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LYAPUNOV FUNCTIONS FOR H-DICHOTOMY OF LINEAR DISCRETE-TIME SYSTEMS

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Abstract. The main aim of this paper is to give characterizations in terms of Lyapunov functions for a general concept of h-dichotomy of noninvertible linear discrete-time systems in Banach spaces.
The results are applied to deduce criteria for the detection of the nonuniform exponential dichotomy respectively nonuniform polynomial dichotomy of linear discrete-time systems.

1. Introduction

In the last few years, the theory of asymptotic behaviors of discrete-time systems has witnessed an impressive development. A number of open problems have been solved and the theory seems to have obtained a certain degree of maturity.

Among the asymptotic behaviors of linear discrete-time systems an important role is played by the dichotomy property. It was studied in an extensive manner from the point of view of both uniform and nonuniform (exponential or polynomial) behavior (see [4], [7], [8], [11], [12], [14], [16]).

A natural generalization of both of the above behaviors is successfully modeled by the (h, k) - dichotomy where a large number of papers containing many interesting results were published, from which we mention [1], [2], [6], [9].

The importance of Lyapunov functions in the study of asymptotic behaviors of linear discrete-time systems is well established. We mention some papers where the dichotomy properties are studied by Lyapunov functions ([3], [5], [10], [13], [15]).

This paper considers a general notion of dichotomy (called h-dichotomy) for nonautonomous linear discrete-time systems in Banach spaces. We remark that we consider nonautonomous linear discrete-time systems having the right side not necessarily invertible. It is important to treat the case of noninvertible systems because of their interest in applications (e.g. random dynamical systems generated by parabolic equations which are not invertible).

The main aim of this paper is to study the relation between a general dichotomy concept with arbitrary growth rate of a linear nonautonomous discrete-time system and a notion of Lyapunov function.

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We note that the results are obtained without making any assumption on the existence of a uniform or nonuniform growth of the system.

As applications, we obtain characterizations in terms of Lyapunov functions for nonuniform exponential dichotomy respectively nonuniform polynomial dichotomy.

2. Growth rates

Definition 2.1. An increasing real sequence $h : \mathbb{N} \to [1, +\infty)$, $h(n) = h_n$ with the property that $\lim\limits_{n\to+\infty} h_n = +\infty$ is called a *growth rate.*

In this paper we consider a particular class of growth rates defined by

Definition 2.2. We say that the growth rate $h = (h_n)$ *satisfies the hypothesis (H)* if there exist $M \in (1, \infty)$ and a growth rate $k = (k_n)$ such that

$$(H_1) \quad \left(\frac{h_n}{k_n}\right) \textit{ is a growth rate}$$

$$(H_2) \quad \sum_{n=0}^{\infty} \frac{k_n}{h_n} \leq M$$

$$(H_3) \quad \sum_{j=n}^{m-1} \frac{h_j}{k_j} \leq M\frac{h_m^2}{k_m^2}$$

for all $(m, n) \in \mathbb{N}^2$ with $m > n$.

In what follows we present three examples of growth rates which satisfy the hypothesis (H).

Example 2.1. If $h_n = e^{n\alpha}$ with $\alpha > 0$ then

$$k_n = e^{n\beta} \quad \textit{with } 0 < \beta < \alpha$$

is a growth rate with the properties

(H_1) $\left(\frac{h_n}{k_n}\right) = e^{n(\alpha-\beta)}$ *is a growth rate*;

(H_2) $\sum\limits_{n=0}^{\infty} \frac{k_n}{h_n} = \sum\limits_{n=0}^{\infty} e^{n(\beta-\alpha)} = \frac{e^{\alpha}}{e^{\alpha}-e^{\beta}} = M \in (1, \infty)$

(H_3) $\sum\limits_{j=n}^{m-1} \frac{h_j}{k_j} = \sum\limits_{j=n}^{m-1} e^{j(\alpha-\beta)} = \frac{e^{(\alpha-\beta)m}-e^{(\alpha-\beta)n}}{e^{\alpha-\beta}-1} \leq \frac{e^{(\alpha-\beta)m+\beta}}{e^{\alpha}-e^{\beta}} \leq \frac{e^{\alpha}}{e^{\alpha}-e^{\beta}} e^{2m(\alpha-\beta)} = M\frac{h_m^2}{k_m^2}$

for all $m > n$.

Example 2.2. If $h_n = (n+1)^{\alpha}$ with $\alpha > 1$ then $k_n = (n+1)^{\beta-1}$ with $1 < \beta < \alpha$ is a growth rate with

(H_1) $\left(\frac{h_n}{k_n}\right) = (n+1)^{\alpha-\beta+1}$ *is a growth rate*;

(H_2) $\sum\limits_{n=0}^{\infty} \frac{k_n}{h_n} = \sum\limits_{n=0}^{\infty} \frac{1}{(n+1)^{\alpha-\beta+1}} = M \in (1, \infty)$

(H_3) $\sum\limits_{j=n}^{m-1} \frac{h_j}{k_j} = \sum\limits_{j=n}^{m-1} (j+1)^{\alpha-\beta+1} \leq (m-n)(m+1)^{\alpha-\beta+1} \leq (m+1)^{2(\alpha-\beta+1)} = \frac{h_m^2}{k_m^2} \leq M\frac{h_m^2}{k_m^2}$

for every $m > n$.

Example 2.3. If $h_n = (n+1)^\alpha e^{n\gamma}$ with $\alpha > 1$ and $\gamma > 0$ then $k_n = (n+1)^{\beta-1} e^{\delta n}$ with $1 < \beta < \alpha$ and $0 < \delta < \gamma$ is a growth rate which satisfies the following properties:

(H_1) $\left(\frac{h_n}{k_n}\right) = (n+1)^{\alpha-\beta+1} e^{n(\gamma-\delta)}$ *is a growth rate*;

(H_2) $\sum\limits_{n=0}^{\infty} \frac{k_n}{h_n} = \sum\limits_{n=0}^{\infty} \frac{e^{n(\delta-\gamma)}}{(n+1)^{\alpha-\beta+1}} \le \sum\limits_{n=0}^{\infty} e^{n(\delta-\gamma)} \le \frac{1}{1-e^{\delta-\gamma}} = \frac{e^\gamma}{e^\gamma - e^\delta} = M \in (1, \infty)$

(H_3) $\sum\limits_{j=n}^{m-1} \frac{h_j}{k_j} = \sum\limits_{j=n}^{m-1} (j+1)^{\alpha-\beta+1} e^{(\gamma-\delta)j} \le (m-n)(m+1)^{\alpha-\beta+1} e^{(\gamma-\delta)m} \le (m+1)^{2(\alpha-\beta+1)} e^{2(\gamma-\delta)m} = \frac{h_m^2}{k_m^2} \le M \frac{h_m^2}{k_m^2}$

for all $m > n$.

3. h-dichotomy

Let X be a real or complex Banach space and let I be the identity operator on X. The norm on X and on $\mathcal{B}(X)$ - the Banach algebra of all bounded linear operators on X, will be denoted by $\|\cdot\|$.

Throughout the paper, we denote by $\mathbb{N}$ the set of natural numbers. We also denote by Δ the set of all pairs of all natural numbers (m, n) with $m \ge n$ and by

$$T = \{(m, n, x) \textit{ with } (m, n) \in \Delta \textit{ and } x \in X\}.$$

We consider the linear nonautonomous difference system

$$x_{n+1} = A_n x_n,\ n \in \mathbb{N} \tag{$\mathcal{A}$}$$

where $A : \mathbb{N} \to \mathcal{B}(X)$, $A(n) = A_n$ is a given sequence.
For $(m, n) \in \Delta$ we define

$$A_m^n := \begin{cases} A_{m-1}...A_n, & m > n \\ I, & m = n. \end{cases}$$

It is obvious that

$$A_m^n A_n^p = A_m^p \quad \textit{for all } (m,n) \textit{ and } (n,p) \in \Delta,$$

and every solution of $(\mathcal{A})$ satisfies

$$x_m = A_m^n x_n \quad \textit{for all } (m,n) \in \Delta.$$

Definition 3.1. A sequence $P : \mathbb{N} \to \mathcal{B}(X)$, $P(n) = P_n$ is called a *projections sequence on* X, if

$$P_n^2 = P_n \quad \textit{for every } n \in \mathbb{N}.$$

Remark 3.1. If (P_n) is a projections sequence on X then

$$Q_n = I - P_n, \quad n \in \mathbb{N}$$

is also a projections sequence on X, which is called the *complementary of projections sequence* (P_n).

Definition 3.2. A projections sequence (P_n) is called *invariant* for the system $(\mathcal{A})$ if

$$A_n P_n = P_{n+1} A_n \quad \textit{for every } n \in \mathbb{N}.$$

Remark 3.2. If (P_n) is invariant for $(\mathcal{A})$ then and its complementary (Q_n) is also invariant for $(\mathcal{A})$.
Furthermore, we have that

$$A_m^n P_n = P_m A_m^n \quad and \quad A_m^n Q_n = Q_m A_m^n \ for \ all \ (m,n) \in \Delta.$$

In what follows we consider a pair $(\mathcal{A}, P)$ where $P = (P_n)$ is a projections sequence which is invariant for $(\mathcal{A})$ and $h = (h_n)$ is a growth rate

Definition 3.3. The pair $(\mathcal{A}, P)$ is called $h-$ *dichotomic*, if there exists a nondecreasing sequence (s_n) with $s_n \geq 1$, such that

$$(hd_1) \quad h_m \|A_m^n P_n x\| \leq h_n s_n \|P_n x\|$$

and

$$(hd_2) \quad h_m \|Q_n x\| \leq h_n s_m \|A_m^n Q_n x\|$$

for all $(m, n, x) \in T$.

In particular, for

- (i) $h_n = e^{n\alpha}$ with $\alpha > 0$ we obtain the property of *(nonuniform) exponential dichotomy* and furthermore if (s_n) is constant it results the concept of *uniform exponential dichotomy*;
- (ii) $h_n = (n+1)^\alpha$ with $\alpha > 1$ we obtain the property of *(nonuniform) polynomial dichotomy* and furthermore if (s_n) is constant it results the concept of *uniform polynomial dichotomy.*

A general example of h-dichotomic pair $(\mathcal{A}, P)$ is presented by

Example 3.1. Let (s_n) be a nondecreasing sequence of positive real numbers $s_n \geq 1$ and let (h_n) be a growth rate.

Let (P_n) be a projections sequence on the Banach space X with the property

$$P_{n+1} P_n = P_n \quad for \ every \ n \in \mathbb{N}.$$

The linear discrete-time system $(\mathcal{A})$ defined by the sequence of linear bounded operators

$$A_n = \frac{h_n s_n}{h_{n+1} s_{n+1}} P_n + \frac{h_{n+1} s_{n+1}}{h_n s_n} Q_n$$

(where $Q_n = I - P_n$) has the property

$$A_n P_n = P_{n+1} A_n = \frac{h_n s_n}{h_{n+1} s_{n+1}} P_n$$

and hence (P_n) is invariant for $(\mathcal{A})$.

Moreover, we have that

$$A_m^n = \frac{h_n s_n}{h_m s_m} P_n + \frac{h_m s_m}{h_n s_n} Q_n,$$

$$h_m \|A_m^n P_n x\| = \frac{h_n s_n}{s_m} \|P_n x\| \leq h_n s_n \|P_n x\|,$$

$$h_n s_m \|A_m^n Q_n x\| = \frac{h_m s_m^2}{s_n} \|Q_n x\| \geq h_m \|Q_n x\|$$

for all $(m, n, x) \in T$.

Finally, it results that the pair $(\mathcal{A}, P)$ is h-dichotomic.

Remark 3.3. The previous example shows that for every projections sequence (P_n) with $P_{n+1}P_n = P_n$ and for every growth rate there exists a linear discrete-time sistem $(\mathcal{A})$ such that (P_n) is invariant for $(\mathcal{A})$ and $(\mathcal{A}, P)$ is h-dichotomic.

4. **h-dichotomy of Datko type**

Let (h_n) be a growth rate which satisfies the hypothesis (H) and let (k_n) be a growth rate given by Definition 2.2.
We consider a pair $(\mathcal{A}, P)$ where $(\mathcal{A})$ is a linear discrete-time system and $P = (P_n)$ a projections sequence invariant for $(\mathcal{A})$.
We denote by $Q_n = I - P_n$.

A necessary condition for h-dichotomy is given by

Theorem 4.1. *If the growth rate h satisfies the hypothesis (H) and the pair $(\mathcal{A}, P)$ is h-dichotomic then there exists a nondecreasing sequence $d_n \geq 1$ such that*

$$(kD_1) \qquad \sum_{j=n+1}^{\infty} k_j \|A_j^n P_n x\| \leq d_n k_n \|P_n x\| \quad for\ every\ (n,x) \in \mathbb{N} \times X$$

and

$$(kD_2) \qquad \sum_{j=n}^{m-1} \frac{\|A_j^n Q_n x\|}{k_j} \leq \frac{d_m \|A_m^n Q_n x\|}{k_m} \quad for\ all\ (m,n,x) \in T\ with\ m > n,$$

where (k_n) is a growth rate given by Definition 2.2.

Proof. Let (d_n) be the sequence defined by

$$d_n = M \frac{h_n s_n}{k_n},$$

where $M > 1$ is given by Definition 2.2 and (s_n) is the sequence from Definition 3.3.
(kD_1) Using (hd_1) we obtain

$$\sum_{j=n+1}^{\infty} k_j \|A_j^n P_n x\| \leq h_n s_n \|P_n x\| \sum_{j=n+1}^{\infty} \frac{k_j}{h_j} \leq d_n k_n \|P_n x\|$$

for all $(n,x) \in \mathbb{N} \times X$.
(kD_2) Similarly, by (hd_2) it results

$$\begin{aligned}
\sum_{j=n}^{m-1} \frac{\|A_j^n Q_n x\|}{k_j} &= \sum_{j=n}^{m-1} \frac{\|Q_j A_j^n x\|}{k_j} \\
&\leq \frac{s_m}{h_m} \|A_m^n Q_n x\| \sum_{j=n}^{m-1} \frac{h_j}{k_j} \\
&\leq M \frac{s_m h_m}{k_m^2} \|A_m^n Q_n x\| \\
&= \frac{d_m \|A_m^n Q_n x\|}{k_m}
\end{aligned}$$

for all $(m,n,x) \in T$ with $m > n$. $\square$

A necessary condition for polinomial dichotomy is given by

Corollary 4.1. *If the pair $(\mathcal{A}, P)$ is polynomially dichotomic then there are a nondecreasing sequence $d_n \geq 1$ and a constant $d > 0$ such that*

$$(pD_1) \qquad \sum_{j=n+1}^{\infty} (j+1)^d \|A_j^n P_n x\| \leq d_n (n+1)^d \|P_n x\| \quad \text{for every } (n,x) \in \mathbb{N} \times X$$

and

$$(pD_2) \qquad \sum_{j=n}^{m-1} (j+1)^{-d} \|A_j^n Q_n x\| \leq d_m (m+1)^{-d} \|A_m^n Q_n x\| \quad \text{for all } (m,n,x) \in T \text{ with } m > n.$$

Proof. It is a particular case of Theorem 4.1, for

$$h_n = (n+1)^\alpha, k_n = (n+1)^d \quad \text{where } d = \beta - 1 \text{ and } 1 < \beta < \alpha$$

(see and Exemple 2.2). □

Now, we introduce the following

Definition 4.1. We say that the pair $(\mathcal{A}, P)$ admits a *h-dichotomy of Datko type* if there exists a nondecreasing sequence $d_n \geq 1$ such that

$$(hD_1) \qquad \sum_{j=n+1}^{\infty} h_j \|A_j^n P_n x\| \leq d_n h_n \|P_n x\| \quad \text{for every } (n,x) \in \mathbb{N} \times X$$

and

$$(hD_2) \qquad \sum_{j=n}^{m-1} \frac{\|A_j^n Q_n x\|}{h_j} \leq \frac{d_m \|A_m^n Q_n x\|}{h_m} \quad \text{for all } (m,n,x) \in T \text{ with } m > n.$$

In particular for

- (i) $h_n = e^{n\alpha}$ with $\alpha > 0$ we obtain the concept of *exponential dichotomy of Datko type*
- (ii) $h_n = (n+1)^\alpha$ with $\alpha > 1$ it results the concept of *polynomial dichotomy of Datko type.*

Remark 4.1. Theorem 4.1 shows that if h satisfies the hypothesis (H) and the pair $(\mathcal{A}, P)$ is h-dichotomic then $(\mathcal{A}, P)$ admits a h-dichotomy of Datko type.

A converse of this implication is given by

Theorem 4.2. *If the pair $(\mathcal{A}, P)$ admits a h-dichotomy of Datko type then it is h-dichotomic.*

Proof. (hd_1) From (hD_1) we obtain

$$h_m \|A_m^n P_n x\| \leq d_n h_n \|P_n x\| \quad \text{for all } (m,n,x) \in T$$

and hence (hd_1) is verified.
(hd_2) The inequality (hD_2) implies (for j=n)

$$h_m \|Q_n x\| \leq h_m d_n \|A_m^n Q_n x\| \quad \text{for all } (m,n,x) \in T \text{ with } m > n.$$

The last inequality is true and for $m = n$ and hence the inequality (hd_2) is satisfied.
Finally, we obtain that $(\mathcal{A}, P)$ is h-dichotomic. □

A sufficient condition for polynomial dichotomy is

Corollary 4.2. *If there exist a nondecreasing sequence $d_n > 1$ and a constant $d > 1$ such that the inequalities (pD_1) respectively (pD_2) from Corollary 4.1 are satisfied for all $(n, x) \in \mathbb{N} \times X$ respectively for every $(m, n, x) \in T$ with $m > n$ then $(\mathcal{A}, P)$ is polynomially dichotomic.*

Proof. It is a particular case of Theorem 4.2 for $h_n = (n+1)^d$ with $d > 1$. □

A characterization of exponential dichotomy is given by

Corollary 4.3. *The pair $(\mathcal{A}, P)$ is exponentially dichotomic if and only if there exist a constant $d > 0$ and a nondecreasing sequence (d_n) such that*

$$(eD_1) \qquad \sum_{j=n+1}^{\infty} e^{jd} \|A_j^n P_n x\| \leq d_n e^{nd} \|P_n x\| \quad \text{for all } (n, x) \in \mathbb{N} \times X$$

and

$$(eD_2) \qquad \sum_{j=n}^{m-1} e^{-jd} \|A_j^n Q_n x\| \leq d_m e^{-md} \|A_m^n Q_n x\| \quad \text{for all } (m, n, x) \in T \text{ with } m > n.$$

Proof. Necesitty. It results from Theorem 4.1 for the particular case

$$h_n = e^{n\alpha}, k_n = e^{nd} \quad \text{with } 0 < d < \alpha$$

(see and Example 2.1).

Sufficiency. By Theorem 4.2 for $h_j = e^{jd}$ it follows that $(\mathcal{A}, P)$ is exponentially dichotomic. □

Remark 4.2. The previous corollary shows that the concepts of exponential dichotomy and exponential dichotomy of Datko type are equivalent.

5. Lyapunov functions for h-dichotomy

We consider a linear discrete-time system $(\mathcal{A})$ on a Banach space X, (P_n) a projections sequence which is a invariant for $(\mathcal{A})$ and $h = (h_n)$ a growth rate. We denote $Q_n = I - P_n$.

Definition 5.1. We say that $L : T \to \mathbb{R}_+$ is a *h-Lyapunov function* for the pair $(\mathcal{A}, P)$ if

$$(hL_1) \quad L(m, p, P_p x) + \sum_{j=n}^{m-1} \frac{h_j}{h_n} \|A_j^p P_p x\| + \sum_{j=p}^{m-1} \frac{h_m}{h_j} \|A_j^p Q_p x\| \leq L(n, p, P_p x)$$

for all $(m, n, p, x) \in \mathbb{N}^3 \times X$ with $m > n \geq p$;

$$(hL_2) \quad L(n, p, Q_p x) + \sum_{j=n}^{m-1} \frac{h_m}{h_j} \|A_j^p Q_p x\| \leq L(m, p, Q_p x)$$

for all $(m, n, p, x) \in \mathbb{N}^3 \times X$ with $m > n \geq p$.

In particular for

- (i) $h_n = e^{n\alpha}$ cu $\alpha > 0$ a h-Lyapunov function is called an *exponential Lyapunov function*
- (ii) $h_n = (n+1)^\alpha$ cu $\alpha > 0$ a h-Lyapunov function is called an *polinomiall Lyapunov function.*

The main result of this section is

Theorem 5.1. *The pair $(\mathcal{A}, P)$ admits a h-dichotomy of Datko-type if and only if there exists a h-Lyapunov function L for $(\mathcal{A}, P)$ and a nondecreasing sequence (d_n) such that*

$$(l_1) \quad L(n, n, P_n x) \leq d_n \|P_n x\|$$

and

$$(l_2) \quad L(m, n, Q_n x) \leq d_m \|A_m^n Q_n x\|$$

for all $(m, n, x) \in T$.

Proof. Necessity. Suppose that $(\mathcal{A}, P)$ admits a h-dichotomy of Datko type.
We define $L : T \to \mathbb{R}_+$ by

$$L(m, n, x) = \sum_{j=m+1}^{\infty} \frac{h_j}{h_n} \|A_j^n P_n x\| + \sum_{j=n}^{m-1} \frac{h_m}{h_j} \|A_j^n Q_n x\| \ \textit{for}\ (m, n, x) \in T \textit{ with } m > n$$

and

$$L(n, n, x) = \sum_{j=n+1}^{\infty} \frac{h_j}{h_n} \|A_j^n P_n x\| \ \textit{for}\ (n, x) \in \mathbb{N} \times X.$$

It remains to verify that L satisfies the properties (hL_1), (hL_2), (l_1) and (l_2).

$$(hL_1) \qquad L(m, p, P_p x) + \sum_{j=n+1}^{m} \frac{h_j}{h_n} \|A_j^p P_p x\| = \sum_{j=m+1}^{\infty} \frac{h_j}{h_p} \|A_j^p P_p x\| + \sum_{j=p}^{m-1} \frac{h_n}{h_j} \|A_j^p P_p x\| + \sum_{j=n+1}^{m} \frac{h_j}{h_n} \|A_j^p P_p x\| \leq$$

$$\leq \sum_{j=n+1}^{\infty} \frac{h_j}{h_p} \|A_j^p P_p x\| + \sum_{j=p}^{m-1} \frac{h_m}{h_j} \|A_j^p Q_p x\| = L(n, p, P_p x)$$

for all $(m, n, p, x) \in \mathbb{N}^3 \times X$ with $m > n \geq p$.

$$(hL_2) \qquad L(n, p, Q_p x) + \sum_{j=n}^{m-1} \frac{h_n}{h_j} \|A_j^p Q_p x\| = \sum_{j=p}^{n-1} \frac{h_n}{h_j} \|A_j^p Q_p x\| + \sum_{j=n}^{m-1} \frac{h_n}{h_j} \|A_j^p Q_p x\| \leq$$

$$\leq \sum_{j=p}^{m-1} \frac{h_m}{h_j} \|A_j^p Q_p x\| = L(m, p, Q_p x)$$

for every $(m, n, p, x) \in \mathbb{N}^3 \times X$ with $m > n \geq p$.
(l_1) Using (hD_1) from Theorem 4.1 we obtain

$$L(n, n, P_n x) \leq \sum_{j=n+1}^{\infty} \frac{h_j}{h_n} \|A_j^n P_n x\| \leq d_n \|P_n x\|$$

for every $(n, x) \in \mathbb{N} \times X$.
(l_2) Similary, by (hD_2) from Theorem 4.2 we obtain

$$L(m, n, Q_n x) = \sum_{j=n}^{m-1} \frac{h_m}{h_j} \|A_j^n Q_n x\| \leq d_m \|A_m^n Q_n x\|$$

for all $(m, n, x) \in T$ with $m > n$.
It is obvious that (l_2) is also true for $m = n$.
Sufficiency. Suppose that there exists a h-Lyapunov function L for $(\mathcal{A}, P)$ with the properties (l_1) and (l_2).

Then from (hL_1) (for $p = n$) and (l_1) it results that there exists a nondecreasing sequence (d_n) such that

$$\sum_{j=n+1}^{m} h_j \|A_j^n P_n x\| \leq L(n, n, P_n x) \leq d_n \|P_n x\|$$

for all $(m, n, x) \in T$ with $m > n$.
For $m \to \infty$, we obtain the inequality (hD_1).

Similarly, from (hL_2) (for $p = n$) and (l_2) it follows

$$\sum_{j=n}^{m-1} h_m \frac{\|A_j^n Q_n x\|}{h_j} \leq L(m, n, Q_n x) \leq d_m \|A_m^n Q_n x\|$$

for all $(m, n, x) \in T$ with $m > n$.
Thus and the inequality (hD_2) is proved.

Finally, we obtain that $(\mathcal{A}, P)$ admits a h-dichotomy of Datko type. □

A sufficient condition for h-dichotomy in terms of h-Lyapunov functions is given by

Corollary 5.1. *If there exists a h-Lyapunov function L for $(\mathcal{A}, P)$ with the properties (l_1) and (l_2) then $(\mathcal{A}, P)$ is h-dichotomic.*

Proof. It is an immediate consequence of Theorems 5.1 and 4.2. □

A necessary condition for h-dichotomy in terms of Lyapunov functions is

Corollary 5.2. *Let h be a growth rate which satisfies the hypothesis (H). If the pair $(\mathcal{A}, P)$ is h-dichotomic then there exists a k-Lyapunov function L for $(\mathcal{A}, P)$ with the properties (l_1) and (l_2), where k is a growth rate given by Definition 2.2.*

Proof. It results from Remark 4.1 and Theorem 5.1. □

A characterization of (nonuniform) exponential dichotomy in terms of Lyapunov functions is given by

Corollary 5.3. *The pair $(\mathcal{A}, P)$ is exponentially dichotomic if and only if there exists an exponential Lyapunov function L for $(\mathcal{A}, P)$ with the properties (l_1) and (l_2).*

Proof. It is a consequence of Remark 4.2 and Theorem 5.1. □

A necessary condition for polynomial dichotomy in terms of Lyapunov function is

Corollary 5.4. *If the pair $(\mathcal{A}, P)$ is polynomially dichotomic then there exists a polynomial Lyapunov function L for $(\mathcal{A}, P)$ with the properties (l_1) and (l_2).*

Proof. It is a particular case of Corollary 5.2 for

$$h_n = (n+1)^\alpha \quad \textit{with } \alpha > 1.$$

□

Corollary 5.5. *If the pair $(\mathcal{A}, P)$ has the property that there exists a h-Lyapunov function L for $(\mathcal{A}, P)$ with the properties (l_1) and (l_2), where*

$$h_n = (n+1)^\alpha \quad and\ \alpha > 1,$$

(i.e. L is a polynomial Lyapunov function for $(\mathcal{A}, P)$ then $(\mathcal{A}, P)$ is polynomially dichotomic.)

Proof. It results from Theorem 4.1 and Theorem 4.2 for the particular case

$$h_n = (n+1)^\alpha \quad with\ \alpha > 1.$$

□

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