).$$

By (7) we have

$$h(x) = c_{q,\nu} \int_0^\infty \mathcal{F}_{q,\nu} g(t) \left[1 + \frac{t^2}{a^2}\right]^{-1} \widetilde{j}_{q,\nu}(tx, q^2) t^{2|\nu|+1} d_q t,$$

So

$$\begin{aligned}
\left[1 - \frac{\widetilde{\Delta}_{q,\nu}}{a^2}\right] h(x) &= c_{q,\nu} \int_0^\infty \mathcal{F}_{q,\nu} g(t) \left[1 + \frac{t^2}{a^2}\right]^{-1} \left[1 - \frac{\widetilde{\Delta}_{q,\nu}}{a^2}\right] \widetilde{j}_{q,\nu}(tx, q^2) t^{2|\nu|+1} d_q t \\
&= c_{q,\nu} \int_0^\infty \mathcal{F}_{q,\nu} g(t) \widetilde{j}_{q,\nu}(tx, q^2) t^{2|\nu|+1} d_q t = g(t).
\end{aligned}$$

d.) Let f be a solution of the q-difference equation

$$\widetilde{\Delta}_{q,\nu} f(x) = [W(x) - \lambda] f(x), \quad \forall x \in \mathbb{R}_q^+. \tag{15}$$

From Remark 1, we have

$$\widetilde{\Delta}_{q,\nu} f(x) = \Delta_{q,|\nu|} f(x) + V(x) f(x),$$

then the last equation (15) is equivalent to:

$$\begin{aligned}
\Delta_{q,|\nu|} f(x) &= [W(x) - V(x) - \lambda] f(x) \\
&= [R(x) - \lambda] f(x),
\end{aligned}$$

where

$$R(x) = W(x) - V(x).$$

From b.) the generalized q-Macdonald function belongs to $\mathcal{L}_{q,2,\nu}$ and we apply Theorem 4 in [6] with $W(x) = 0$ and $\lambda = -a^2$. This give the result. □

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