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GENERALIZED COMPOSITE VECTOR EQUILIBRIUM PROBLEMS

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Abstract. In this paper, we study the weak and strong versions of generalized composite vector equilibrium problem in Hausdorff topological vector spaces. The generalized KKM-theorem due to [3] is applied to solve both the problems in non-compact setting. Some special cases are listed and examples are constructed.

1. Introduction

Equilibrium problems studied by Blum and Oettli [2] have wast applications in many branches of basic, pure and applied sciences. The equilibrium problems are more general than many known problems such as mathematical programming problems, variational inequalities problems, complementarity problems and fixed point problems etc., see for example [12, 13].

Let X and Y be two nonempty sets and $f : X \times Y \to \mathbb{R}$ be a given function. The scalar equilibrium problem is to find $x \in X$ such that

$$f(x, y) \geq 0, \ \textit{for all } y \in Y. \tag{1.1}$$

Problem (1.1) is studied extensively by many authors, see for example [2, 4, 5, 6, 11].

The vector variational inequality was introduced and studied by Giannessi [10] and further extended by Chen and Yang [8]. An inspiration is developed by the applications of notion of vector variational inequalites to study vector equilibrium problems.

If the scalar function f in problem (1.1) is replaced by a vector-valued function, say $f : X \times Y \to Z$, where Z is a topological vector space, partially ordered by a convex cone $C \subseteq Z$ with $intC \neq \phi$, then vector equilibrium problem is considered in two ways:

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$$\textit{Find } x \in X \textit{ such that } f(x,y) \notin -C \setminus \{0\}, \textit{ for all } y \in Y, \tag{1.2}$$

and

$$\textit{Find } x \in X \textit{ such that } f(x,y) \notin -intC, \textit{ for all } y \in Y. \tag{1.3}$$

Problem (1.2) is called strong vector equilibrium problem, while the problem (1.3) is called weak vector equilibrium problem.

Motivated by the applications of vector equilibrium problems, in this work, we introduce and study the strong as well as weak versions of generalized composite vector equilibrium problems in non-compact setting. Some existence results are established for both the problems and some examples are constructed.

2. Preliminaries and formulation of the problems

Let X and Y be two Hausdorff topological vector spaces, K a nonempty, closed and convex subset of X and C a pointed, closed and convex cone in Y with $intC \neq \phi$. The partial order induced by the cone C in Y is defined as

$$y_1 \leq y_2 \Leftrightarrow y_2 - y_1 \in C, \textit{ for all } y_1, y_2 \in Y.$$

Let us denote the space of all continuous linear operators from X into Y by $L(X,Y)$ and for $q \in L(X,Y)$, $\langle q, x\rangle$ means the value of q at $x \in X$.

Now, we state two problems to be studied in this paper.
Let $f, g : K \to K$, $S, T : K \to L(X,Y)$, $h : K \times K :\to Y$ and $A : K \times K \to X$ be the mappings.

Find $f(x) \in K$ such that for all $f(y), g(y) \in K$, we have

$$h(f(x), g(y)) + \langle T(f(x)) + S(g(x)), A(f(y), g(x))\rangle \notin -C \setminus \{0\}, \tag{2.1}$$

and

$$h(f(x), g(y)) + \langle T(f(x)) + S(g(x)), A(f(y), g(x))\rangle \notin -intC. \tag{2.2}$$

Problem (2.1) is called strong generalized composite vector equilibrium problem, while problem (2.2) is called weak generalized composite vector equilibrium problem.

Special cases:

(i) If $S = 0$ and $f, g = I$, the identity mapppings, then problem (2.1) and (2.2) reduces to the following problems considered by Rahaman et al. [16].

Find $x \in K$ such that

$$\begin{cases} h(x,y) + \langle T(x), \eta(y,x)\rangle \notin -C \setminus \{0\}, \ \forall y \in K, \\ h(x,y) + \langle T(x), \eta(y,x)\rangle \notin -intC, \ \forall y \in K. \end{cases}$$

(ii) If $S = 0$, $f, g = I$ and $A(y,x) = y - x$, then problem (2.1) and (2.2) reduces to the following problems considered by Rahaman and Ahmad [15].

Find $x \in K$ such that

$$\begin{cases} h(x,y) + \langle T(x), y - x\rangle \notin -C \setminus \{0\}, \forall y \in K, \\ h(x,y) + \langle T(x), y - x\rangle \notin -intC, \ \forall y \in K. \end{cases}$$

Let us recall some known definitions and results that are essential to prove the main results of this paper.

Definition 2.1. *A set-valued mapping $g : X \to 2^Y$ is said to be*

(i) *lower semicontinuous with respect to C at a point $x_0 \in K$, if for any neighbourhood V of $g(x_0)$ in Y, there exists a neighbourhood U of x_0 in X such that*

$$g(U \cap K) \subseteq V + C.$$

(ii) *upper semicontinuous with respect to C at a point $x_0 \in K$, if*

$$g(U \cap K) \subseteq V - C.$$

(iii) *continuous with respect to C at a point $x_0 \in K$, if it is lower semicontinuous and upper semicontinuous with respect to C at that point.*

Remark. *If g is lower semicontinuous, upper semicontinuous and continuous with respect to C at any arbitrary point of K, then g is lower semicontinuous, upper semicontinuous and continuous with respect to C on K, respectively.*

Definition 2.2. *Let $f, g : K \to K$, $T, S : K \to L(X, Y)$ and $A : K \times K \to X$ be the mappings. Then*

(i) *T and S are said to be mixed C-A-pseudomonotone with respect to f and g, if for any $x, y \in K$,*

$$\begin{aligned} \langle T(f(x)) + S(g(x)), A(f(y), g(x))\rangle &\notin -intC \\ \textit{implies } \langle T(f(y)) + S(g(y)), A(f(y), g(x))\rangle &\notin -intC. \end{aligned}$$

(ii) *T and S are said to be mixed strongly C-A-pseudomonotone with respect to f and g, if for any $x, y \in K$,*

$$\begin{aligned} \langle T(f(x)) + S(g(x)), A(f(y), g(x))\rangle &\notin -C \setminus \{0\} \\ \textit{implies } \langle T(f(y)) + S(g(y)), A(f(y), g(x))\rangle &\in C. \end{aligned}$$

(iii) *A is said to be affine in the first argument with respect to f, if for any $x_i \in K$, $\lambda_i \geq 0$, $1 \leq i \leq n$ with $\sum_{i=1}^{n} \lambda_i = 1$ and any $y \in K$, we have*

$$A(\sum_{i=1}^{n} \lambda_i f(x_i), g(y)) = \sum_{i=1}^{n} \lambda_i A(f(x_i), g(y)).$$

(iv) *T and S are said to be mixed A-hemicontinuous with respect to f and g, if for any $x, y \in K$ and $\lambda \in (0, 1]$, the mapping $\lambda \to \langle T(f(x) + \lambda(f(y) - f(x))) + S(g(x) + \lambda(g(y) - g(x))), A(f(y), g(x))\rangle$ is continuous at 0^+.*

Definition 2.3. [3] *Consider a subset K of a topological vector space X and a topological space Y. A family $\{(C_i, Z_i)\}_{i \in I}$ of pair of sets is said to be coercing for a mapping $F : K \to 2^Y$ if and only if*

(i) *for each $i \in I$, C_i is contained in a compact convex subset of K and Z_i is a compact subsets of Y,*

(ii) *for each $i, j \in I$, there exists $k \in I$ such that $C_i \bigcup C_j \subseteq C_k$,*

(iii) *for each $i \in I$, there exists $k \in I$ with $\bigcap_{x \in C_k} F(x) \subseteq Z_i$.*

Remark. *In case, where the coercing family reduced to single element, condition* (iii) *of Definition* 2.3 *appeared first in this generality (with two sets* C *and* Z*) in* [7] *and generalizes the condition of Karamardian* [14] *and Allen* [1]*. Condition* (iii) *is also an extension of coercivity condition given by Fan* [9].

Definition 2.4. *Let* K *be a nonempty convex subset of a topological vector space* X*. A set-valued mapping* $F : K \to 2^X$ *is said to be KKM-mapping, if for any finite* $\{x_1, x_2, \cdots, x_n\} \subseteq K$, $Co\{x_1, x_2, \cdots, x_n\} \subseteq \bigcup_{i=1}^{n} F(x_i)$, *where* $Co\{x_1, x_2, \cdots, x_n\}$ *denots the convex hull of* $\{x_1, x_2, \cdots, x_n\}$.

Theorem 2.1. [3] *Let* X *be a Hausdorff topological vector space,* Y *a convex subset of* X, K *a nonempty subset of* Y *and* $F : K \to 2^Y$ *is a KKM-mapping with compactly closed values in* Y *(i.e., for all* $x \in K, F(x) \bigcap Z$ *is closed for every compact set* Z *of* Y*). If* F *admits a coercing family, then*

$$\bigcap_{x \in K} F(x) \neq \phi.$$

Lemma 2.2. [17] *Let* X *be a Hausdorff topological space and* $\{A_i\}_{i \in I}$ *a nonempty compact convex subsets of* X*. Then* $Co\{A_i : i \in I\}$ *is compact.*

In support of (i) and (ii) of Definition 2.2, we construct the following example.

Example 2.1. *Let* $X = \mathbb{R}, K = \mathbb{R}_+, Y = \mathbb{R}^2, C = \mathbb{R}^2_+$*. Let* $f, g : K \to K$, $T, S : K \to L(X, Y)$ *and* $A : K \times K \to X$ *be the mappings defined by*

$$f(x) = 2x,\ g(x) = \frac{x}{2},\ T(f(x)) = (f(x), 2f(x)),$$
$$S(g(x)) = (2g(x), 6g(x)) \text{ and } A(f(y), g(x)) = f(y) - 2g(x).$$

Then

$$\begin{aligned} \langle T(f(x)) + S(g(x)), A(f(y), g(x))\rangle &= \langle (f(x), 2f(x)) \\ &+ (2g(x), 6g(x)), (f(y) - 2g(x))\rangle \\ &= \langle (2x, 4x) + (x, 3x), (2y - x)\rangle \\ &= \langle (3x, 7x), (2y - x)\rangle \\ &= (3x, 7x)(2y - x) \notin -intC, \end{aligned}$$

implies that $2y > x$*, so it follows that*

$$\begin{aligned} \langle T(f(y)) + S(g(y)), A(f(y), g(x))\rangle &= \langle (f(y), 2f(y)) \\ &+ (2g(y), 6g(y)), (f(y) - 2g(x))\rangle \\ &= \langle (2y, 4y) + (y, 3y), (2y - x)\rangle \\ &= \langle (3y, 7y), (2y - x)\rangle \\ &= (3y, 7y)(2y - x) \notin -intC, \end{aligned}$$

which shows that

$$\begin{aligned} &\langle T(f(x)) + S(g(x)), A(f(y), g(x))\rangle \notin -intC \\ \text{implies}\quad &\langle T(f(y)) + S(g(y)), A(f(y), g(x))\rangle \notin -intC, \end{aligned}$$

That is, T and S are mixed C-A-pseudomonotone with respect to f and g. Further,

$$\begin{aligned}\langle T(f(x)) + S(g(x)), A(f(y), g(x))\rangle &= \langle (f(x), 2f(x)) \\ &+ (2g(x), 6g(x)), (f(y) - 2g(x))\rangle \\ &= \langle (2x, 4x) + (x, 3x), (2y - x)\rangle \\ &= \langle (3x, 7x), (2y - x)\rangle \\ &= (3x, 7x)(2y - x) \notin -C \setminus \{0\},\end{aligned}$$

implies that $2y > x$, so it follows that

$$\begin{aligned}\langle T(f(y)) + S(g(y)), A(f(y), g(x))\rangle &= \langle (f(y), 2f(y)) \\ &+ (2g(y), 6g(y)), (f(y) - 2g(x))\rangle \\ &= \langle (2y, 4y) + (y, 3y), (2y - x)\rangle \\ &= \langle (3y, 7y), (2y - x)\rangle \\ &= (3y, 7y)(2y - x) \in C,\end{aligned}$$

which shows that

$$\begin{aligned} &\langle T(f(x)) + S(g(x)), A(f(y), g(x))\rangle \notin C \setminus \{0\} \\ \textit{implies}\quad &\langle T(f(y)) + S(g(y)), A(f(y), g(x))\rangle \in C.\end{aligned}$$

That is, T and S are mixed strongly C-A-pseudomonotone with respect to f and g.

3. Existence result

In this section, we establish some existence results for strong generalized composite vector equilibrium problem (2.1) and weak generalized composite vector equilibrium problem (2.2).

Theorem 3.1. *Let $f, g : K \to K$, $T, S : K \to L(X, Y)$, $h : K \times K \to Y$ and $A : K \times K \to X$ be the mappings satisfying the following conditions:*

- (i) *f and g are linear, continuous and converges pointwise,*
- (ii) *h is affine in second argument with respect to g and continuous in first argument with respect to f,*
- (iii) *$h(f(x), g(x)) = 0$, for all $x \in K$,*
- (iv) *$A(f(x), g(x)) = 0$ and $A(f(x), g(y)) + A(f(y), g(x)) = 0$, for all $x, y \in K$,*
- (v) *A is affine in both the arguments with respect to f and g, and continuous in second argument with respect to g,*
- (vi) *T and S are mixed A-hemicontinuous, mixed C-A-pseudomonotone and continuous.*

Then, the following two problems are equivalent:

$$\begin{aligned}(I)\quad &\textit{Find } f(x) \in K \textit{ such that for all } f(y), g(y) \in K, \textit{ we have} \\ &h(f(x), g(y)) + \langle T(f(x)) + S(g(x)), A(f(y), g(x))\rangle \notin -intC, \\ (II)\quad &\textit{Find } f(x) \in K \textit{ such that for all } f(y), g(y) \in K, \textit{ we have} \\ &h(f(x), g(y)) - \langle T(f(y)) + S(g(y)), A(f(x), g(y))\rangle \notin -intC.\end{aligned}$$

Proof. Suppose that (I) holds. That is,

$$h(f(x), g(y)) + \langle T(f(x)) + S(g(x)), A(f(y), g(x))\rangle \notin -intC. \tag{3.1}$$

Since T and S are mixed C-A-pseudomonotone with respect to f and g, we have

$$h(f(x), g(y)) + \langle T(f(y)) + S(g(y)), A(f(y), g(x))\rangle \notin -intC. \tag{3.2}$$

Also from assumptions (iv) and (3.2), we get

$$h(f(x), g(y)) - \langle T(f(y)) + S(g(y)), A(f(x), g(y))\rangle \notin -intC, \tag{3.3}$$

that is, (II) holds.
Conversely, suppose that (II) holds. That is,

$$h(f(x), g(y)) - \langle T(f(y)) + S(g(y)), A(f(x), g(y))\rangle \notin -intC. \tag{3.4}$$

For any fixed $y \in K$, letting $x_\lambda = \lambda y + (1-\lambda)x$, for $\lambda \in [0,1]$. Obviously $x_\lambda \in K$ and it follows that

$$h(f(x), g(x_\lambda)) - \langle T(f(x_\lambda)) + S(g(x_\lambda)), A(f(x), g(x_\lambda))\rangle \notin -intC. \tag{3.5}$$

Multiplying (3.5) by $(1-\lambda)$, we have

$$(1-\lambda)h(f(x), g(x_\lambda)) - (1-\lambda)\langle T(f(x_\lambda)) + S(g(x_\lambda)), A(f(x), g(x_\lambda))\rangle \notin -intC. \tag{3.6}$$

Since f and g are linear, A is affine in both arguments with respect to f and g, and $A(f(x), g(x)) = 0$, we have

$$\begin{aligned} 0 &= \langle T(f(x_\lambda)) + S(g(x_\lambda)), A(f(x_\lambda), g(x_\lambda))\rangle \\ &= \langle T(f(x_\lambda)) + S(g(x_\lambda)), A(f(y + (1-\lambda)x), g(x_\lambda))\rangle \\ &= \lambda\langle T(f(x_\lambda)) + S(g(x_\lambda)), A(f(y), g(x_\lambda))\rangle \\ &\quad +(1-\lambda)\langle T(f(x_\lambda)) + S(g(x_\lambda)), A(f(x), g(x_\lambda))\rangle, \end{aligned}$$

which implies that

$$\begin{aligned} &-(1-\lambda)\langle T(f(x_\lambda)) + S(g(x_\lambda)), A(f(x), g(x_\lambda))\rangle \\ &\quad = \lambda\langle T(f(x_\lambda)) + S(g(x_\lambda)), A(f(y), g(x_\lambda))\rangle. \end{aligned} \tag{3.7}$$

Since $(1-\lambda)h(f(x), g(x_\lambda)) \in Y$, adding $(1-\lambda)h(f(x), g(x_\lambda))$ on both side of (3.7), we obtain

$$\begin{aligned} &(1-\lambda)h(f(x), g(x_\lambda)) - (1-\lambda)\langle T(f(x_\lambda)) + S(g(x_\lambda)), A(f(x), g(x_\lambda))\rangle \\ &\quad = (1-\lambda)h(f(x), g(x_\lambda)) + \lambda\langle T(f(x_\lambda)) + S(g(x_\lambda)), A(f(y), g(x_\lambda))\rangle. \end{aligned} \tag{3.8}$$

Combining (3.6) and (3.8), we get

$$(1-\lambda)h(f(x), g(x_\lambda)) + \lambda\langle T(f(x_\lambda)) + S(g(x_\lambda)), A(f(y), g(x_\lambda))\rangle \notin -intC. \tag{3.9}$$

Since h is affine in second argument with respect to g and $h(f(x), g(x)) = 0$, (3.9) becomes

$$\lambda(1-\lambda)h(f(x), g(y)) + \lambda\langle T(f(x_\lambda)) + S(g(x_\lambda)), A(f(y), g(x_\lambda))\rangle \notin -intC. \tag{3.10}$$

Using (iv) and (v), (3.10) implies that

$$\lambda(1-\lambda)h(f(x), g(y)) + \lambda(1-\lambda)\langle T(f(x_\lambda)) + S(g(x_\lambda)), A(f(y), g(x))\rangle \notin -intC. \tag{3.11}$$

Dividing $\lambda(1-\lambda)$ of (3.11), we have

$$h(f(x), g(y)) + \langle T(f(x_\lambda)) + S(g(x_\lambda)), A(f(y), g(x))\rangle \notin -intC.$$

Using mixed A-hemicontinuity of T and S with respect to f and g, we obtain

$$h(f(x), g(y)) + \langle T(f(x)) + S(g(x)), A(f(y), g(x))\rangle \notin -intC.$$

That is, (I) holds. □

We prove an existence result for weak generalized composite vector equilibrium problem (2.2), using the concept of coercing family of pair of sets.

Theorem 3.2. *Let* $f, g : K \to K$, $T, S : K \to L(X, Y)$, $h : K \times K \to Y$ *and* $A : K \times K \to X$ *be the mappings such that all the assumptions* (i)-(vi) *of Theorem* 3.1 *are satisfied. In addition, assume that the following conditions are satisfied:*

- (vii) *the mapping* $W : K \to 2^Y$ *defined by* $W = Y/\{-intC\}$ *is upper semicontinuous on* K,
- $(viii)$ *there a family* $\{C_i, Z_i\}_{1\le i\le n}$ *satisfying condition* (i) *and* (ii) *of Definition* 2.3 *and the following condition for each* $1 \le i \le n$, *there exist* $1 \le k \le n$ *such that*

$$\{f(x) \in K : h(f(x), g(y)) + \langle T(f(x)) \\ +S(g(x)), A(f(y), g(x))\rangle \notin -intC, \quad \forall \quad g(y) \in C_k\} \subseteq Z_i.$$

Then, there exists $f(x) \in K$ *such that*

$$h(f(x), g(y)) + \langle T(f(x)) + S(g(x)), A(f(y), g(x))\rangle \notin -intC, \forall g(y) \in K.$$

Proof. For all $g(y) \in K$, consider the sets

$$M(g(y)) = \{f(x) \in K : h(f(x), g(y)) - \langle T(f(y)) + S(g(y)), A(f(x), g(y))\rangle \notin -intC\},$$
$$N(g(y)) = \{f(x) \in K : h(f(x), g(y)) + \langle T(f(x)) + S(g(x)), A(f(y), g(x))\rangle \notin -intC\}.$$

Then, $M(g(y))$ and $N(g(y))$ are nonempty sets as $g(y) \in M(g(y))$ and $g(y) \in N(g(y))$. First, we will prove that M is a KKM-mapping. Suppose that $M(g(y))$ is not a KKM-mapping, then there exists a finite subset $\{g(y_1), g(y_2), g(y_3), \cdots, g(y_n)\} \subseteq K$, $\lambda_i \ge 0$, for each $1 \le i \le n$ with $\sum_{i=1}^{n} \lambda_i = 1$ such that

$$f(w) = \sum_{i=1}^{n} \lambda_i g(y_i) \notin \bigcup_{i=1}^{n} M(g(y_i)).$$

That is,

$$h(f(w), g(y_i)) - \langle T(f(y_i)) + S(g(y_i)), A(f(w), g(y_i))\rangle \in -intC, \forall\ 1 \le i \le n.$$

As $intC$ is convex, therefore

$$\sum_{i=1}^{n} \lambda_i h(f(w), g(y_i)) - \sum_{i=1}^{n} \lambda_i \langle T(f(y_i)) + S(g(y_i)), A(f(w), g(y_i))\rangle \in -intC. \quad (3.12)$$

Since h is affine in the second argument with respect to g and A is affine in both the arguments with respect to f and g, form (3.12), we have

$$\begin{aligned} & h(f(w), g(w)) - \langle T(f(y_i)) + S(g(y_i)), A(f(w), g(w))\rangle \quad (3.13) \\ = \ & h(f(w), \sum_{i=1}^{n} \lambda_i g(y_i)) - \langle T(f(y_i)) + S(g(y_i)), A(f(w), \sum_{i=1}^{n} \lambda g(y_i))\rangle \\ = \ & \sum_{i=1}^{n} \lambda_i h(f(w), g(y_i)) - \sum_{i=1}^{n} \lambda_i \langle T(f(y_i)) + S(g(y_i)), A(f(w), g(y_i))\rangle \\ \in \ & -intC. \end{aligned}$$

By assumption (iii) and (iv), we know that $A(f(x), g(x)) = 0 = h(f(x), g(x))$. Then (3.13) implies that $0 \in -intC$, which is contradicts the pointedness of C and hence M is a KKM-mapping. Further, we prove that

$$\bigcap_{g(y)\in K} M(g(y)) = \bigcap_{g(y)\in K} N(g(y)). \tag{3.14}$$

Let $f(x) \in M(g(y))$, so that

$$h(f(x), g(y)) - \langle T(f(y)) + S(g(y)), A(f(x), g(y))\rangle \notin -intC. \tag{3.15}$$

Since T and S are mixed C-A-pseudomonotone and $A(f(x), g(y)) + A(f(y), g(x)) = 0$, then (3.15) implies that

$$h(f(x), g(y)) + \langle T(f(x)) + S(g(x)), A(f(y), g(x))\rangle \notin -intC \tag{3.16}$$

and so $f(x) \in N(g(y))$, for each $g(y) \in K$, that is $M(g(y)) \subseteq N(g(y))$ and hence

$$\bigcap_{g(y)\in K} M(g(y)) \subseteq \bigcap_{g(y)\in K} N(g(y)). \tag{3.17}$$

Conversely, suppose that $f(x) \in \bigcap_{g(y)\in K} N(g(y))$. Then

$$h(f(x), g(y)) + \langle T(f(x)) + S(g(x)), A(f(y), g(x))\rangle \notin -intC.$$

It follows from Theorem 3.1 that

$$h(f(x), g(y)) - \langle T(f(y)) + S(g(y)), A(f(x), g(y))\rangle \notin -intC, \tag{3.18}$$

that is, $f(x) \in M(g(y))$ and so

$$\bigcap_{g(y)\in K} N(g(y)) \subseteq \bigcap_{g(y)\in K} M(g(y)). \tag{3.19}$$

Combining (3.17) and (3.19), we obtain

$$\bigcap_{g(y)\in K} N(g(y)) = \bigcap_{g(y)\in K} M(g(y)). \tag{3.20}$$

Now, since M is KKM-mapping, for any finite subset $\{g(y_1), g(y_2), \cdots, g(y_n)\} \subseteq K$, we have

$$Co\{g(y_1), g(y_2), \cdots, g(y_n)\} \subseteq \bigcup_{i=1}^{n} M(g(y_i)) \subseteq \bigcup_{i=1}^{n} N(g(y_i)).$$

This implies that N is also a KKM-mapping. Next, we show that $N(g(y))$ is closed for all $g(y) \in K$. Let $\{f_\alpha(x)\}_{\alpha\in\Lambda}$ be a net in $N(g(y))$ such that f_α converges pointwise to f i.e., $\lim_{\alpha\to\infty} f_\alpha(x) \to f(x)$. Then

$$h(f_\alpha(x), g(y)) + \langle T(f_\alpha(x)) + S(g_\alpha(x)), A(f(y), g_\alpha(x))\rangle \notin -intC.$$

Since g is continuous, h is continuous in the first argument with respect to f, A is continuous in the second argument with respect to g, T and S are continuous with respect to f and g, we have

$$\begin{aligned} & h(f_\alpha(x), g(y)) + \langle T(f_\alpha(x)) + S(g_\alpha(x)), A(f(y), g_\alpha(x))\rangle \\ \to \quad & h(f(x), g(y)) + \langle T(f(x)) + S(g(x)), A(f(y), g(x))\rangle. \end{aligned}$$

As $W = Y/\{-intC\}$ is upper semicontinuous, we obtain

$$h(f(x), g(y)) + \langle T(f(x)) + S(g(x)), A(f(y), g(x))\rangle \in W.$$

Thus, we have

$$h(f(x), g(y)) + \langle T(f(x)) + S(g(x)), A(f(y), g(x))\rangle \notin -intC.$$

Therefore $f(x) \in N(g(y))$. Hence N is closed. By assumption $(viii)$ it follows that N has compactly closed values in K.

In view of assumption $(viii)$, we see that the family $\{(C_i, Z_i)\}_{1\leq i\leq n}$ satisfies the condition that is for all $1 \leq i \leq n$, there exists $1 \leq k \leq n$ such that

$$\bigcap_{g(y)\in C_k} N(g(y)) \subseteq Z_i,$$

and consequently, it is coercing family for N. Finally, we conclude that N satisfies all the conditions of Theorem 2.1 and thus, we have

$$\bigcap_{g(y)\in K} N(g(y)) \neq \phi.$$

Hence, there exists $f(x) \in \bigcap_{g(y)\in K} N(g(y))$ such that for all $g(y) \in K$,

$$h(f(x), g(y)) + \langle T(f(x)) + S(g(x)), A(f(y), g(x))\rangle \notin -intC.$$

This complete the proof. □

The following example ensures that all the conditions of Theorem 3.2 are fulfilled.

Example 3.1. *Let* $X = \mathbb{R}, K = \mathbb{R}_+, Y = \mathbb{R}^2, C = \mathbb{R}^2_+$. *Let* $f, g : K \to K$, $T, S : K \to L(X, Y)$, $h : K \times K :\to Y$ *and* $A : K \times K \to X$ *be the mappings defined by*

$$f(x) = 2x,\ \ g(x) = \frac{x}{2},\ \ T(f(x)) = (f(x), 2f(x)),\ \ S(g(x)) = (2g(x), 6g(x)),$$

$$h(f(x), g(y)) = (2f(x) - 8g(y), \frac{f(x)}{2} - 2g(y)) \text{ and } A(f(y), g(x)) = 4g(x) - f(y).$$

It is easy to check that

- (i) f *and* g *are linear and continous,*
- (ii) h *is affine in second argument with respect to* g *and continuous in first argument with respect to* f,
- (iii) $h(f(x), g(x)) = 0$, *for all* $x \in K$,
- (iv) $A(f(x), g(x)) = 0$ *and* $A(f(x), g(y)) + A(f(y), g(x)) = 0$, *for all* $x, y \in K$,
- (v) A *is affine in both arguments with respect to* f *and* g, *and continuous in second argument with respect to* g,
- (vi) T *and* S *are mixed* A*-hemicontinuous and continuous in first argument with respect to* f.

Now,

$$\begin{aligned}\langle T(f(x)) + S(g(x)), A(f(y), g(x))\rangle &= \langle (f(x), 2f(x)) \\ &\quad + (2g(x), 6g(x)), (4g(x) - f(y))\rangle \\ &= \langle (2x, 4x) + (x, 3x), (2x - 2y)\rangle \\ &= \langle (3x, 7x), (2x - 2y)\rangle \\ &= (3x, 7x)(2x - 2y) \notin -intC,\end{aligned}$$

implies that $x > y$, *and*

$$\begin{aligned}\langle T(f(y)) + S(g(y)), A(f(y), g(x))\rangle &= \langle (f(y), 2f(y)) \\ &\quad +(2g(y), 6g(y)), (4g(x) - f(y))\rangle \\ &= \langle (2y, 4y) + (y, 3y), (2x - 2y)\rangle \\ &= \langle (3y, 7y), (2x - 2y)\rangle \\ &= (3y, 7y)(2x - 2y) \notin -intC,\end{aligned}$$

which shows that

$$\begin{aligned}&\langle T(f(x)) + S(g(x)), A(f(y), g(x)) \notin -intC \\ \text{implies} \quad &\langle T(f(y)) + S(g(y)), A(f(y), g(x)) \notin -intC.\end{aligned}$$

Therefore, T *and* S *are mixed* C*-*A*-pseudomonotone with respect to* f *and* g. *Suppose* $W : K \to 2^Y$ *is a mapping defined by*

$$W(x) = (x, 2x), \forall x \in K.$$

It is easy to check that W *is upper semicontinous on* K. *Choosing* $C_i = [0, 2i]$ *and* $Z_i = [-2i, 2i] \times [-3i, 3i]$, $1 \le i \le n$. *It is easy to verify condition* $(viii)$ *of Theorem* 3.2 *Hence, all the conditions of Theorem* 3.2 *are fulfilled.*

Now, we show the existence of solution of weak generalized composite vector equilibrium problem (2.2). *Let* $f(x) = 2x \in K$ *such that*

$$\begin{aligned}h(f(x), g(y)) \quad &+ \quad \langle T(f(x)) + S(g(x)), A(f(y), g(x))\rangle \\ &= \quad (f(x) - 4g(y), \frac{f(x)}{2} - 2g(y)) + \langle (f(x), 2f(x)) \\ &+ \quad (2g(x), 6g(x)), (4g(x) - f(y))\rangle \\ &= \quad (4x - 4y, x - y) + \langle (2x, 4x) + (x, 3x), (2x - 2y)\rangle \\ &= \quad (4x - 4y, x - y) + \langle (3x, 7x), (2x - 2y)\rangle \\ &= \quad (4x - 4y, x - y) + (3x, 7x)(2x - 2y) \notin -intC, \text{ with } x > y.\end{aligned}$$

That is,

$$h(f(x), g(y)) + \langle T(f(x)) + S(g(x)), A(f(y), g(x)) \notin -intC.$$

Therefore, $f(x) = 2x$ *is a solution of weak generalized composite vector equilibrium problem* (2.2).

The following two results are related to the strong generalized composite vector equilibrium problem (2.1).

Theorem 3.3. *Let* $f, g : K \to K$, $T, S : K \to L(X, Y)$, $h : K \times K \to Y$ *and* $A : K \times K \to X$ *be the mappings such that all the assumptions* (i)-(v) *of Theorem* 3.1 *are satisfied. In addition, we assume that* T *and* S *are mixed strongly* C*-*A*-pseudomonotone and mixed* A*-hemicontinuous with respect to* f *and* g. *Then, the following two problems are equivalent:*

(I) *Find* $f(x) \in K$ *such that for all* $f(y), g(y) \in K$, *we have*
$$h(f(x), g(y)) + \langle T(f(x)) + S(g(x)), A(f(y), g(x))\rangle \notin -C \setminus \{0\}.$$

(II) *Find* $f(x) \in K$ *such that for all* $f(y), g(y) \in K$, *we have*
$$h(f(x), g(y)) + \langle T(f(y)) + S(g(y)), A(f(y), g(x))\rangle \in C.$$

Proof. As T and S are mixed strong C-A-pseudomonotone with respect to f and g, it follows that $(I) \Rightarrow (II)$. Conversely, suppose that (II) holds. That is, find $f(x) \in K$ such that

$$h(f(x), g(y)) + \langle T(f(y)) + S(g(y)), A(f(y), g(x))\rangle \in C. \tag{3.21}$$

Putting $x_\lambda = \lambda y + (1-\lambda)x$, for $\lambda \in [0,1]$ in (3.21), we obtain

$$h(f(x), g(x_\lambda)) + \langle T(f(x_\lambda)) + S(g(x_\lambda)), A(f(x_\lambda), g(x))\rangle \in C. \tag{3.22}$$

As A is affine in both arguments with respect to f and g, and $A(f(x), g(x)) = 0$, (3.22) implies that

$$h(f(x), g(x_\lambda)) + \lambda\langle T(f(x_\lambda)) + S(g(x_\lambda)), A(f(y), g(x))\rangle \in C. \tag{3.23}$$

Since h is affine in second argument with respect to g and $h(f(x), g(x)) = 0$, from (3.23), we get

$$\lambda h(f(x), g(y)) + \lambda\langle T(f(x_\lambda)) + S(g(x_\lambda)), A(f(y), g(x))\rangle \in C. \tag{3.24}$$

As C is a cone, therefore

$$h(f(x), g(y)) + \langle T(f(x_\lambda)) + S(g(x_\lambda)), A(f(y), g(x))\rangle \in C. \tag{3.25}$$

On contrary suppose that

$$\{h(f(x), g(y)) + \langle T(f(x_\lambda)) + S(g(x_\lambda)), A(f(y), g(x))\rangle\} \bigcap (Y \setminus C) \neq \emptyset.$$

As T and S are mixed A-hemicontinuous, we have

$$\{h(f(x), g(y)) + \langle T(f(x)) + S(g(x)), A(f(y), g(x))\rangle\} \bigcap (Y \setminus C) \neq \emptyset,$$

for sufficiently small λ, which contradicts (3.25). Therefore, we have

$$h(f(x), g(y)) + \langle T(f(x)) + S(g(x)), A(f(y), g(x))\rangle \notin -C \setminus \{0\},$$

and hence (I) holds. This complete the proof. □

Theorem 3.4. *Let $f, g : K \to K$, $T, S : K \to L(X, Y)$, $h : K \times K \to Y$ and $A : K \times K \to X$ be the mappings such that all the assumptions (i)-(v) of Theorem 3.1 are satisfied. In addition, assume that the following conditions are satisfied:*

- $(vi)'$ *for each $g(y) \in K$, the set $\{f(x) \in K : h(f(x), g(y)) + \langle T(f(x)) + S(g(x)), A(f(y), g(x))\rangle \in -C \setminus \{0\}\}$ is open in K,*
- $(vii)'$ *there exists a nonempty compact and convex subset D of K and for each $f(x) \in K \setminus D$, there exists $g(u) \in D$ such that*

$$\{f(x) \in K : h(f(x), g(u)) + \langle T(f(x)) + S(g(x)), A(f(u), g(x))\rangle \in -C \setminus \{0\}\},$$

- $(viii)'$ *there exists a family $\{C_i, Z_i\}_{1 \le i \le n}$ satisfying conditions (i) and (ii) of Definition 2.3 and the following condition for each $1 \le i \le n$, there exist $1 \le k \le n$ such that*

$$\{f(x) \in K : h(f(x), g(y)) + \langle T(f(x)) + S(g(x)), A(f(y), g(x))\rangle \notin -C \setminus \{0\}, \quad \forall \quad g(y) \in C_k\} \subseteq Z_i.$$

Then, there exists $f(x) \in K$ such that

$$h(f(x), g(y)) + \langle T(f(x)) + S(g(x)), A(f(y), g(x))\rangle \notin -C \setminus \{0\}, \forall g(y) \in K.$$

Proof. Let $F : K \to 2^D$ be defined by

$$\begin{aligned} F(g(y)) &= \{f(x) \in D : h(f(x), g(y)) + \langle T(f(x)) \\ &\quad +S(g(x)), A(f(y), g(x))\rangle \notin -C \setminus \{0\}\}, \ \textit{for all } g(y) \in K. \end{aligned}$$

Obviously, for all $g(y) \in K$

$$\begin{aligned} F(g(y)) &= \{f(x) \in K : h(f(x), g(y)) + \langle T(f(x)) \\ &\quad +S(g(x)), A(f(y), g(x))\rangle \notin -C \setminus \{0\}\} \bigcap D. \end{aligned}$$

As $F(g(y))$ is closed subset of D and D is compact, therefore $F(g(y))$ is compactly closed.

Now, we show that, for any finite subset $\{g(y_1), g(y_2), g(y_3), \cdots, g(y_n)\}$ of K, $\bigcap_{1\leq i\leq n} F(g(y_i)) \neq \phi$. Let $E = Co\{D \bigcup \{g(y_i)\}_{1\leq i\leq n}\}$. Then, by Lemma 2.1, E is a compact and convex subset of K.

Let $H : E \to 2^E$ be defined by

$$\begin{aligned} F(g(y)) &= \{f(x) \in E : h(f(x), g(y)) + \langle T(f(x)) \\ &\quad +S(g(x)), A(f(y), g(x))\rangle \notin -C \setminus \{0\}\}, \ \textit{for all } g(y) \in K. \end{aligned}$$

First, we show that H is a KKM-mapping. On contrary, suppose that H is not a KKM-mapping, then there exists $f(v) \in Co\{g(y_i)\}_{1\leq i\leq n}$ such that for $\lambda_i \geq 0$, $1 \leq i \leq n$ with $\sum_{i=1}^{n} \lambda_i = 1$ such that $v = \sum_{i=1}^{n} \lambda_i y_i$, we have

$$\sum_{i=1}^{n} \lambda_i g(y_i) \notin \bigcup_{i=1}^{n} g(y_i),$$

which implies that

$$h(f(v), g(y_i)) + \langle T(f(v)) + S(g(v)), A(f(y_i), g(v))\rangle \notin -C \setminus \{0\}. \tag{3.26}$$

Since h and A are affine in the second argument with respect to g, (3.26) implies that

$$\begin{aligned} & h(f(v), g(v)) + \langle T(f(v)) + S(g(v)), A(f(v), g(v))\rangle \\ = \ & h(f(v), \sum_{i=1}^{n} \lambda_i g(y_i)) + \langle T(f(v)) + S(g(v)), A(\sum_{i=1}^{n} \lambda_i f(y_i), g(v))\rangle \\ = \ & \sum_{i=1}^{n} \lambda_i h(f(v), g(y_i)) + \sum_{i=1}^{n} \lambda_i \langle T(f(v)) + S(g(v)), A(f(y_i), g(v))\rangle \\ = \ & \sum_{i=1}^{n} \lambda_i \{h(f(v), g(y_i)) + \langle T(f(v)) + S(g(v)), A(f(y_i), g(v))\rangle\} \\ \in \ & -C \setminus \{0\}. \end{aligned} \tag{3.27}$$

Since $h(f(x), g(x)) = 0 = A(f(x), g(x))$, (3.27) implies that $0 \in C \setminus \{0\}$, which is a contradiction. Hence, H is a KKM-maping.

As $H(y)$ is closed subset of E, therefore it is compactly closed. From assumptions $(viii)'$, it is clear that the family $\{(C_i, Z_i)\}_{1\leq i\leq n}$ satisfies the condition

$\bigcap_{g(y)\in C_k} H(g(y)) \subseteq Z_i$ and therefore it is a coercing family for H. By Theorem 2.1, we have

$$\bigcap_{g(y)\in E} H(g(y)) \neq \phi.$$

Thus, we conclude that there exists $g(y_0) \in \bigcap_{g(y)} H(g(y))$. To prove that $g(y_0) \in D$, on contrary suppose that $g(y_0) \in E \setminus D$. Then condition $(vii)'$ implies that there exists $g(u) \in D$ such that

$$h(f(y_0), g(u)) + \langle T(f(y_0)) + S(g(y_0)), A(f(u), g(y_0))\rangle \in -C \setminus \{0\},$$

which contradicts the fact that $g(y_0) \in H(g(y))$ and hence $g(y_0) \in D$. Since $F(g(y_i)) = H(g(y_i)) \bigcap D$, for each $g(y_i) \in E$. It follows that $g(y_0) \in \bigcap_{i=1}^{n} F(g(y_i))$, that is, $\bigcap_{i=1}^{n} F(g(y_i)) \neq \phi$, for a finite subset $\{g(y_i)\}_{1\leq i\leq n} \subset K$. As $F(g(y))$ is closed and compact, it follows that, for each $g(y) \in K$, there exists $f(x) \in D$ such that $f(x) \in \bigcap_{g(y)\in K} F(g(y))$. Hence, there exists $f(x) \in K$ such that for all $g(y) \in K$,

$$h(f(x), g(y)) + \langle T(f(x)) + S(g(x)), A(f(y), g(x)) \notin -C \setminus \{0\}.$$

This complete the proof. □

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References

[1] G. Allen: *Variational inequalities, complementarity problems, and duality theorems*, J. Math. Anal. Appl., **58 (1)**, (1977) 1–10.

[2] E. Blum and W. Oettli: *From optimization and variational inequalities to equilibrium problems*, Math. Stud., **63 (1-4)**, (1994), 123–145.

[3] H. Ben-El-Mechaiekh, S. Chebbi and M. Florenzano : *A generalized KKMF principle*, J. Math. Anal. Appl., **309 (2)**, (200), 583–590.

[4] M. Bianchi and R. Pini: *A note on equilibrium problems with properly quasimonotone bifunctions*, J. Global Optim., **20** (2001), 67–76.

[5] M. Bianchi and R. Pini: *Coercivity conditions for equilibrium problems*, J. Optim. Theory Appl., **124** (2005), 79–92.

[6] M. Bianchi and S. Schaible: *Generalized monotone bifunctions and equilibrium problems*, J. Optim. Theory Appl., **90** (1996), 31–43.

[7] H. Ben-El-Mechaiekh, P. Deguire and A. Granas: *Une alternative non lineaire en analyse convexe et applications*, Comptes Rendus des Seances de l'Academie des Sciences. Serie I. Mathematique, **295 (3)**, (1982), 257-259.

[8] G.-Y. Chen and X.-Q. Yang: *The vector complementary problem and its equivalences with the weak minimal element in ordered space*, J. Math. Anal. Appl., **153 (1)** (1990), 136–158.

[9] K. Fan: *Some properties of convex sets related to fixed point theorems*, Mathematische Annalen, **266 (4)**, (1984) 519-537.

[10] F. Giannessi: *Theorems of alternative, quadratic programs and complementarity problems in Variational inequalities and complementarity problems*, R.W. Cottle, F. Giannessi and L. J. Lions, Eds., **151-186**, John Wiely & Sons, New York, USA, 1980.

[11] A. N. Iusem, G. Kassay and W. Sosa: *On certain conditions for the existence of solutions of equilibrium problems*, Mathematical Programming, Ser. B, **116** (2009), 259-273.

[12] S. Kum and M. M. Wong: *An extension of a generalized equilibrium problem*, Taiwanese J. Math., **15 (4)**, (2011), 1667-1675.

[13] G. Kassay : *The equilibrium problem and related topics*, Risoprint Cluj-Napoca, (2000).

[14] S. Karamardian: *Generalized complementarity problem*, J. Optim. Theory Appl., **8**, (1971), 161–168.
[15] M. Rahaman, R. Ahmad: *Weak and strong mixed vector equilibrium problems on non-compact domain*, J. Egyptian Math. Soci., **23**, 352-355, DOI:10.1016/j.joems.2014.06.007 (2015).
[16] M. Rahaman, A. Kilicman and R. Ahmad: *Extended mixed vector equilibrium problems*, A.A.A., 2014, Article ID 376759, 6 pages, DOI:10.1155/2014/376759 (2014).
[17] A. E. Taylor: *An introduction to functional analysis*, John Wiley and Sons, New York, USA, (1963).

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